Efficient recovery of non-periodic multivariate functions from few scattered samples

Nicolas Nagel

Joint work with Felix Bartel, Kai Lüttgen and Tino Ullrich

UNIVERSITY OF TECHNOLOGY
IN THE EUROPEAN CAPITAL OF CULTURE
CHEMNITZ

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Setting

- Given: $f : [-1,1]^d \to \mathbb{C}$ from some function space (Sobolev space)
- Goal: approximation of f via samples $f(\mathbf{x}^1), ..., f(\mathbf{x}^n)$
- Requirements:
  - Universality: $\mathbf{x}^1, ..., \mathbf{x}^n$ should work for the whole function space
  - Bound on the approximation error (depending on the dimension d, the number of samples n and the smoothness s of f)

It has been observed: if $\mathbf{x}^1, ..., \mathbf{x}^n$ follow a **Chebyshev distribution** and one uses **Chebyshev polynomials** one obtains near optimal approximations.

We give a theoretical explanation of this phenomenon.

Periodic functions

- $f(\mathbf{x}) \approx \sum_{\mathbf{k}\in\Lambda} c_{\mathbf{k}} \exp(\pi i \mathbf{k} \cdot \mathbf{x})$
  - $\Lambda \subset \mathbb{Z}^d$ finite set of frequencies
  - $c_{\mathbf{k}}$ determined from the samples $f(\mathbf{x}^i)$
- Questions:
  - Which frequencies Λ to use?
  - How to choose the sample nodes?
  - How to determine the coefficients $c_{\mathbf{k}}$?

Which frequencies Λ to use?

- If f is s-smooth with $s > 1/2$, i.e. $f \in H^s_{\text{mix}}(\mathbb{T}^d)$, then the Fourier coefficients

$$f(\mathbf{x}) = \sum_{\mathbf{k}\in\mathbb{Z}^d} \hat{f}_{\mathbf{k}} \exp(\pi i \mathbf{k}\cdot\mathbf{x})$$

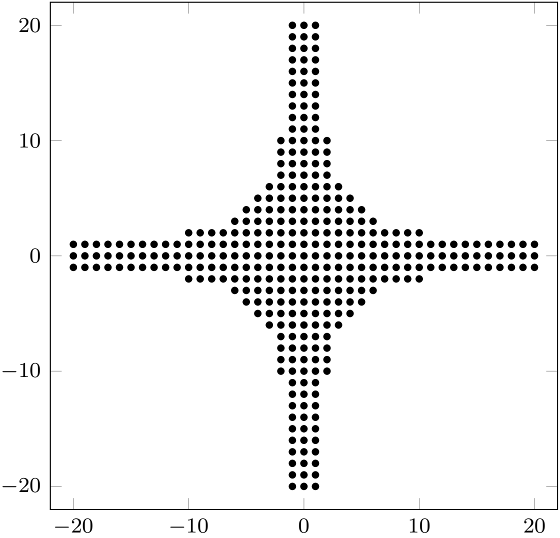
decay like

$$\underbrace{\sum_{\mathbf{k}\in\mathbb{Z}^d} |\hat{f}_{\mathbf{k}}|^2 \prod_{\ell=1}^{d} (1+|k_\ell|^2)^s}_{\asymp \|f\|^2_{H^s_{\text{mix}}}} < \infty$$

- Thus: frequencies near 0 are the most important ones
- Use **hyperbolic cross**

$$\Lambda = \{\mathbf{k}\in\mathbb{Z}^d : \prod_{\ell=1}^{d} \max\{1, |k_\ell|\} \le R\}$$

Hyperbolic cross $d = 2, R = 20, \#\Lambda = 345$



How to determine the coefficients $c_{\mathbf{k}}$?

- $m = \#\Lambda$ number of frequencies, N number of sample nodes,

$$\begin{bmatrix} \eta_{\mathbf{k}_1}(\mathbf{x}^1) & \cdots & \eta_{\mathbf{k}_m}(\mathbf{x}^1) \\ \vdots & & \vdots \\ \eta_{\mathbf{k}_1}(\mathbf{x}^N) & \cdots & \eta_{\mathbf{k}_m}(\mathbf{x}^N) \end{bmatrix} \begin{bmatrix} c_{\mathbf{k}_1} \\ \vdots \\ c_{\mathbf{k}_m} \end{bmatrix} \approx \begin{bmatrix} f(\mathbf{x}^1) \\ \vdots \\ f(\mathbf{x}^N) \end{bmatrix}$$

  with $\eta_{\mathbf{k}}(\mathbf{x}) = \prod_{\ell=1}^{d} e^{\pi i k_\ell x_\ell}$
- Approximate $f(\mathbf{x}) \approx \sum_{\mathbf{k}\in\Lambda} c_{\mathbf{k}} \eta_{\mathbf{k}}(\mathbf{x})$

How to choose the sample nodes?

- Symmetry of $\mathbb{T}^d$: $\mathbf{x}^1, ..., \mathbf{x}^n$ should be uniformly distributed
- Choose $\mathbf{x}^i, i = 1, ..., N$ uniformly at random
- To get a good approximation: need a logarithmic oversampling of $N = O(m \log m)$ due to randomness

Theorem (Krieg, M. Ullrich '21)

For $s > 1/2$ and using N samples, the above described procedure yields (with high probability) an approximation $\tilde{f}$ with

$$\|f - \tilde{f}\|_{L_2} \lesssim N^{-s} (\log N)^{ds} \|f\|_{H^s_{mix}(\mathbb{T}^d)}.$$

Subsampling

- Wish: $n = O(m)$ samples should be enough
- Indeed: Possible by the Kadison-Singer problem via Weaver's KS_2-conjecture

Kadison-Singer problem:

- Posed by Richard Kadison and Isadore Singer in 1959
- Rather abstract problem from functional analysis (motivated by quantum physics)
- Many equivalent formulations:

$$\begin{aligned}\text{Kadison-Singer} &\Leftrightarrow \text{Anderson's paving conjecture}\\ &\Leftrightarrow \text{Weaver's } KS_2\text{-conjecture}\\ &\Leftrightarrow \text{Feichtinger conjecture}\\ &\Leftrightarrow \text{Bourgain-Tzafriri conjecture}\end{aligned}$$

- Solved by Marcus, Spielman, Srivastava 2015

Consider $\mathbb{C}^m$ as a Hilbert space

- A **frame** is a sequence $\mathbf{u}^1, ..., \mathbf{u}^N \in \mathbb{C}^m$ such that

$$A\|\mathbf{w}\|_2^2 \leq \sum_{i=1}^{N} |\langle \mathbf{w}, \mathbf{u}^i \rangle|^2 \leq B\|\mathbf{w}\|_2^2$$

  for all $\mathbf{w} \in \mathbb{C}^m$, where A and B are constants (the **frame bounds**)
- B/A the **condition** of the frame

Theorem (Weaver's KS_2-conjecture; Marcus, Spielman, Srivastava '15)

If $\mathbf{u}^1, ..., \mathbf{u}^N \in \mathbb{C}^m$ *with* $\|\mathbf{u}^i\|_2 = 1$ *for all* i *and*

$$\sum_{i=1}^{N} |\langle \mathbf{w}, \mathbf{u}^i \rangle|^2 = 18\|\mathbf{w}\|_2^2$$

for all $\mathbf{w} \in \mathbb{C}^m$*, then one can partition* $S_1 \dot\cup S_2 = [N]$ *such that*

$$2\|\mathbf{w}\|_2^2 \leq \sum_{i \in S_j} |\langle \mathbf{w}, \mathbf{u}^i \rangle|^2 \leq 16\|\mathbf{w}\|_2^2$$

for all $\mathbf{w} \in \mathbb{C}^m$ *and* $j = 1, 2$*.*

Need: extract subframes from large frames with guaranties on their condition

- NITZAN, OLEVSKII, ULANOVSKII 2014: 1-tight frame $\mathbf{u}^1, ..., \mathbf{u}^N \in \mathbb{C}^m$ with $\|\mathbf{u}^i\|_2^2 = m/N$, there is a $J \subseteq [N]$ with $\#J = O(m)$ and resulting frame bounds $c\frac{m}{N}$ and $C\frac{m}{N}$
- TEMLYAKOV/N, SCHÄFER, T. ULLRICH 2020: Non-tight frames and only upper bound on the norms
- DOLBEAULT, KRIEG, M. ULLRICH 2022: Infinite-dimensional version in ℓ_2

Apply to $\mathbf{u}^i = [\eta_{\mathbf{k}}(\mathbf{x}^i)]_{\mathbf{k}\in\Lambda}$ to get well-conditioned subframe on $J \subseteq [N]$ of size $n = O(m)$ (down from $N = O(m \log m)$) with almost asymptotically equal approximation properties

Theorem (N, Schäfer, T. Ullrich '22)

For $s > 1/2$ and using n samples, the algorithm together with the subsampling step yields (with high probability) an approximation $\tilde{f}$ with

$$\|f - \tilde{f}\|_{L_2} \lesssim n^{-s} (\log n)^{(d-1)s+1/2} \|f\|_{H^s_{mix}(\mathbb{T}^d)}$$

- Problems:
  - Non-algorithmic: Kadison-Singer only gives existence of a subframe
  - The oversampling factor $n = bm$ might be huge (e.g. $b = 6000$)
- Polynomial time algorithm with small oversampling factor $b = 1 + \varepsilon$: Based on Batson, Spielman, Srivastava 2009 and developed further by Bartel, Schäfer, T. Ullrich 2023

Theorem (Batson, Spielman, Srivastava '09/Bartel, Schäfer, T. Ullrich '23)

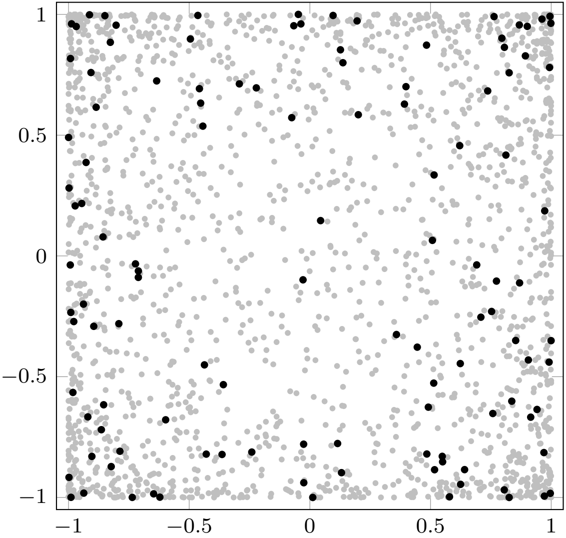
Let $\mathbf{u}^1, \ldots, \mathbf{u}^N \in \mathbb{C}^m$ *(arbitrary), choose* $b > 1 + \frac{1}{m}$ *and assume* $N \geq bm$. *There is a polynomial time algorithm to construct a* $J \subseteq [N]$ *with* $\#J \leq \lceil bm \rceil$ *and*

$$\frac{1}{N} \sum_{i=1}^{N} |\langle \mathbf{w}, \mathbf{u}^i \rangle|^2 \leq 89 \frac{(b+1)^2}{(b-1)^3} \cdot \frac{1}{m} \sum_{j \in J} |\langle \mathbf{w}, \mathbf{u}^j \rangle|^2.$$

Note: only lower bound, algorithm with guaranty on the upper bound unknown
Still: resulting sample nodes $\mathbf{x}^1, \ldots, \mathbf{x}^n$ random, their quality is measured by the condition of the matrix

$$\begin{bmatrix} \eta_{\mathbf{k}_1}(\mathbf{x}^1) & \cdots & \eta_{\mathbf{k}_m}(\mathbf{x}^1) \\ \vdots & & \vdots \\ \eta_{\mathbf{k}_1}(\mathbf{x}^n) & \cdots & \eta_{\mathbf{k}_m}(\mathbf{x}^n) \end{bmatrix}$$

Subsampling ($d = 2, R = 20, m = 107, N = 2000, n = 117$)



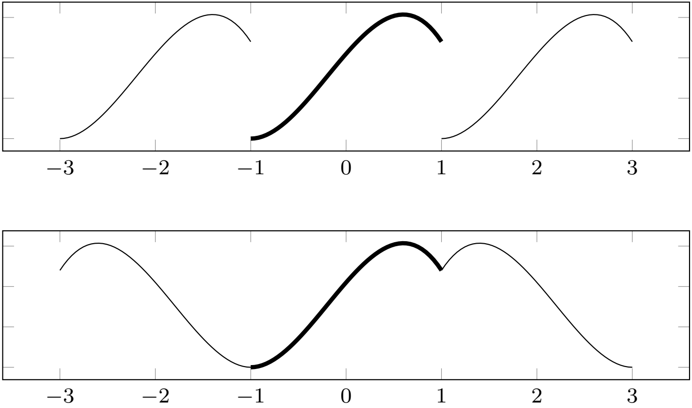
Final algorithm (periodic):

- Choose size of hyperbolic cross $\Lambda = \{\mathbf{k} \in \mathbb{Z}^d : \prod_{\ell=1}^d \max\{1, |k_\ell|\} \leq R\}$, $m = \#\Lambda$
- Set $N = \lceil 4m \log m \rceil$
- Choose N nodes $\mathbf{x}^i \in [-1, 1]^d$ uniformly at random
- Subsampling gives nodes $\{\mathbf{x}^j : j \in J\}$ with $n = \#J \leq \lceil 1.1m \rceil$ using the basis functions $\eta_{\mathbf{k}}(\mathbf{x}) = \prod_{\ell=1}^d e^{\pi i k_\ell x_\ell}$ (works for all of $H^s_{\text{mix}}(\mathbb{T}^d)$)
- Determine the coefficients $c_{\mathbf{k}}$ from the $\mathbf{x}^j, j \in J$ via the least-squares system with the basis functions $\eta_{\mathbf{k}}(\mathbf{x})$

Non-periodic functions

Applying the above procedure to a more general function $f : [-1, 1]^d \to \mathbb{C}$ treats f like a periodized version on $\mathbb{R}^d$

$\Rightarrow$ may introduce non-regularities



To apply the algorithm for periodic functions, we need to periodize f in a way that preserves regularity

- Periodic extension: May introduce discontinuities
- Tent transform: Preserves continuity, might destroy smoothness (kinks)

We will use a **cosine composition** $T_{\cos}$ defined by

$$(T_{\cos} f)(x_1, ..., x_d) = f(\cos \pi x_1, ..., \cos \pi x_d)$$

Theorem (Bartel, Lüttgen, N, T. Ullrich)

The operator $T_{\cos}$ is continuous as

$$T_{\cos} : H^s_{mix}([-1,1]^d) \to H^s_{mix}(\mathbb{T}^d)$$

for $s > 1/2$.

More general versions over Besov spaces are possible

Strategy: Approximate $T_{\cos} f$ with the Fourier basis and undo the periodization

	$f(\cos \pi x_1, \ldots, \cos \pi x_d)$	$f(x_1, \ldots, x_d)$
Sample nodes	$\mathbf{x}^i \sim \mathcal{U}[-1,1]^d$	$\mathbf{x}^i = \cos(\pi \mathbf{U}^i), \mathbf{U}^i \sim \mathcal{U}[-1,1]^d$ $\mathrm{d}\varrho(\mathbf{x}) = \prod_{\ell=1}^d \left(\pi\sqrt{1-x_\ell^2}\right)^{-1} \mathrm{d}\mathbf{x}$
Basis functions	$\prod_{\ell=1}^d \cos(\pi k_\ell x_\ell)$	$\prod_{\ell=1}^d \cos(k_\ell \arccos x_\ell)$

- $\mathbf{x}^i$ Chebyshev distributed
- $\eta_{\mathbf{k}}(\mathbf{x}) = \prod_{\ell=1}^d T_{k_\ell}(x_\ell), \mathbf{k} \in \mathbb{N}_0^d$ with $T_k(x) = \sqrt{2}^{\min\{1,k\}} \cos(k \arccos x)$ ($L_2(\varrho)$-normalized Chebyshev polynomials)

Final algorithm (non-periodic):

- Choose size of hyperbolic cross $\Lambda = \{\mathbf{k} \in \mathbb{N}_0^d : \prod_{\ell=1}^d \max\{1, k_\ell\} \leq R\}$ (in the positive orthant), $m = \#\Lambda$
- Set $N = \lceil 4m \log m \rceil$
- Choose N nodes $\mathbf{x}^i \in [-1,1]^d$ Chebyshev distributed
- Subsampling gives nodes $\{\mathbf{x}^j : j \in J\}$ with $n = \#J \leq \lceil 1.1m \rceil$ using the basis functions $\eta_{\mathbf{k}}(\mathbf{x}) = \prod_{\ell=1}^d T_{k_\ell}(x_\ell)$ (works for all of $H^s_{\text{mix}}([-1,1]^d)$)
- Determine the coefficients $c_{\mathbf{k}}$ from the $\mathbf{x}^j, j \in J$ via the least-squares system with the basis functions $\eta_{\mathbf{k}}(\mathbf{x})$

Theorem (Bartel, Lüttgen, N, T. Ullrich)

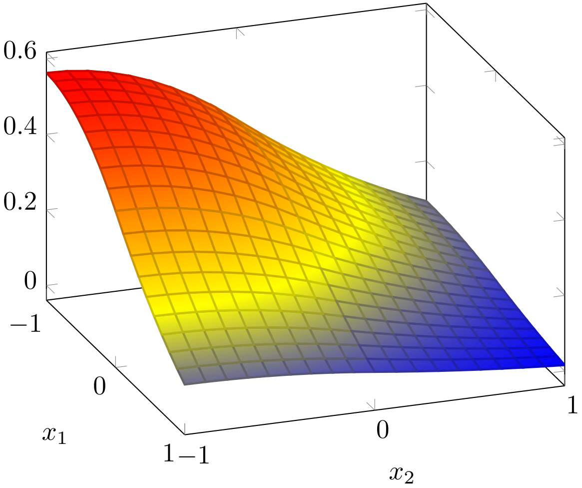
For $s > 1/2$ and using n samples, the above algorithm yields (with high probability) an approximation $\tilde{f}$ with

$$\|f - \tilde{f}\|_{L_2(\varrho)} \lesssim n^{-s} (\log n)^{(d-1)s+1/2} \|f\|_{H^s_{mix}([-1,1]^d)}.$$

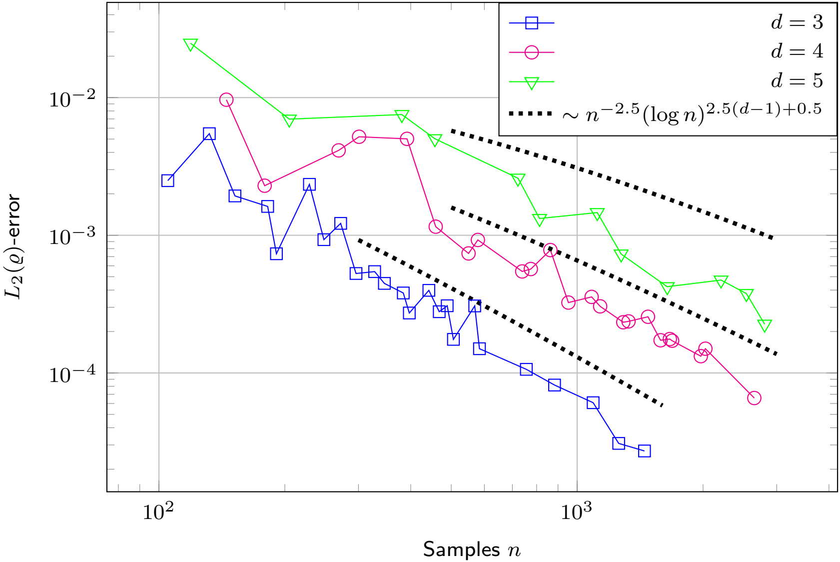
- $L_2(\varrho)$-norm stronger than usual (Lebesgue) L_2-norm
- $\sqrt{\log n}$ could be removed, but no know (efficient) subsamplings algorithm

Numerical experiment

Test function: Tensored cutout of a quadratic B-spline (smoothness $s = 2.5$)



Approximation error for an $f \in H^{2.5-\varepsilon}_{\text{mix}}([-1,1]^d)$



The end

- "Deterministic Kadison-Singer"? (remove the $\sqrt{\log}$-factor)
- Deterministic constructions for good samples nodes?

Thank you for your attention!