The L_2-discrepancy of Latin hypercubes

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Inspired by conversations with Dmitriy Bilyk and Ian Ruohoniemi (University of Minnesota)

ICERM

Recent Progress on Optimal Point Distributions and Related Fields

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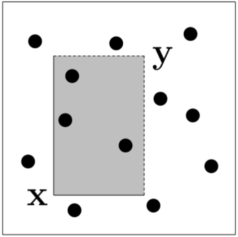
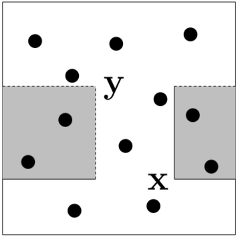




Extremal and periodic L_2-discrepancy

$X \subseteq [0,1)^d, \#X = N$

$$L_2^{\text{extr}}(X)^2 := \iint_{\mathbf{x}<\mathbf{y}} \big(\#(X \cap [\mathbf{x},\mathbf{y})) - N|[\mathbf{x},\mathbf{y})|\big)^2 \,\mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{y}$$

$$= \frac{N^2}{12^d} - \frac{N}{2^{d-1}} \sum_{\mathbf{x}\in X} \prod_{k=1}^{d} x_k(1-x_k)$$

$$+ \sum_{\mathbf{x},\mathbf{y}\in X} \prod_{k=1}^{d} (\min\{x_k, y_k\} - x_k y_k)$$

$$L_2^{\text{per}}(X)^2 := \iint \big(\#(X \cap [\mathbf{x},\mathbf{y})) - N|[\mathbf{x},\mathbf{y})|\big)^2 \,\mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{y}$$

$$= -\frac{N^2}{3^d} + \sum_{\mathbf{x},\mathbf{y}\in X} \prod_{k=1}^{d} \left(\frac{1}{2} - |x_k - y_k| + |x_k - y_k|^2\right)$$

What's known?

[Hinrichs, Kritzinger, Pillichshammer]:
For **rational lattices** and **Hammersley sets**:

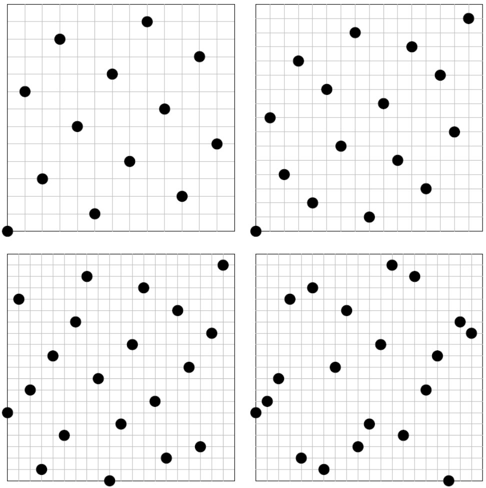
$$L_2^{\mathrm{per}}(X)^2 - 4L_2^{\mathrm{extr}}(X)^2 = \frac{N^2+1}{18N^2}$$

What's new? [N.]

Holds for all **permutation sets**

$$X = \left\{ \left(\frac{m}{N}, \frac{\sigma(m)}{N} \right) : m = 0, 1, \dots, N-1 \right\},$$

$\sigma : \{0, 1, \dots, N-1\} \to \{0, 1, \dots, N-1\}$ an arbitrary permutation.



Latin hypercubes

$\mathcal{H} \subseteq G := \frac{1}{M}\{0, 1, \ldots, M-1\}^d$, one point from each row of the **discretized torus** G, $\#\mathcal{H} = N = M^{d-1}$.

Theorem 1: Precise relation extremal vs periodic [N.]

$$L_2^{\text{per}}(\mathcal{H})^2 - 2^d L_2^{\text{extr}}(\mathcal{H})^2 = \frac{(2M^2+1)^d + (M^2-1)^d - (1+2^d)M^{2d}}{6^d M^2} = \Theta\left(M^{2(d-2)}\right)$$

Theorem 2: Formula for Latin hypercubes [N.]

$$L_2^{\text{per}}(\mathcal{H})^2 = -\frac{N^2}{3^d} + \sum_{\mathbf{f}\in G} \mu_{\mathbf{f}} \left| \sum_{\mathbf{x}\in\mathcal{H}} \exp\left(2\pi \mathrm{i} M \cdot \mathbf{f}^\top \mathbf{x}\right) \right|^2,$$

$$\mu_{\mathbf{f}} = \prod_{k=1}^{d} \begin{cases} \frac{1}{3} + \frac{1}{6M^2} & , f_k = 0 \\ \frac{1}{2M^2 \sin^2(\pi f_k)} & , f_k \neq 0 \end{cases}$$

Lower bound [N.]

$$L_2^{\text{per}}(\mathcal{H}) \geq \left(\frac{d}{2\cdot 3^d}\right)^{1/2} N^{\frac{d-2}{d-1}}, \quad L_2^{\text{extr}}(\mathcal{H}) \geq \left(\frac{d}{12^d}\right)^{1/2} (1-o(1))N^{\frac{d-2}{d-1}}$$

$d=2$ (permutation sets): Candidates for (approximate) global minimizers.
$d=3$: Same order as random $O(\sqrt{N})$.

Upper bound [N.]

$$\mathbb{E}L_2^{\text{per}}(\mathcal{H})^2 = \frac{(M-1)(M+1)^d + (2M^2+1)^d - 2^d M^{2d}}{6^d M^2} = \Theta\left(M^{\max\{d-1,2(d-2)\}}\right)$$

For $d \geq 4$:

$$\exists \mathcal{H}: \quad L_2^{\text{per}}(\mathcal{H}) \leq \left(\frac{d}{2\cdot 3^d}\right)^{1/2} (1+o(1))N^{\frac{d-2}{d-1}}$$

Asymptotically tight!