Vertex Algebras and moduli

Definition A *vertex algebra* is a tuple $\left(V = \bigoplus_{i=0}^{\infty} V_i \,,\ |0\rangle \in V_0 \,,\right.$

$T : V \xrightarrow{\text{linear deg } 1} V \,,\quad \left. Y(-, z) : V \longrightarrow (\operatorname{End} V)[[z^{\pm 1}]] \right).$

such that

- $Y(|0\rangle, z) = \mathrm{id}$, $Y(A,z)|0\rangle = A z^0 +$ higher order terms
- $[T, Y(A,z)] = \partial_z Y(A,z)$, $T|0\rangle = 0$.
- $Y(A,z)$ is a field & all $Y(A,z)$ are local wrt each other.

field $B(z) \in (\operatorname{End} V)[[z^{\pm 1}]]$ st $B(z) \cdot v \in V((z))$

local $A(z), B(w)$ local wrt each other if $\exists N$

$$(z-w)^N [A(z), B(w)] = 0.$$

(Vertex Algebras & Algebraic Curves)

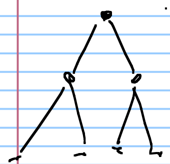
Example 1 recall $\mathfrak{g}$ ss $\rightsquigarrow \hat{\mathfrak{g}}$.

$$0 \to \mathbb{C}\mathbb{1} \to \hat{\mathfrak{g}} \to \mathfrak{g}((t)) \to 0$$

$k \in \mathbb{Z}$. Define $V_k(\hat{\mathfrak{g}}) = U(\hat{\mathfrak{g}}) \otimes_{U(\mathfrak{g}[[t]] \oplus \mathbb{C}\mathbb{1})} \mathbb{C}_k$

$\mathbb{C}_k$ repn of $\mathfrak{g}[[t]] \oplus \mathbb{C}\mathbb{1}$ where $\mathfrak{g}[[t]] \curvearrowright 0$ & $\mathbb{1}$ acts as k.

rmk $V_k(\mathfrak{g}) \simeq \bigoplus_{\lambda \in \hat{P}^k_+} L_\lambda$



$$|0\rangle = 1 \otimes 1 \otimes 1$$

Choose basis $\{J^a\}$ of $\mathfrak{g}$. $J^a_n := J^a \otimes t^n \in \mathfrak{g}((t))$

$$Y(|0\rangle, z) = id, \quad Y(J^a_{-1}|0\rangle, z) := J^a(z) := \sum_{n \in \mathbb{Z}} J^a_n z^{-n-1}$$

reconstruction theorem (Thm 2.3.11) to determine rest of Y.

$T|0\rangle = 0$, $[T, J^a_n] = -n J^a_{n-1}$

Thm $V_k(\mathfrak{g})$ is a vertex algebra.

Example 2 (Virasoro algebra)

$0 \to \mathbb{C}C \to Vir \to Der\,\mathcal{K} \to 0$.

$\mathcal{K} = \mathbb{C}((t))$, $\mathcal{O} = \mathbb{C}[[t]]$

$[f\partial_t, g\partial_t] = (fg' - f'g)\partial_t + \frac{-1}{12}\left(\mathrm{Res}_{t=0}\, f g''' dt\right)C$

Vir gen by C & $L_n := -t^{n+1}\partial_t$

$c \in \mathbb{Z}$ $\quad Vir_c = \underset{U(Der\,\mathcal{O} \oplus \mathbb{C}C)}{U(Vir) \otimes} \mathbb{C}_c$ "central charge c"

$|0\rangle = 1 \otimes 1 \otimes 1$, $T = L_{-1}$, $Y(L_{-2}|0\rangle, z) := T(z)$

$:= \sum L_n z^{-n-2}$.

Thm Vir_c is a VA.

§2. conformality

defn A VA V is <u>conformal of cent charge c</u> if $\exists \neq 0$ homo

$$\mathrm{Vir}_c \longrightarrow V. \qquad (\text{i.p. } \mathrm{Der}\,\mathcal{O} \curvearrowright V)$$

defn A VA V is <u>quasi-conformal</u> if $\exists$ action $\mathrm{Der}\,\mathcal{O} \curvearrowright V$

st $-t\partial_t$ acts as gradation & $-\partial_t$ acts as T.

note qcVA $\Rightarrow$ cVA. **note** for V qcVA, $T(z) := \sum_{n \geq -1} L_n z^{-n-2}$.

eg Vir_c & $V_k(\mathfrak{g})$ qcVA.

<u>$k \neq -h^\vee$</u> $V_k(\mathfrak{g})$ is conformal. $c = \dfrac{k \dim \mathfrak{g}}{k+h^\vee}$

$$T(z) = \frac{1}{2(k+h^\vee)} \sum :J_a(z) J^a(z):$$

<u>$k = -h^\vee$</u> $V_k(\mathfrak{g})$ is <u>not</u> conformal!

corrn fns $\langle 0 | Y(A_1, z_1) \ldots Y(A_n, z_n) | 0 \rangle \in \mathbb{C}((z_1, \ldots, z_n))$.

defn for V $\mathfrak{g}$-VA,

$X = (X, x_1, \ldots, x_n)$ $\quad H_V(M_1, \ldots, M_n)_{\hat{X}}$ & $C_V(M_1, \ldots, M_n)_{\hat{X}}$

(§8.9) $\quad M_i$ irred $\mathfrak{g}$-modules on V. $\quad \left(H_V = \dfrac{\otimes M_i}{\mathfrak{g}_{stab} \cdot \otimes M_i} \right)$

§3. moduli & VAs (motivation/introduction)

$\mathcal{Y}_{x_i}$ is End $\mathcal{V}_{x_i}$ valued section of $\mathcal{V}_{X \setminus x_i}$

fix $V, M_1, \ldots, M_n$.

$\rightsquigarrow$ $\mathcal{O}$-mods $\quad \mathcal{C}_V(M_1, \ldots, M_n), \mathcal{H}_V(M_1, \ldots, M_n) \longrightarrow M_{g,n}$

$\langle 0 | \mathcal{Y}_{x_1}(A_1) \ldots \mathcal{Y}_{x_n}(A_n) | 0 \rangle$ section of $\mathcal{C}_V \otimes \overline{\mathcal{C}_V}$.

V $\mathfrak{g}$VA

Theorem, $\exists!$ unitary connection on $\mathcal{H}_V(M_1, \ldots, M_n)$ compatible w/ some natural ip on $\mathcal{H}_V$.

moreover this connection is projectively flat (not flat!) ($\Rightarrow$ twisted $\mathcal{D}$-mod)

also corrn fcns are horizontal wrt this connection.

Bun$_G$ (G ss Lie group) $V_{\mathfrak{g}}$VA

Can construct

$$\mathcal{C}_V(M_1, \ldots, M_n), \quad \mathcal{L}_V(M_1, \ldots, M_n) \longrightarrow \mathrm{Bun}_G$$

Thm V $\mathfrak{g}$VA, $\exists$ twisted $\mathcal{D}$-module $\Delta(V, M_\cdot)$ whose underlying $\mathcal{O}$-module is $\mathcal{H}_V(M_\cdot)$.

Rmk for $k = -h^\vee$ $\Delta(V, L_?)$ satisfies the Hecke property. (ie $= \mathrm{Aut}_E$)