GLC _ Chemnitz

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§ Last time.

- Chronecker_Weber theorem [KW]: the max ab. extension $\mathbb{Q}^{ab}$ of $\mathbb{Q}$ is obtained by adjoining all roots of unity.

$$\Longrightarrow \quad \mathrm{Gal}(\mathbb{Q}^{ab}/\mathbb{Q}) \simeq (\hat{\mathbb{Z}})^{*} = \prod_{p \in \mathrm{Spec}\,\mathbb{Z}} (\mathbb{Z}_p)^{*}$$

$$\text{with } \hat{\mathbb{Z}} := \varprojlim \mathbb{Z}/N\mathbb{Z} = \prod_{p \in \mathrm{Spec}\,\mathbb{Z}} \mathbb{Z}_p$$

w.r.t syst. of surjections $P_{NM} : (\mathbb{Z}/N\mathbb{Z})^{*} \to (\mathbb{Z}/M\mathbb{Z})^{*}$ when $M \mid N$.

$\nexists$ [KW] theorem describing F^{ab} for a gen. # field F.

But: ACFT in the case $F = \mathbb{Q}$ gives:

$$\mathrm{Gal}(\mathbb{Q}^{ab}/\mathbb{Q}) \simeq \text{connected components of } \mathbb{Q}^{*}\backslash \mathbb{A}_{\mathbb{Q}}^{*}$$

$$\text{where } \mathbb{Q}^{*}\backslash \mathbb{A}_{\mathbb{Q}}^{*} \simeq \prod_{p \in \mathrm{Spec}\,\mathbb{Z}} (\mathbb{Z}_p)^{*} \times \mathbb{R}_{>0}.$$

and ACFT gives more generally for a field F:

$$\boxed{\begin{array}{ccc} \mathrm{Gal}(F^{ab}/F) & \cong & \left(F^* \backslash \mathbb{A}_F^*\right)_{c.c.} \\ \| & & \\ \left(\mathrm{Gal}(\bar{F}/F)\right)^{ab} & & \end{array}}$$

What is $\mathbb{A}_F$?

$F \rightsquigarrow \mathcal{O}_F \rightsquigarrow \mathrm{Spec}\, \mathcal{O}_F$

$\mathfrak{p} \in \mathrm{Spec}\, \mathcal{O}_F \rightsquigarrow$ valuation (norm) v

archimedean

$v(x+y) \leq v(x) + v(y)$

triang. ineq.

non-archimedean

$v(x+y) \geq \min\{v(x), v(y)\}$

non-arch. property

$\rightsquigarrow$ completion : $F_v \cong \mathbb{R}$ or $\mathbb{C}$ $\qquad$ $F_v \cong$ finite extension of $\mathbb{Q}_p$.

Ostrowski theorem

$$\mathbb{A}_F := \prod_{\substack{v \in \mathcal{O}_F \\ v \text{ non-arch.}}} F_v \times \prod_{\substack{v \in \mathcal{O}_F \\ v \text{ arch.}}} \mathcal{O}_v$$

§ Rigidity ACFT : $p \in \mathrm{Spec}\, \mathbb{Z}$

$q = p^m, \ m \in \mathbb{Z}_{\geq 1}$

F : a # fiel

$$\mathrm{Gal}\, \mathbb{F}_q / \mathbb{F}_p \cong \mathbb{Z}/m\mathbb{Z}$$

$$Fr : \mathbb{F}_q \longrightarrow \mathbb{F}_q \qquad \longmapsto 1 \bmod m$$

$$x \mapsto x^p$$

$x \in \mathbb{F}_q$

$$Fr^m(x) = (x^p)^m = x^{p^m}$$

$$= x^q = x$$

$$\langle Fr \rangle = \mathrm{Gal}(\mathbb{F}_q / \mathbb{F}_p)$$

$\forall\, m' \in \mathbb{Z}_{\geq 1}$, $q' := q^{m'}$

$\rightsquigarrow \mathbb{F}_q \hookrightarrow \mathbb{F}_{q'} \rightsquigarrow$ system of inclusions I

$$\overline{\mathbb{F}_p} = \bigcup_{\substack{q = p^m \\ m > 0}} \mathbb{F}_q = \varinjlim_m \mathbb{F}_{p^m} \quad \text{w.r.t. } I.$$

$$\rightsquigarrow \mathrm{Gal}(\overline{\mathbb{F}}_p / \mathbb{F}_p) \cong \varprojlim_m \mathrm{Gal}(\mathbb{F}_{p^m} / \mathbb{F}_p),$$

$$= \varprojlim \mathbb{Z}/m\mathbb{Z} =: \hat{\mathbb{Z}}$$

similarly, $\boxed{\mathrm{Gal}(\overline{\mathbb{F}}_q / \mathbb{F}_q) \cong \hat{\mathbb{Z}}}$ ☆

$$\forall m \in \mathbb{Z}_{\geq 1}, \quad \exists\ \hat{\mathbb{Z}} \longrightarrow \mathbb{Z}/m\mathbb{Z}$$
$$\xi \longmapsto 1 \bmod m$$

$$\rightsquigarrow \langle \xi \rangle \cong \mathbb{Z} \underset{\text{complete}}{\rightsquigarrow} \hat{\mathbb{Z}}$$

so ξ is "topological generator" of $\hat{\mathbb{Z}}$, called the Frobenius automorphism of $\mathbb{F}_q$.

(closure of $\langle \xi \rangle$ is $= \hat{\mathbb{Z}}$)

Q: In view of ☆, can we relate $\mathrm{Gal}(\overline{F}/F)$ to $\mathrm{Gal}(\overline{\mathbb{F}}_q / \mathbb{F}_q)$?

A: yes!

Let K/F be a finite ext. of # fields.

Let $v \in \operatorname{Spec} \mathcal{O}_F$.

$v \subset \mathcal{O}_F \subset \mathcal{O}_K \implies v \subset \mathcal{O}_K$

$\implies v = w_1 \cdot \ldots \cdot w_g$, $w_i \in \operatorname{Spec} \mathcal{O}_K$, $i \in 1, \ldots, g$.

$\mathcal{O}_K$ is Dedekind ring

$w := w_i$ for some $i \in \{1, \ldots, g\}$

Look at residue fields:

$\mathcal{O}_F / v \cong \mathbb{F}_q$, for some q

$\mathcal{O}_K / w \cong \mathbb{F}_{q'}$, $q' = q^m$ for some $m \in \mathbb{Z}_{\geq 1}$

$$\begin{array}{cccc} w & & \mathcal{O}_K & \subset K \\ | & & & \\ v & \in & \mathcal{O}_F & \subset F \\ | & & & \\ p & \in & \mathbb{Z} & \subset \mathbb{Q} \end{array}$$

$\underbrace{\mathcal{O}_F/v}_{\text{finite ext of } \mathbb{F}_p}$ contains $\mathbb{F}_p$.

$\mathcal{O}_K/w \,\big/\, \mathcal{O}_F/v$ is a finite ext. of finite fields of degree n.

$\rightsquigarrow \mathrm{Gal}\left(\mathcal{O}_K/w \,\big/\, \mathcal{O}_F/v\right) \cong \mathbb{Z}/n\mathbb{Z}$.

Def: Decomposition group D_w of w is:

$$D_w : \{ \sigma \in \mathrm{Gal}(K/F) \mid \sigma(w) \subseteq w \}$$

Since $\sigma \in \mathrm{Gal}(K/F)$ preserve F, preserve v,

$$\rightsquigarrow d_w : D_w \twoheadrightarrow \underbrace{\mathrm{Gal}\left(\mathcal{O}_K/w \,\big/\, \mathcal{O}_F/v\right)}_{\cong \mathbb{Z}/n\mathbb{Z}}$$

Def: • The inertia grp I_w of w is $I_w := \ker d_w$.

- Call K/F unramified if $I_w = \{1\}$

Let K/F be unramified ext.

$$\rightsquigarrow \quad D_w \cong \mathrm{Gal}\left(\mathcal{O}_K/w \,\middle/\, \mathcal{O}_F/v\right) \cong \mathbb{Z}/n\mathbb{Z}$$

$$\mathrm{Fr}[w] \longmapsto * \longmapsto 1 \bmod n$$

Let $w' = w_i$ for some $i \in \{1, \dots, g\}$

Then:
$$\left.\begin{aligned} D_{w'} &= s\, D_w\, s^{-1} \\ I_{w'} &= s\, I_w\, s^{-1} \\ \mathrm{Fr}[w'] &= s\, \mathrm{Fr}[w]\, s^{-1} \end{aligned}\right\} \text{for some } s \in \mathrm{Gal}(K/F)$$

$\rightsquigarrow$ conj class of $\mathrm{Fr}[w]$ in $\mathrm{Gal}(K/F)$ depend only on v. (not dep. on the choice of w_i)

so $\mathrm{Fr}(v) := \mathrm{Fr}[w]$

<u>Rem:</u> If K/F is ramified then $\mathrm{Fr}[w]$ would just be a coset in D_w/I_w and will not be

an element of D_w.

<u>E.g:</u> $\mathcal{O}_F = \mathbb{Z}$, $\mathcal{O}_K = \mathbb{Z}[\xi_N]$, $\mathrm{Gal}(K/F) = (\mathbb{Z}/N\mathbb{Z})^*$
$F = \mathbb{Q}$

Fact: $\mathbb{Q}(\xi_N)$ is unramified at $p \in \operatorname{Spec} \mathbb{Z}$ iff $p \nmid N$. (Exercise)

If K/F is unramif. then: $(p) = \mathfrak{P}_1 \cdots \mathfrak{P}_r$

$\mathfrak{P}_i \in \operatorname{Spec} \mathcal{O}_K$

Now $\mathcal{O}_F / (p) = \mathbb{F}_p \implies$ Frobenius autom. corresponds to raising to the power p.

$\rightsquigarrow \mathrm{Fr}(p) : K \to K$
$\xi_N \mapsto \xi_N^p$.

<u>Q:</u> How to define $\mathrm{Fr}(p)$ for $\mathrm{Gal}(\mathbb{Q}^{ab}/\mathbb{Q})$?

$\mathbb{Q}^{ab,p} :=$ maximal abelian ext unramified at p.

$$\rightsquigarrow \mathrm{Gal}(\mathbb{Q}^{ab,p}/\mathbb{Q}) \cong \mathrm{Gal}(\mathbb{Q}^{ab}/\mathbb{Q})/\underbrace{I_p}$$

inertia group of p.

$$[KW]: \quad \mathbb{Q}^{ab} = \mathbb{Q}(\text{all primitive roots of unity})$$

$$\mathbb{Q}^{ab,p} = \mathbb{Q}\left(\text{all primitive roots of } 1,\ \xi_N, \text{ s.t. } p \nmid N\right)$$

$$\rightsquigarrow \mathrm{Gal}(\mathbb{Q}^{ab,p}/\mathbb{Q}) \cong \prod_{\substack{p' \in \mathrm{Spec}\,\mathbb{Z} \\ p' \neq p}} \mathbb{Z}_{p'}$$

$$\cong \underbrace{\left(\mathbb{Q}^* \backslash \mathbb{A}^*_{\mathbb{Q}} / \mathbb{Z}_p^*\right)_{c.c.}}$$

connected components

geometric Froben. auto.

$$\overbrace{(Fr(p))^{-1}} \longmapsto (1, \dots, 1, p, 1, \dots \dots)$$

$\uparrow$ factor $\mathbb{Q}_p$.

(nice interpretation of the reciprocity laws

$$p = a^2 + b^2 \iff p \equiv 1\ [4] \,)$$

(modern proof of reciprocity law follows from ACFT)

<u>Rem:</u>

$$\left.\begin{array}{l} \omega = \omega_1 \cdots\cdots \omega_g \\ Fr(p)^f = 1 \end{array}\right\} \Rightarrow \deg K/F = f \cdot g$$

<u>Q:</u> How to define $Fr(p)$ for $Gal(F^{ab}/F)$?

ACFT: $Gal(F^{ab}/F) \cong F^* \backslash \mathbb{A}_F^*$

taking completion

$$I_v \longmapsto (\widehat{\mathcal{O}_v})^* =: \mathcal{O}_v$$

$$\rightsquigarrow Gal(F^{ab,v}/F) \cong \left(F^* / \mathbb{A}_F^* \backslash \mathcal{O}_v^*\right)_{c.c.}$$

$$(Fr(v))^{-1} \longmapsto (1, \dots, 1, \underset{\substack{\uparrow \\ Fr_v\text{ - factor}}}{t_v}, 1, \dots\dots)$$

t_v: generator of the maximal ideal of $\mathcal{O}_v$

Chebotarev theorem: Frobenius conj classes generate a dense subset of the Galois grp

Chebotaref $\Longrightarrow$ Rigidification of ACFT, i.e., $\exists!$ isomorphism that "encompasses all possible reciprocity laws that one can write in K/F."

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