WZW models and connections on conformal blocks

§1. QFT & CFTs

Physics dictionary

physics	(1+1)d spacetime	space of states	symmetries of our system	observables
maths	X RS (for us!)	$\mathbb{H}$ Hilbert space	for each $U \subseteq X$, a "von Newmann" algebra acting on $\mathbb{H}$.	linear operators on $\mathbb{H}$. invariant under symmetry algebras

aim completely understand the observables.

in QFT we often use "correlation functions". (Ward identities)

Defn M $(1+1)d$ Minkowski space.

A 2d QFT is a pair $(\mathbb{H}, \mathcal{A})$ with $\mathbb{H}$ a Hilbert space and A a functor $A : \left\{ \begin{matrix} \text{open subsets} \\ \text{of } M \end{matrix} \right\} \longrightarrow \left\{ \begin{matrix} \text{von Neumann algebras} \\ \text{acting on } \mathbb{H} \end{matrix} \right\}$.

note $\exists \Omega \in \mathbb{H}$ ground state.

A 2dCFT is a 2dQFT st the symmetry algebras contain all conformal transformations.

$\hookrightarrow$ preserves angles.

("in string theory our states are circles instead of points so invariant under rotation" ?!)

§2. WZW model

2.1 Kac-Moody algebra

$\Delta \longrightarrow \mathfrak{g}$

Defn $\mathfrak{g}$ s.s. Lie algebra. Write $\mathcal{L}(\mathfrak{g}) = \mathbb{C}((t)) \otimes_{\mathbb{C}} \mathfrak{g}$.

st $[P \otimes x, Q \otimes y] := PQ \otimes [x,y]$. loop algebra.

Recall Killing form $K : \mathfrak{g} \times \mathfrak{g} \longrightarrow \mathbb{C}$; $(x,y) \longmapsto \mathrm{Tr}(\mathrm{ad}_x \mathrm{ad}_y)$

Defn $\nu : \mathcal{L}(\mathfrak{g}) \times \mathcal{L}(\mathfrak{g}) \longrightarrow \mathbb{C}$; $(f,g) \longmapsto \mathrm{Res}\left(\left(\frac{d}{dt} f, g\right)\right)$.

$\mathrm{Res}: \mathbb{C}((t)) \longrightarrow \mathbb{C}\ ;\quad t^r \longmapsto \delta_{r,-1}$

$(P\otimes x, Q\otimes y) \longmapsto PQ \otimes \check{K}(x,y)$

Defn the affine Kac-Moody algebra of $\mathfrak{g}$ is $\hat{\mathfrak{g}} = \mathcal{L}(\mathfrak{g}) \oplus \mathbb{C}c$ where c is a formal central element "central extension"
and $(f, g \in \mathcal{L}(\mathfrak{g}))$ $[f,g]_{\hat{\mathfrak{g}}} = [f,g]_{\mathcal{L}(\mathfrak{g})} + \nu(f,g)c$.

Defn a repn V of $\hat{\mathfrak{g}}$ has level $k \in \mathbb{C}$ if c acts on V via multiplication by k.

2.2 WZW model (local)

the WZW model of level k (on $\Delta^\times$) is the 2dCFT with Hilbert space

$$\mathbb{H}_k := \bigoplus_{\lambda \in \hat{P}^k_+} L_\lambda \otimes \overline{L_\lambda}$$

(here, $\hat{P}^k_+$ is set of highest weights of level k

L_λ irred rep of $\hat{\mathfrak{g}}$ level k & weight λ.)

and symmetry algebra $\hat{\mathfrak{g}} \oplus \overline{\hat{\mathfrak{g}}}$

chiral symmetries

anti-chiral symmetries

<u>question</u> what is (are) the ground state(s) of the system?

<u>2.3 WZW model (global)</u>

X RS, $x_1, \dots, x_n \in X$. $\hat{X} = (X, x_1, \dots, x_n)$.

Choose coordinates $t_1, \dots, t_n$ respectively.

Lie algebra $\mathfrak{g}_{out} := \mathfrak{g} \otimes \mathbb{C}[X \setminus \{x_1, \dots, x_n\}]$ "$=$" $\{X \setminus \{x_1, \dots, x_n\} \to \mathfrak{g}\}$

The embedding $\mathfrak{g}_{out} \hookrightarrow \hat{\bigoplus_{i=1}^{n}} \mathcal{L}_i(\mathfrak{g}) = \hat{\bigoplus_{i=1}^{n}} \mathbb{C}((t_i)) \otimes \mathfrak{g}$

lifts to the central extensions, so $\mathfrak{g}_{out} \curvearrowright \bigotimes_{i=1}^{n} L_{\lambda_i}$ for any λ_i

<u>defn</u> the <u>WZW model for $\hat{X}$</u> is

$$\mathbb{H}_{k,\hat{X}} = \mathbb{H}_k^n \quad \text{and symmetry algebra } \mathfrak{g}_{out} \oplus \overline{\mathfrak{g}_{out}}.$$

<u>defn</u> the <u>space of conformal blocks</u> for $\hat{X}$

$$C_{\mathfrak{g}}(L_{\lambda_1}, \dots, L_{\lambda_n})_{\hat{X}} = \text{space of linear maps } \bigotimes_{i=1}^{n} L_{\lambda_i} \to \mathbb{C}$$

invariant under the action of $\mathfrak{g}_{out}$.

(<u>note</u> i.e. satisfying "chiral Ward identities")

dual space, <u>space of coinvariants</u> is

$$H_{\mathfrak{g}}(L_{\lambda_1}, \dots, L_{\lambda_n})_{\hat{X}} = \frac{\bigotimes_{i=1}^{n} L_{\lambda_i}}{\mathfrak{g}_{out} \cdot \bigotimes_{i=1}^{n} L_{\lambda_i}}$$

<u>Def^n</u> Given $\Phi_i \in L_{\lambda_i} \otimes \overline{L_{\lambda_i}}$, $\Phi_i(x_i)$ corresponding element in $\mathbb{H}_{\kappa, \hat{X}}$. The associated <u>correlation function</u> is $\langle \Phi_1(x_1) \ldots \Phi_n(x_n) \rangle := \langle \Omega, \Phi_1(x_1) \otimes \ldots \otimes \Phi_n(x_n) \rangle$.

(Ω ? a ground state).

(element of $C_{\mathfrak{g}}(L_{\lambda_1}, \ldots, L_{\lambda_n})_{\hat{X}} \otimes \overline{C_{\mathfrak{g}}(L_{\lambda_1}, \ldots, L_{\lambda_n})_{\hat{X}}}$)

(satisfy chiral & anti-chiral Ward identities.)

<u>note</u> $\exists$ "natural" Hermitian inner product on $H_{\mathfrak{g}}(L_{\lambda_1}, \ldots, L_{\lambda_n})_{\hat{X}}$. Writing ϕ for the dual of $\Phi_1(x_1) \otimes \ldots \otimes \Phi_n(x_n)$

$$\langle \Phi_1(x_1) \ldots \Phi_n(x_n) \rangle = \|\phi\|^2.$$