# Efficient recovery of non-periodic multivariate functions from few scattered samples

Nicolas Nagel

Joint work with Felix Bartel, Kai Lüttgen and Tino Ullrich

UNIVERSITY OF TECHNOLOGY
IN THE EUROPEAN CAPITAL OF CULTURE
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# Setting

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- Given: $f : [-1,1]^d \to \mathbb{C}$ from some function space (Sobolev space)
- Goal: approximation of $f$ via samples $f(\mathbf{x}^1), ..., f(\mathbf{x}^n)$
- Requirements:
  - Universality: $\mathbf{x}^1, ..., \mathbf{x}^n$ should work for the whole function space
  - Bound on the approximation error (depending on the dimension $d$, the number of samples $n$ and the smoothness $s$ of $f$)

It has been observed: if $\mathbf{x}^1, ..., \mathbf{x}^n$ follow a **Chebyshev distribution** and one uses **Chebyshev polynomials** one obtains near optimal approximations.

We give a theoretical explanation of this phenomenon.

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# Periodic functions

- $f(\mathbf{x}) \approx \sum_{\mathbf{k}\in\Lambda} c_{\mathbf{k}} \exp(\pi i \mathbf{k}\cdot\mathbf{x})$
  - $\Lambda \subset \mathbb{Z}^d$ finite set of frequencies of size $m$
  - $c_{\mathbf{k}}$ determined from the samples $f(\mathbf{x}^i)$
- Questions:
  - Which frequencies $\Lambda$ to use? $\rightarrow$ Hyperbolic cross ($f \in H^s_{\text{mix}}(\mathbb{T}^d)$ for $s > 1/2$)
  - How to choose the sample nodes? $\rightarrow$ Uniformly random with logarithmic oversampling $N = O(m \log m)$
  - How to determine the coefficients $c_{\mathbf{k}}$? $\rightarrow$ Ideally interpolate, in general as a least squares system

**Theorem (Krieg, M. Ullrich '21)**

*For $s > 1/2$ and using $N$ samples, the above described procedure yields (with high probability) an approximation $\tilde{f}$ with*

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# Subsampling

- Wish: $n = O(m)$ samples should be enough
- Indeed: Possible by the Kadison-Singer problem via Weaver's $KS_2$-conjecture
- Posed by Richard Kadison and Isadore Singer in 1959
- Rather abstract problem from functional analysis (motivated by quantum physics)
- Many equivalent formulations:

$$\begin{aligned} \text{Kadison-Singer} &\Leftrightarrow \text{Anderson's paving conjecture} \\ &\Leftrightarrow \text{Weaver's } KS_2\text{-conjecture} \\ &\Leftrightarrow \text{Feichtinger conjecture} \\ &\Leftrightarrow \text{Bourgain-Tzafriri conjecture} \end{aligned}$$

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Consider $\mathbb{C}^m$ as a Hilbert space

- A **frame** is a sequence $\mathbf{u}^1, ..., \mathbf{u}^N \in \mathbb{C}^m$ such that

$$A\|\mathbf{w}\|_2^2 \leq \sum_{i=1}^{N} |\langle \mathbf{w}, \mathbf{u}^i \rangle|^2 \leq B\|\mathbf{w}\|_2^2$$

  for all $\mathbf{w} \in \mathbb{C}^m$, where $A$ and $B$ are constants (the **frame bounds**)
- $B/A$ the **condition** of the frame

**Theorem (Weaver's $KS_2$-conjecture; Marcus, Spielman, Srivastava '15)**

*If* $\mathbf{u}^1, ..., \mathbf{u}^N \in \mathbb{C}^m$ *with* $\|\mathbf{u}^i\|_2 = 1$ *for all* $i$ *and*

$$\sum_{i=1}^{N} |\langle \mathbf{w}, \mathbf{u}^i \rangle|^2 = 18\|\mathbf{w}\|_2^2$$

*for all* $\mathbf{w} \in \mathbb{C}^m$*, then one can partition* $S_1 \dot{\cup} S_2 = [N]$ *such that*

$$2\|\mathbf{w}\|_2^2 \leq \sum_{i \in S_j} |\langle \mathbf{w}, \mathbf{u}^i \rangle|^2 \leq 16\|\mathbf{w}\|_2^2$$

*for all* $\mathbf{w} \in \mathbb{C}^m$ *and* $j = 1, 2$.

Need: extract subframes from large frames with guarantees on their condition

- Nitzan, Olevskii, Ulanovskii 2014: 1-tight frame $\mathbf{u}^1, \ldots, \mathbf{u}^N \in \mathbb{C}^m$ with $\|\mathbf{u}^i\|_2^2 = m/N$, there is a $J \subseteq [N]$ with $\#J = O(m)$ and resulting frame bounds $c\frac{m}{N}$ and $C\frac{m}{N}$
- Temlyakov/N, Schäfer, T. Ullrich 2020: Non-tight frames and only upper bound on the norms
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Apply to $\mathbf{u}^i = [\eta_{\mathbf{k}}(\mathbf{x}^i)]_{\mathbf{k}\in\Lambda}$ to get well-conditioned subframe on $J \subseteq [N]$ of size $n = O(m)$ (down from $N = O(m \log m)$) with almost asymptotically equal approximation properties

## Theorem (N, Schäfer, T. Ullrich '22)

*For $s > 1/2$ and using $n$ samples, the algorithm together with the subsampling step yields (with high probability) an approximation $\tilde{f}$ with*

$$\|f - \tilde{f}\|_{L_2} \lesssim n^{-s}(\log n)^{(d-1)s+1/2}\|f\|_{H^s_{mix}(\mathbb{T}^d)}$$

Note: Can get rid of the $\sqrt{\log n}$ factor by Dolbeault, Krieg, M. Ullrich

- Problems:
  - Non-algorithmic: Kadison-Singer only gives existence of a subframe
  - The oversampling factor $n = bm$ might be huge (e.g. $b = 6000$)
- Polynomial time algorithm with small oversampling factor $b = 1 + \varepsilon$: Based on Batson, Spielman, Srivastava 2009 and developed further by Bartel, Schäfer, T. Ullrich 2023

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*Let* $\mathbf{u}^1, ..., \mathbf{u}^N \in \mathbb{C}^m$ *(arbitrary), choose* $b > 1 + \frac{1}{m}$ *and assume* $N \geq bm$. *There is a polynomial time algorithm to construct a* $J \subseteq [N]$ *with* $\#J \leq \lceil bm \rceil$ *and*

$$\frac{1}{N}\sum_{i=1}^{N} |\langle \mathbf{w}, \mathbf{u}^i \rangle|^2 \leq 89\frac{(b+1)^2}{(b-1)^3} \cdot \frac{1}{m}\sum_{j\in J} |\langle \mathbf{w}, \mathbf{u}^j \rangle|^2.$$

Note: only lower bound, algorithm with guarantee on the upper bound unknown

Still: resulting sample nodes $\mathbf{x}^1, ..., \mathbf{x}^n$ random, their quality is measured by the condition of the matrix

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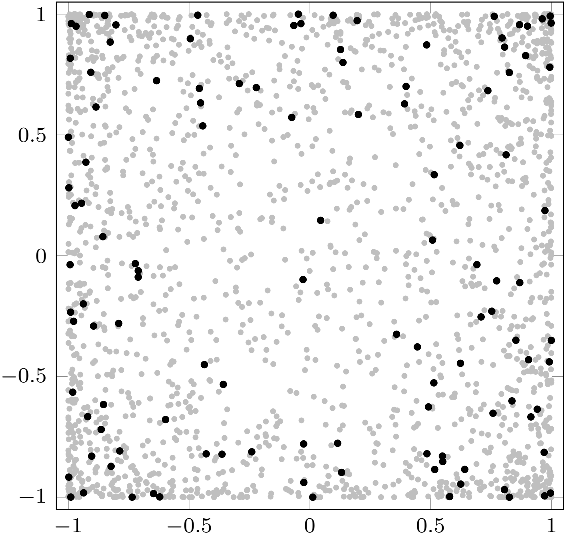
Subsampling ($d = 2, R = 20, m = 107, N = 2000, n = 117$)



Final algorithm (periodic):

- Choose size of hyperbolic cross $\Lambda = \{\mathbf{k} \in \mathbb{Z}^d : \prod_{\ell=1}^{d} \max\{1, |k_\ell|\} \leq R\}$, $m = \#\Lambda$
- Set $N = \lceil 4m \log m \rceil$
- Choose $N$ nodes $\mathbf{x}^i \in [-1,1]^d$ uniformly at random
- Subsampling gives nodes $\{\mathbf{x}^j : j \in J\}$ with $n = \#J \leq \lceil 1.1m \rceil$ using the basis functions $\eta_{\mathbf{k}}(\mathbf{x}) = \prod_{\ell=1}^{d} e^{\pi i k_\ell x_\ell}$ (works for all of $H^s_{\text{mix}}(\mathbb{T}^d)$)
- Determine the coefficients $c_{\mathbf{k}}$ from the $\mathbf{x}^j, j \in J$ via the least-squares system with the basis functions $\eta_{\mathbf{k}}(\mathbf{x})$

# Non-periodic functions

Applying the above procedure to a more general function $f : [-1, 1]^d \to \mathbb{C}$ treats $f$ like a periodized version on $\mathbb{R}^d$

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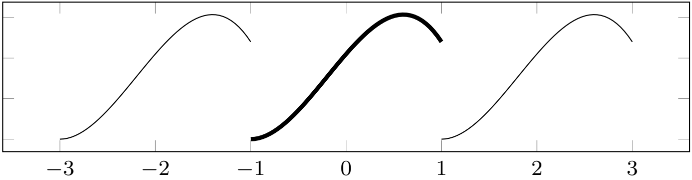

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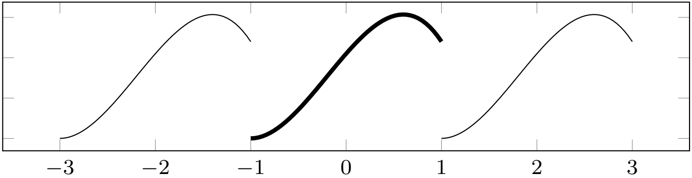


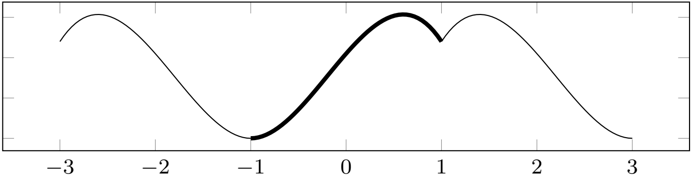

To apply the algorithm for periodic functions, we need to periodize $f$ in a way that preserves regularity

- Periodic extension: May introduce discontinuities
- Tent transform: Preserves continuity, might destroy smoothness (kinks)

We will use a **cosine composition** $T_{\cos}$ defined by

$$(T_{\cos} f)(x_1, ..., x_d) = f(\cos \pi x_1, ..., \cos \pi x_d)$$

**Theorem (Bartel, Lüttgen, N, T. Ullrich)**

*The operator $T_{\cos}$ is continuous as*

$$T_{\cos} : H^s_{mix}([-1,1]^d) \to H^s_{mix}(\mathbb{T}^d)$$

*for $s > 1/2$.*

More general versions over Besov spaces are possible (to be published in a future paper by Lüttgen, T. Ullrich)

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Strategy: Approximate $T_{\cos} f$ with the Fourier basis and undo the periodization

| | $f(\cos \pi x_1, \ldots, \cos \pi x_d)$ | $f(x_1, \ldots, x_d)$ |
|---|---|---|
| Sample nodes | $\mathbf{x}^i \sim \mathcal{U}[-1,1]^d$ | $\mathbf{x}^i = \cos(\pi \mathbf{U}^i), \mathbf{U}^i \sim \mathcal{U}[-1,1]^d$ <br> $\mathrm{d}\varrho(\mathbf{x}) = \prod_{\ell=1}^d \left(\pi\sqrt{1-x_\ell^2}\right)^{-1} \mathrm{d}\mathbf{x}$ |
| Basis functions | $\prod_{\ell=1}^d \cos(\pi k_\ell x_\ell)$ | $\prod_{\ell=1}^d \cos(k_\ell \arccos x_\ell)$ |

- $\mathbf{x}^i$ Chebyshev distributed
- $\eta_{\mathbf{k}}(\mathbf{x}) = \prod_{\ell=1}^d T_{k_\ell}(x_\ell), \mathbf{k} \in \mathbb{N}_0^d$ with $T_k(x) = \sqrt{2}^{\min\{1,k\}} \cos(k \arccos x)$ ($L_2(\varrho)$-normalized Chebyshev polynomials)

Strategy: Approximate $T_{\cos} f$ with the Fourier basis and undo the periodization

| | $f(\cos \pi x_1, \ldots, \cos \pi x_d)$ | $f(x_1, \ldots, x_d)$ |
|---|---|---|
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Final algorithm (non-periodic):

- Choose size of hyperbolic cross $\Lambda = \{\mathbf{k} \in \mathbb{N}_0^d : \prod_{\ell=1}^d \max\{1, k_\ell\} \leq R\}$ (in the non-negative orthant), $m = \#\Lambda$
- Set $N = \lceil 4m \log m \rceil$
- Choose $N$ nodes $\mathbf{x}^i \in [-1,1]^d$ Chebyshev distributed
- Subsampling gives nodes $\{\mathbf{x}^j : j \in J\}$ with $n = \#J \leq \lceil 1.1m \rceil$ using the basis functions $\eta_{\mathbf{k}}(\mathbf{x}) = \prod_{\ell=1}^d T_{k_\ell}(x_\ell)$ (works for all of $H^s_{\text{mix}}([-1,1]^d)$)
- Determine the coefficients $c_{\mathbf{k}}$ from the $\mathbf{x}^j, j \in J$ via the least-squares system with the basis functions $\eta_{\mathbf{k}}(\mathbf{x})$

**Theorem (Bartel, Lüttgen, N, T. Ullrich)**

*For $s > 1/2$ and using $n$ samples, the above algorithm yields (with high probability) an approximation $\tilde{f}$ with*

$$\|f - \tilde{f}\|_{L_2(\varrho)} \lesssim n^{-s} (\log n)^{(d-1)s+1/2} \|f\|_{H^s_{\text{mix}}([-1,1]^d)}.$$

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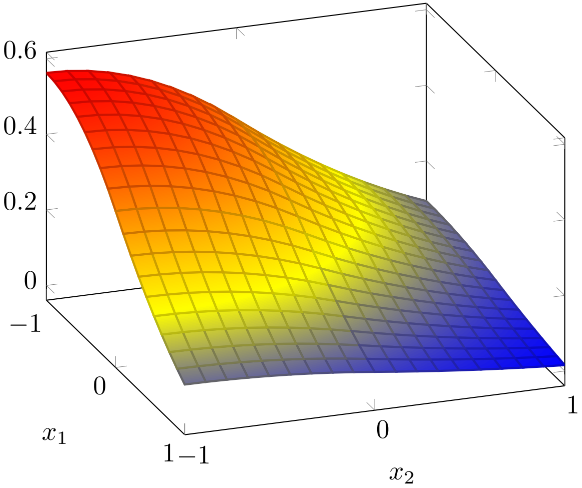
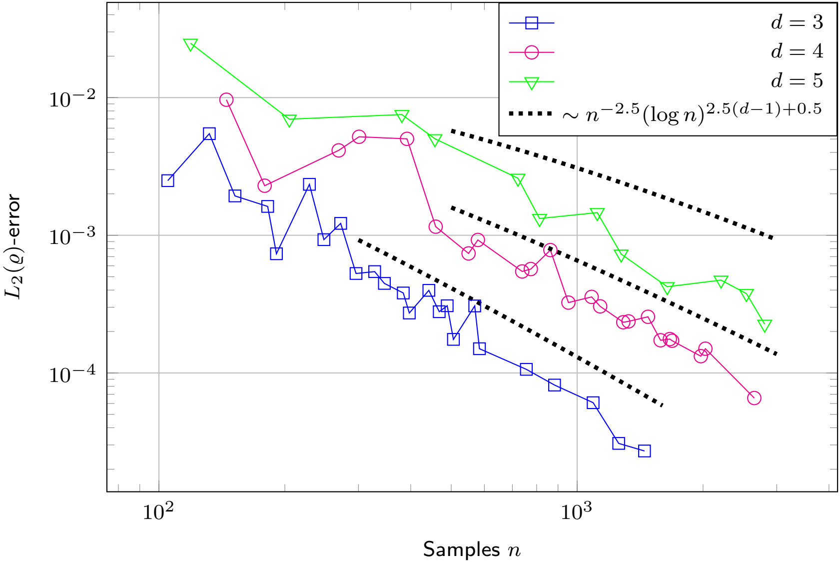
# Numerical experiment

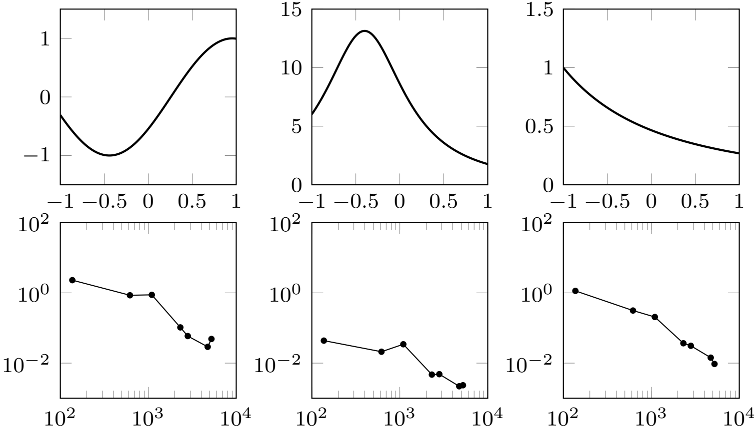
Test function: Tensored cutout of a quadratic B-spline (smoothness $s = 2.5$)



Approximation error for an $f \in H^{2.5-\varepsilon}_{\text{mix}}([-1,1]^d)$



Further test functions (7-dimensional), error measured in $L_\infty$ (Barthelmann, Novak, Ritter 2000)



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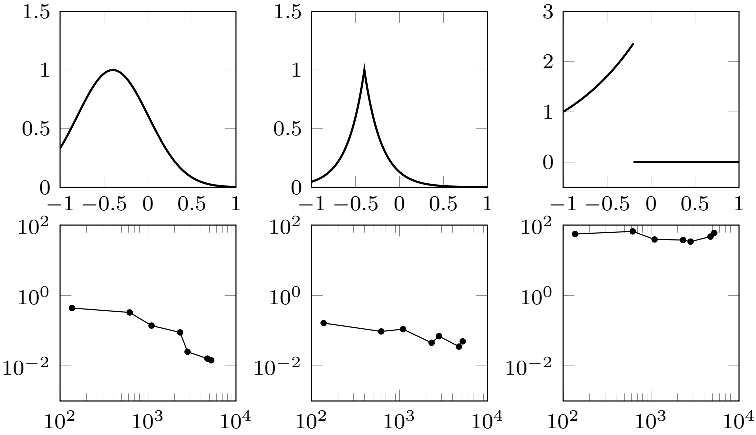

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- Deterministic constructions for good samples nodes?

Thank you for your attention!

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