Langlands correspondence and modularity conjecture.

1.5 Langlands + rigidity

1.6 RHS $n=2$ + modular forms

1.7 LHS $n=2$ + elliptic curves

recall (ACFT)

$$\mathrm{Gal}(F^{ab}/F) \simeq \left(F^\times \backslash \mathbb{A}_F^\times\right)_{c.c.}$$

$$\mathrm{Gal}(F^{ab,v}/F) \simeq \left(F^\times \backslash \mathbb{A}_F^\times / \mathcal{O}_v^\times\right)_{c.c.}$$

$$\begin{array}{ccc} \cup & & \cup \\ Fr_v & \longleftrightarrow & (1,\dots,1,t_v,1,\dots,1) \end{array}$$

e.g. $\mathrm{Gal}(\overline{F}/F)$?

"Tannakian philosophy"

group $\longleftrightarrow$ category of representations

1-dim'l $\mathbb{C}$-linear reps of $\mathrm{Gal}(F^{ab}/F)$	$\overset{\text{ACFT}}{\simeq}$	1-dim'l $\mathbb{C}$-linear reps of $F^\times \backslash \mathbb{A}_F^\times$
$\parallel$		$\parallel$
1-dim'l $\mathbb{C}$-linear reps of $\mathrm{Gal}(\bar{F}/F)$	$\simeq$	repns of $\mathbb{A}_F^\times$ in fcns on $F^\times \backslash \mathbb{A}_F^\times$.

Langlands correspondence

n-dim'l $\mathbb{C}$-linear reps of $\mathrm{Gal}(\bar{F}/F)$	$\longleftrightarrow$	repns of $GL_n(\mathbb{A}_F)$ in fcns on $GL_n(F) \backslash GL_n(\mathbb{A}_F)$

$\underline{n=1}$ ACFT

<u>$n=2$</u> unproven (?)
some specific cases are known.
e.g. Taniyama-Shimura conjecture

<u>rigidity?</u> $v \subseteq \mathcal{O}_F$, $\bar{v} \subseteq \mathcal{O}_{\bar{F}}$ lying over it

$D_{\bar{v}}$:= elmts of $Gal(\bar{F}/F)$ fixing v.

$I_{\bar{v}} := \ker(D_{\bar{v}} \xrightarrow{\sim} Gal(\bar{F}_v/F_v) \to Gal(\bar{k}_v/k_v))$

an n-diml $\mathbb{C}$ linear repn σ of $Gal(\bar{F}/F)$ is <u>unramified</u> if

$$I_{\bar{v}} \subseteq \ker \sigma$$

in this case we have a well-defined conjugacy class $Fr(v, \sigma)$ in GL_n.
(in Frenkel)

Hecke eigenvalues are our RHS objects.

rigidity: there is some kind of "matching condition" between Fr evalues + Hecke evalues. (?)

source: Bernstein + Gelbart. [27].

1.6 automorphic representations of $GL_2(\mathbb{A}_{\mathbb{Q}})$

from now on, $F = \mathbb{Q}$.

Def^n $f : GL_2(\mathbb{Q}) \backslash GL_2(\mathbb{A}) \to \mathbb{C}$

is smooth if it is locally constant on $GL_2(\mathbb{A}^f)$, $\mathbb{A}^f := \prod_p{}' \mathbb{Q}_p$, and smooth on $GL_2(\mathbb{R})$.

<u>Def^n</u> $\chi : Z(GL_2(\mathbb{A}))(\simeq \mathbb{A}^\times) \to \mathbb{C}^\times$,

$\rho \in \mathbb{C}$

$\mathcal{C}_{\chi,\rho}(GL_2(\mathbb{Q})\backslash GL_2(\mathbb{A}))$ is space of smooth fcns f s.t.

- translates of f by elmts of $K := \prod_p GL_2(\mathbb{Z}_p) \times O_2$ form fdim vspace
- <u>central character condition</u>

$$f(gz) = \chi(z) f(g) \quad \forall z \in Z(GL_2 \mathbb{A})$$

$C \cdot f = \rho f$, C <u>Casimir element</u> of $GL_2\mathbb{C}$.

- f is bounded.

  ($\underline{qn}$: in other sources "slow growth", weaker condition. ?)

- <u>cuspidality</u>

$$\int_{\mathbb{Q}\backslash\mathbb{A}} f\left(\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} g\right) du = 0$$

$$\forall g \in GL_2(\mathbb{Q})\backslash GL_2(\mathbb{A})$$

$GL_2(\mathbb{A}_f)$ representation.

and also repn $(\mathfrak{gl}_2, O_2)$.

<u>Fact</u> cuspidality + central character condition

$\Rightarrow$ $\mathcal{C}_{\chi,\rho}$ is a direct sum of distinct irreducible representations

These are called <u>cuspidal automorphic representation</u>.

<u>How to associate a modular form?</u>

$\boxed{\chi = 1}$ $\pi = \bigotimes' \pi_p \otimes \pi_\infty$

By first condition, for all but finitely many p, $\exists!\, v_p \in \pi_p$ s.t. v_p is invariant under action of $GL_2(\mathbb{Z}_p)$

("π_p is unramified")

Fact for π_p ramified, $\exists$ vector v_p invariant under

$$K_p' = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \,\middle|\, c \equiv 0 \bmod p^{n_p}\mathbb{Z}_p \right\}$$

for some n_p.

Then the space of K'-invariants in π is

$$\tilde{\pi}_\infty := \bigotimes_p \mathbb{C} v_p \otimes \pi_\infty .$$

this is a repn of $GL_2(\mathbb{A})$ on space of fcns on $GL_2(\mathbb{Q}) \backslash GL_2(\mathbb{A}) / K'$

strong approximation theorem

$$GL_2(\mathbb{Q})\backslash GL_2(\mathbb{A})/K' \simeq \Gamma_0(N)\backslash GL_2^+(\mathbb{R})$$

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \;\middle|\; c \equiv 0 \bmod N\mathbb{Z} \right\}$$

$$N = \prod p^{n_p}$$

$\rightsquigarrow$ (central character condition)

$\tilde{\pi}_\infty$ rep of $GL_2(\mathbb{A})$ w/ coeffs

in fcns on $\Gamma_0(N)\backslash SL_2(\mathbb{R})$.

Assume $\rho = \frac{k(k-2)}{2}$ $k \in \mathbb{Z}_{\geq 2}$.

Fact as $\mathfrak{sl}_2$-module, π_∞ is isom

to direct sum of irred highest weight $-k$ & irred lowest weight k repn former gen. by v_∞ ($\swarrow$ weight $-k$.) Then v_∞ generates π_∞ as $\mathfrak{gl}_2$-module.

$\phi_\pi : SL_2(\mathbb{R}) \longrightarrow \mathbb{C}$ fcn associated to v_∞.

$\rightsquigarrow \phi_\pi(\gamma g) = \phi_\pi(g) \quad \forall \gamma \in \Gamma_0(N)$.

$$\rightsquigarrow \phi_\pi\left(g\begin{pmatrix}\cos\theta & \sin\theta \\ -\sin\theta & \cos\theta\end{pmatrix}\right) = e^{ik\theta}\phi_\pi(g)$$

(since $X_0 v_\infty = -k v_\infty$)

$$f_\pi : \mathbb{H} \simeq SL_2(\mathbb{R})/SO_2 \longrightarrow \mathbb{C}$$

$$g = \begin{pmatrix} a & b \\ c & d\end{pmatrix} \longmapsto \phi_\pi(g)(ci+d)^k$$

well-defined, holomorphic. $(X_+ \cdot v_\infty = 0)$

$\rightsquigarrow$ f_π is a <u>modular form</u> of <u>weight k and level N</u>.

$$f_\pi = \sum a_n q^n.$$

<u>cuspidality</u> $\Rightarrow$ $a_0 = 0$.

normalise so that $a_1 = 1$.

<u>Hecke operators</u>

Assume π_p is unramified, $v_p \in \pi_p$

Define $H_{1,p}$, $H_{2,p}$ oprs on π_p

$$H_{1,p} \cdot v_p = \int_{M_2^1(\mathbb{Z}_p)} g \cdot v_p \, dg$$

$$H_{2,p} \cdot v_p = \int_{M_2^2(\mathbb{Z}_p)} g \cdot v_p \, dg$$

$$M_2^1(\mathbb{Z}_p) = GL_2(\mathbb{Z}_p) \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} GL_2(\mathbb{Z}_p)$$

$$M_2^2(\mathbb{Z}_p) = GL_2(\mathbb{Z}_p) \begin{pmatrix} p & 0 \\ 0 & p \end{pmatrix} GL_2(\mathbb{Z}_p)$$

normalise so that $\mathrm{vol}(GL_2(\mathbb{Z}_p)) = 1$.

now, if $\chi = 1$, then $H_{2,p} \cdot v_p = v_p$.

by uniqueness of v_p,

$$H_{1,p} \cdot v_p = h_{1,p} v_p \text{ some } h_{1,p} \in \mathbb{C}$$

Hecke eigenvalues.

<u>fact</u> $H_{1,p}$ = "classical" Hecke operator.

So in particular $h_{1,p} = a_p$.

1.7 elliptic curves

$E/\mathbb{Q}$. <u>Fact</u> $H^1_{\acute{e}t}(E_{\overline{\mathbb{Q}}}, \mathbb{Q}_\ell) \simeq \mathbb{Q}_\ell^2$

ℓ prime.

Since $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ acts on this,

we get a 2-dim'l $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ repn.

$\sigma_{E,\ell}$.

unramified at primes of good reduction.

calculation $(\ell \neq p)$

(indep of ℓ)

$$\mathrm{Tr}\, \sigma_{E,\ell}(\mathrm{Fr}_p) = p + 1 - \# E(\mathbb{F}_p)$$

Langlands correspondence:

$$\mathrm{Tr}\, \sigma_{E,\ell}(\mathrm{Fr}_p) = h_{1,p} \quad \forall \text{ unramified } p.$$

Taniyama-Shimura conjecture

$E/\mathbb{Q}$. There exists a cuspidal modular form $f_E(q) = \sum a_n q^n$, $a_1 = 1$, with $a_p = p+1 - \# E(\mathbb{F}_p)$ for all but finitely many p.

<u>Proven</u>. by A. Wiles.

implies Fermat's last theorem.