Universal Dichotomy for Dynamical Systems with Variable Delay

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(Received 27 March 2015; revised manuscript received 10 November 2016; published 27 January 2017)

We show that the dynamics of systems with a time-dependent delay is fundamentally affected by the functional form of the retarded argument. Associating with the latter an iterated map, the access map, and a corresponding Koopman operator, we identify two universality classes. Members in the first are equivalent to systems with a constant delay. The new, second class is characterized by the mode-locking behavior of their access maps and by an asymptotically linear, instead of a logarithmic, scaling of the Lyapunov spectrum. The membership depends in a fractal manner only on the parameters of the delay.

DOI: 10.1103/PhysRevLett.118.044104

Equivalence, classification, and invariant properties are of great interest in the analysis of dynamical systems [1]. Different representations of one phenomenon help to understand basic principles, uncover unknown connections, and improve models. Examples for equivalent dynamics can be found for iterated maps [2,3], ordinary differential equations [4], and delay differential equations (DDEs) [5–11]. DDEs arise in various fields [12], as, for example, in engineering [13], climate dynamics [14], life science [15–17], control theory [18], and synchronization of networks [19]. In these fields, systems with fluctuating delays were also studied [7,8,10,20–22]. They can be described by DDEs of the form

$$\hat{y}(t) = f(t, y(t), y(t - \tau(t)))$$ (1)

where the variable delay $\tau(t)$ is typically used for more realistic models or for a more effective control. We focus on the practically most relevant case, where the retarded argument $t - \tau(t)$ is strictly increasing; i.e., the condition $\dot{\tau}(t) < 1$ holds for all $t$ (cf. [9]). In general, the presence of a variable delay rather than a constant delay increases significantly the complexity of the dynamics of time delay systems [9,22,23]. This motivates us to study the equivalence of DDEs with a variable delay [Eq. (1)] and DDEs with a constant delay.

We will show that, on the one hand, there are DDEs with a variable delay, which can be transformed to a DDE with a constant delay. We call the associated delays conservative delays. However, it has not been recognized yet that, on the other hand, there exists a second class of systems, where a transformation to a constant delay fails even if the condition $\dot{\tau}(t) < 1$ is fulfilled. We call the associated delays dissipative delays. We will show in the following that this dichotomy is fundamental, because it exists independently of the specific form of the function $f$ in Eq. (1) and depends only on the properties of the map $t' = R(t) := t - \tau(t)$ with $t, t' \in \mathbb{R}$. We call this map the access map, because $R(t)$ is just the retarded argument from Eq. (1). For the sake of brevity, we present our results for periodic delays in a suitably rescaled time with $\tau(t) = \tau(t - 1)$. Our main results, however, are valid for quasiperiodic delays. We define the reduced access map $t' = r(t) = R(t) \text{ mod } 1$ with $t, t' \in [0, 1]$, which belongs to the family of circle maps and was extensively studied in the literature [1,2,24]. The dichotomy is universal, insofar as it depends solely on the mode-locking behavior of $r$. For quasiperiodic dynamics under iterations of the map $r$, a transformation to a constant delay is possible, whereas attracting so-called mode-locking dynamics in $r$ is associated with dissipative delays. Consequently, we can show that it depends extremely sensitively on the delay parameters whether the variable delay represents a conservative delay or a dissipative delay. Moreover, we will show that, for systems with a dissipative delay, the DDE inherits also an additional dissipation leading to an asymptotically linear scaling of the Lyapunov spectrum, in contrast to the known logarithmic scaling for systems with a constant delay [25]. This will become clear below by decomposing the solution operator of a DDE into an integral operator known from the solution of DDEs with a constant delay and a composition operator also known as the Koopman operator [26,27]. The Koopman operator is defined by a composition of the observable $y$ with the access map $R$. The linear scaling of the Lyapunov exponents for DDEs with a dissipative delay can be derived from the spectrum of the Koopman operator. In this context, it will also become clear why the dynamics of DDEs with time-varying delay is influenced by the dynamic behavior of the access map $R$ under iterations.

The two different delay classes can be identified by considering a nonlinear time scale transformation $\varphi = \Phi(t)$ of Eq. (1) and its inverse $t = \Phi^{-1}(\varphi)$. The system in the new time scale $\varphi$ with the new state variable $z(\varphi) = y[\Phi^{-1}(\varphi)]$ can be written as [5–11]

$$z'(\varphi) = [\Phi^{-1}(\varphi)]'f(\Phi^{-1}(\varphi), z(\varphi), z(\Phi[R[\Phi^{-1}(\varphi)])].$$ (2)
The function $\Phi(t)$ is assumed to be bijective and differentiable almost everywhere to ensure a one-to-one mapping between Eqs. (1) and (2). A constant delay $c$ appears in the transformed system [Eq. (2)] if and only if

$$\Phi \circ R \circ \Phi^{-1} = R_c,$$  

(3)

where $\circ$ denotes composition and $R_c(\phi) := \phi - c$. Similar to the maps $R$ and $r$ for the variable delay $\tau(t)$, the maps $\phi' = R_c(\phi)$, with $\phi, \phi' \in \mathbb{R}$, and $\phi' = r_c(\phi) := R_c(\phi) \mod 1$, with $\phi, \phi' \in [0, 1]$, are defined for the constant delay $c$. Equation (3) is the well-known relation for the topological conjugacy between the two access maps $R$ and $R_c$. We define conservative delays as the delays where

$$\Phi(\phi(t)) = \phi(t) = \phi(\tau(t)),$$

(4)

where $\Phi'(t)$ denotes the derivative of $\Phi(t)$. The equivalence of Eqs. (3) and (4) can be seen by integrating Eq. (4) yielding $\Phi(i) - \Phi[R(i)] = c$ and inserting $i = \Phi^{-1}(\phi)$. Without recognizing its character as a conjugacy relation, the definition [Eq. (4)] is often found in applications in engineering and nature. For example, if $\phi = \Phi(t)$ specifies a distance for a transport over time $t$, the delay $\tau(t)$ in $R_c(\phi)$, which is the traveling time for a transport with variable velocity $\Phi'(t)$ over the constant distance $c$. In the literature, these delays are called variable transport delays [11,28], pipe delays [6], or threshold-type delays [7,16,29]. On the other hand, there are variable delays, which are not defined by Eq. (4) (cf. [6,11]). In this case, it depends on the dynamics of the map $R$ under iterations whether the variable delay $\tau(t)$ is a conservative delay or not. The dynamics of $R$ can be characterized, for example, by its Lyapunov exponent $\mu$ and by the rotation number $\omega = \lim_{k \to \infty} \frac{1}{k} [R^k(t) - R^0(t)],$

(5)

where $R^k = R \circ R^{k-1}$ and $R^0(t) = t$. We call $\mu$ the access map Lyapunov exponent. For map parameters leading to irrational rotation numbers, quasiperiodic dynamics appears in the reduced map $r$ with zero access map Lyapunov exponent $\mu = 0$ [24]. It is well known [1] that in this case a smooth conjugacy $\Phi$ between the maps $R$ and $R_c$ exists [30]. As a result, variable delays with an irrational $\omega$ are conservative delays. On the other hand, for rational rotation numbers, attracting periodic motion with $\mu < 0$ appears (apart from exceptions of Lebesgue measure zero [31]). In this case, a conjugacy $\Phi$ between the maps $R$ and $R_c$ does not exist [1]. We call these delays dissipative delays.

We chose two examples of periodic delays for a more detailed investigation:

$$\tau(t) = \tau_0 + A(t \mod 1) - 0.5,$$

(6a)

$$\tau(t) = \tau_0 + A \sin(2\pi t)/(2\pi),$$

(6b)

where $\tau_0 > 0$ and $0 < A < 1$ specifies the mean delay and the amplitude, respectively. $A = 1$ represents the boundary between invertible and noninvertible dynamics of the access map $R$. The parameter regions of conservative and dissipative sawtooth-shaped delays [Eq. (6a)] are illustrated in Fig. 1. Black regions correspond to conservative delays with irrational rotation numbers and quasiperiodic dynamics, and white regions represent dissipative delays with rational rotation numbers $\omega$ of the corresponding access map $R$. They are equivalent to the so-called mode-locking regimes or Arnold tongues [2,24]. For sinusoidal delays [Eq. (6b)], which are often used in systems with variable delays [20,32], the reduced access map $r$ is equivalent to Arnold’s circle map. In this case, an analogous behavior can be observed; only the shape of the Arnold tongues is different from the tongues in Fig. 1 [24]. In general, for invertible access maps $R$, the integrated density of irrational rotation numbers $\omega$ increases with the mean delay $\tau_0$ as an incomplete devil’s staircase; i.e., the irrational rotation numbers have nonzero Lebesgue measure but form a fat fractal [24]. The latter implies that rational rotation numbers are stable against generic perturbations of the delay function, whereas irrational rotation numbers are unstable [1]. This means that, for conservative delays, a typical continuous parameter change in an arbitrary dimensional parameterization of the delay function changes the rotation number of the access map and, therefore, passes through infinitely many parameter regions of finite size corresponding to dissipative delays (white). Note that in Fig. 1, due to the necessarily finite resolution,
the densely lying, finest fractal structures cannot be resolved. In the definition of conservative delays via Eq. (4), one sees that a change of the velocity \( \dot{\Phi}'(t) \) corresponds to a change of the access map \( R \) on a manifold with constant rotation number \( \omega = -c \). In contrast, a typical continuous parameter change in the delay \( \tau(t) \) always has components transverse to this manifold. For the latter case, our results imply that one passes inevitably through infinitely many situations with an unphysical dynamic behavior if by physical reasons the existence of a velocity \( \dot{\Phi}'(t) \) is presupposed.

In the following, we show numerically and analytically that the dynamics of the access map \( R \) affects fundamentally the dynamics of the delay system. The latter can be characterized by the Lyapunov spectrum of the DDE, which is the set of all Lyapunov exponents \( \lambda_i \) arranged in decreasing order with \( \lambda_i > \lambda_{i+1} \) not to be confused with the access map Lyapunov exponent \( \mu \) of \( R \). Since DDEs are infinite-dimensional systems, infinitely many Lyapunov exponents \( \lambda_i \) exist [25]. We consider the DDEs

\[
\begin{align*}
\dot{y}(t) &= \frac{10y[t - \tau(t)]}{1 + y[t - \tau(t)]^{10}} - 5y(t), \quad (7a) \\
\dot{y}(t) + \dot{y}(t) + 4\pi^2 y(t) &= 8y[t - \tau(t)], \quad (7b) \\
\dot{y}(t) &= 2y(t)\{1 - y[t - \tau(t)]\}. \quad (7c)
\end{align*}
\]

For constant delay \( \tau(t) = \tau_0 \), Eq. (7a) is the Mackey-Glass equation, a model for blood production [25], Eq. (7b) is a delayed oscillator, a model for self-excited chatter vibrations in metal cutting [20], and Eq. (7c) is the Hutchinson equation, a model for population dynamics [15–17]. Two sinusoidally varying delays [Eq. (6b)] with \((\tau_0, A) = (1.54, 0.9)\) and \((\tau_0, A) = (1.51, 0.9)\) are chosen, corresponding to a conservative and a dissipative delay, respectively. For these parameters, the attractors of the systems are a chaotic attractor for Eq. (7a), a stable equilibrium for Eq. (7b), and a limit cycle for Eq. (7c). The Lyapunov spectra for the three systems with the two delays are shown in Fig. 2(a). For a conservative delay (crosses), the Lyapunov exponents decrease logarithmically, \( \lambda_i \sim -\log l \), for \( l \to \infty \). In Fig. 2(b), the logarithmic part \( \lambda_{l,l} \sim -(C/\omega) \log l \) is subtracted from the Lyapunov spectra, where \( C = 1 \) for Eqs. (7a) and (7c) and \( C = 2 \) for Eq. (7b). \( \lambda_{l,l} \) can be computed using the method of Farmer [25] by assuming a constant delay \( c = -\omega \). Since conservative delays are equivalent to constant delays, the asymptotic behavior of the spectrum is logarithmic and equals \( \lambda_{l,1} \). In contrast, for a dissipative delay, a linear behavior \( \lambda_i \sim -\alpha l \) remains for \( l \to \infty \) (dots). The dichotomy in the Lyapunov spectra as well as the asymptotic slope \( \alpha \) of the spectra for the dissipative delay is independent of the attractor and independent of the specific form of \( f \) in Eq. (1), which can be explained as follows.

The Lyapunov exponents \( \lambda_i \) of an attractor of Eq. (1) are obtained from its linearization

\[
\dot{x}(t) = A(t)x(t) + B(t)x[t - \tau(t)],
\]

where \( x = y - y_\ast \) are small perturbations of a trajectory of the original system and \( A(t) \) and \( B(t) \) are the Jacobians of \( \dot{x} \) with respect to \( y(t) \) and \( y[t - \tau(t)] \), respectively. Let us define the intervals \( T_i := [t_{i-1}, t_i] \), with \( R(t_i) = t_{i-1} \) and \( t_{0} = 0 \), and denote \( x_i(t) \) the solution segment of Eq. (8) with \( t \in T_i \). The method of steps can be used to solve Eq. (8), which becomes a nonhomogeneous ordinary differential equation (ODE) in one solution segment [5,33]

\[
x_i(t) = \hat{I}_i \hat{C}_i x_{i-1}(t).
\]

We have split the solution operator that maps \( x_{i-1}(t) \) to \( x_i(t) \) into two parts [34] (see Fig. 3). The operator \( \hat{C}_i \) is defined by \( \hat{C}_i x_{i-1}(t) = \hat{x}_i(t) \), where \( \hat{x}_i(t) \) is the segment of \( \hat{x}(t) = x[R(t)] \) with \( t \in T_i \). Thus, the operator \( \hat{C}_i \) is the operator that maps \( x_{i-1}(t) \) to \( \hat{x}_i(t) \) on the next interval \( T_i \). The method of steps can be used to solve Eq. (8), which becomes a nonhomogeneous ordinary differential equation (ODE) in one solution segment [5,33]

\[
x_i(t) = \hat{I}_i \hat{C}_i x_{i-1}(t).
\]
composition of the observable \(\mathbf{x}\) with the access map \(R\) and is known as the Koopman operator \([26,27]\). It characterizes the properties of the delay access in the DDE \([\text{Eq. (8)}]\). The operator \(\hat{I}_i\) is equivalent to the known solution operator for linear DDEs with a constant delay and has the structure of the variation of constants formula

\[
\hat{I}_i \mathbf{x}_i(t) = \mathbf{M}(t, t_{i-1}) \mathbf{x}_i(t_i) + \int_{t_{i-1}}^{t_i} d\theta \mathbf{M}(t, \theta) \mathbf{B}(\theta) \mathbf{x}_i(\theta),
\]

where \(\mathbf{M}(t, \theta)\) denotes the fundamental matrix solution of the ODE part \(\dot{\mathbf{x}} = \mathbf{A} \mathbf{x}\) of \(\text{Eq. (8)}\). From \(\text{Eq. (9)}\), one can see that the operator \(\hat{C}_i\) and, therefore, the access map \(R\) can affect the dynamics of the time delay system. In particular, the effect on the scaling of the Lyapunov exponents can be analyzed by a generalization of the Farmer method \([25]\) to DDEs with a variable delay, where the divergence \(\delta_N\) of an \(N\)-dimensional approximation \(I_i^{(N)} C_i^{(N)}\) of the solution operator in \(\text{Eq. (9)}\) is analyzed \([34,35]\):

\[
\delta_N = \lim_{n \to \infty} \frac{1}{t_n - t_0} \sum_{i=1}^{n} (\log |\det I_i^{(N)}| + \log |\det C_i^{(N)}|).
\]

The matrices \(I_i^{(N)}\) and \(C_i^{(N)}\) associated with the operators \(\hat{I}_i\) and \(\hat{C}_i\) are obtained by expanding the function segments \(\mathbf{x}_i(t)\) and \(\hat{\mathbf{x}}_i(t)\) into \(N\) basis functions \(\alpha\). Consequently, the Lyapunov exponents \(\lambda_i \approx \delta_i - \delta_{i-1}\) for large \(l\) are composed of a contribution \(\lambda_{l,j}\) from the matrices \(I_i^{(N)}\) with the known logarithmic behavior \([25]\) and a contribution \(\lambda_{C,j}\) from the matrices \(C_i^{(N)}\). For systems with constant delay \(c\), the Koopman operators \(\hat{C}_i\) are simple shift operators with \(\det C_i^{(N)} = 1\), resulting in \(\lambda_{C,j} = 0\). The same holds for all systems with a conservative delay, because they are equivalent to systems with a constant delay, which means that in this case the Koopman operator \(\hat{C}_i\) has no significant effect on the tangent space dynamics of the delay system. In contrast, for systems with dissipative delays, i.e., access maps with rational rotation number \(\omega = -\frac{p}{q}\) and \(\mu < 0\), the contribution of the matrices \(C_i^{(N)}\) to the divergence \(\delta_N\) does not vanish and can be characterized by studying the eigenvalues of the operator \(\hat{D} = \hat{C}_q \hat{C}_{q-1} \ldots \hat{C}_1\). In particular, if \(e^{\hat{a}}\) is an eigenvalue of the operator \(\hat{D}\), then \(e^{\hat{a}l}\) with \(l \in \mathbb{Z}^+\) is also an eigenvalue of \(\hat{D}\) \([27]\). If the whole spectrum of an \(N\)-dimensional approximation \(D^{(N)}\) of the operator \(\hat{D}\) is specified only by different powers \(e^0, e^{\hat{a}}, \ldots, e^{\hat{a}(N-1)}\) of one eigenvalue, the contribution of the Koopman operators to the divergence can be characterized by \(\log |\det D^{(N)}| \sim \hat{a} N^2/2\), and a linear scaling of the Lyapunov exponents appears \(\lambda_{C,j} \approx \hat{a} l\). In fact, such a linear scaling has been identified in \(\text{Fig. 2(b)}\) and can be explicitly calculated for systems with dissipative sawtooth-shaped delays \([\text{Eq. (6a)}]\) \([35]\). Consequently, for DDEs with a dissipative delay, the scaling of the Lyapunov exponents \(\lambda_l = \lambda_{l,j} + \lambda_{C,j}\) for large \(l\) is dominated by the linear part \(\lambda_{C,j} \approx al\) associated with the Koopman operator \(\hat{C}_i\) and the access map \(R\). This is verified numerically in \(\text{Fig. 4}\), where for a DDE with dissipative sinusoidal delays the slope \(\hat{a}\) calculated from the determinant of the matrix \(D^{(N)}\) is plotted vs the asymptotic slope \(\alpha\) measured in the Lyapunov spectrum of the DDE similar to \(\text{Fig. 2(b)}\). As a result, there is an additional dissipation due to the retarded access with the variable delay in systems with dissipative delays, which is completely independent of the specific form \(f\) of the DDE.

In conclusion, we have shown the existence of a fundamental dichotomy in systems with time-dependent delays, which depends solely on the mode-locking behavior of the access map defined via the delayed argument. In a second step, by considering analytically and numerically the spectral properties of the Koopman operator associated with this map, we found a new universal behavior of the Lyapunov spectrum of DDEs. Apart from providing a new interesting connection between delay equations, iterated maps, and the theory of Koopman operators, our findings are expected to have consequences in many research fields. First, one should mention that, by some modification of the method in Ref. \([36]\), it should be possible to detect the dichotomy in the Lyapunov spectra also experimentally from scalar time series, e.g., from coupled semiconductor lasers \([37]\). Furthermore, in systems with a large delay and correspondingly high-dimensional chaotic attractors, which are of much interest by themselves \([38]\), or in connection with synchronization \([39]\), the fractal switching between linear and logarithmic decays should affect also the fractal dimension of the chaotic attractors: The Kaplan-Yorke dimension \([24]\) is expected to fluctuate strongly as some parameter of the oscillatory delay is varied, because the

\[\text{FIG. 4. Asymptotic slope } \hat{a} \text{ calculated from approximations } D^{(N)} \text{ of the operator } \hat{D} \text{ expanded in the eigenbasis of the DDE vs measured slope } \alpha \text{ of the Lyapunov spectrum of DDEs with various dissipative sinusoidal delays. The slopes of the Lyapunov spectra are identified as illustrated in } \text{Fig. 2(b)}\text{. One finds } \hat{a} = \alpha, \text{ implying that the Koopman operator determines the asymptotics of the Lyapunov spectrum.}\]
positive Lyapunov exponents are compensated by the negative ones at much lower indices for linearly decaying spectra than for a logarithmic decay. Finally, because our findings address the metric properties of the infinite-dimensional stable manifolds in DDEs, one may in the future expect interesting new consequences where the stable manifolds play a role, be it for boundaries of basins of attractions or for homo- and heteroclinic intersections with its consequences for fractal chaotic saddles. These are topics reasonably well understood for finite-dimensional systems, but for time delay systems with its infinite-dimensional stable manifolds in DDEs, one may in the future expect interesting new consequences where the stable manifolds play a role, be it for boundaries of basins of attractions or for homo- and heteroclinic intersections with its consequences for fractal chaotic saddles. These are topics reasonably well understood for finite-dimensional systems, but for time delay systems with its infinite-dimensional state space only a few results are available (see, e.g., [40,41]), and when it comes to a variable delay basically, nothing is known in such respects.
for concrete examples with dissipative sawtooth-shaped delays.


