The Connection Between the Kadison-Singer Problem and Frame Theory

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TECHNISCHE UNIVERSITÄT
CHEMNITZ

- Frames crucial for approximation theory.
- $(\mathcal{H}, \langle \cdot, \cdot \rangle)$ complex Hilbert space, $\| \cdot \|$ its norm.
- **Frame** is a sequence $(f_i)_{i \in I}$ in $\mathcal{H}$ with absolute constants $0 < c \leq C$ such that
$$c\|f\|^2 \leq \sum_{i \in I} |\langle f, f_i \rangle|^2 \leq C\|f\|^2 \quad \forall f \in \mathcal{H}.$$
- Question: When does a frame have a "good" subframe? (*subsampling*)
- Application: Lower budget while keeping good approximation properties.
- Answer: *Weaver's KS_2-conjecture*.

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Interlude: Kadison-Singer problem.

- Motivated by quantum mechanics.
- $\mathcal{H} = \ell_2(\mathbb{N})$ Hilbert space of complex, square-summable sequences.
- $\mathfrak{B} = \mathcal{L}(\mathcal{H})$ space of bounded, linear operators $\mathcal{H} \to \mathcal{H}$.
- $\mathfrak{D} \subseteq \mathfrak{B}$ space of *diagonal* operators.
- **State** a continuous, linear functional $\varphi : \mathfrak{D} \to \mathbb{C}$ with
  - (i) $\varphi(I) = 1$ (normalization);
  - (ii) $\varphi(P) \geq 0$ for all *positive* operators $P \in \mathfrak{D}$ (positivity).
- Set of all states $\mathcal{S} \subseteq \mathfrak{D}'$ in the dual of $\mathfrak{D}$ convex and weak*-compact, thus $\mathcal{S} \subseteq \mathfrak{D}'$ the convex hull of its *extreme points*.
- Extreme points of $\mathcal{S}$ are called **pure states** (i.e. cannot be written as a proper convex combination of two other states).

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- Every state $\varphi : \mathfrak{D} \to \mathbb{C}$ can be extended to a state $\psi : \mathfrak{B} \to \mathbb{C}$ (restrict to entries on the main diagonal).
- **Kadison, Singer (1959):** Is the extension of a pure state unique?

Theorem (Marcus, Spielman, Srivastava 2015; Weaver 2004; Anderson 1979)

YES!

Kadison-Singer $\Leftrightarrow$ Anderson paving $\Leftrightarrow$ Weaver KS_2($\Leftrightarrow$ Feichtinger)

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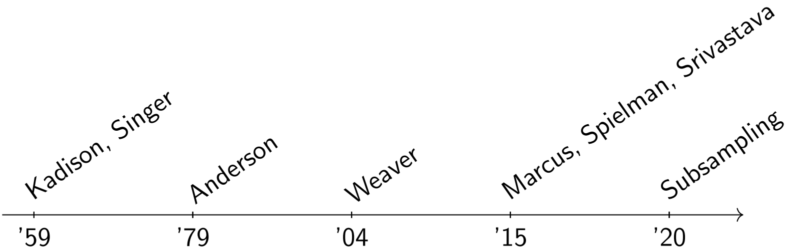
MSS $\Rightarrow$ Anderson $\Rightarrow$ Kadison-Singer
MSS $\Rightarrow$ Weaver $\Rightarrow$ Subsampling



Overview of the proof of the Kadison-Singer problem.

Theorem (Marcus, Spielman, Srivastava 2015)

Let $\varepsilon > 0$ *and* $v_1, ..., v_m \in \mathbb{C}^n$ *be independent random vectors with finite support fulfilling*

$$\sum_{i=1}^{m} \mathbb{E} v_i v_i^* = I$$

and $\mathbb{E}\|v_i\|^2 \leq \varepsilon$ *for all* i. *Then*

$$\mathbb{P}\left(\left\|\sum_{i=1}^{m} v_i v_i^*\right\| \leq (1+\sqrt{\varepsilon})^2\right) > 0.$$

- Algebraically: there is an assignment fulfilling the stated bound.

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Corollary

Let $r \in \mathbb{N}$ *and* $u_1, \dots, u_m \in \mathbb{C}^n$ *vectors fulfilling*

$$\sum_{i=1}^{m} u_i u_i^* = I$$

and $\|u_i\|^2 \leq \delta$ *for all* i. *Then there exists a partition* $\{S_1, \dots, S_r\}$ *of* $[m]$ *with*

$$\left\| \sum_{i \in S_j} u_i u_i^* \right\| \leq \left(\frac{1}{\sqrt{r}} + \sqrt{\delta} \right)^2 \quad \forall j = 1, \dots, r.$$

- Apply the Theorem to the uniformly random vectors $v_i = \sqrt{r}[\mathbf{0} \dots u_i \dots \mathbf{0}]^\top \in \mathbb{C}^{rn}$ (randomness in the position of u_i).
- Get partition from the position of the u_i in the assignment of v_i.

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A matrix $T \in \mathbb{C}^{n\times n}$ is (r,ε)-**pavable** if there are coordinate projections $P_1, ..., P_r \in \{0,1\}^{n\times n}$ with $\sum_{i=1}^r P_i = I$ and $\|P_i T P_i\| \le \varepsilon \|T\|$ for all i.

Theorem (Anderson's Paving Conjecture)

For every $0 < \varepsilon < 1$ there is an $r = r(\varepsilon) \in \mathbb{N}$, so that any Hermitian $T \in \mathbb{C}^{n\times n}$ with zero diagonal is (r,ε)-pavable.

- The concrete projections can depend on T, but their number can be bounded by ε alone.
- Uses the previous Corollary and even yields the bound $r \le 136/\varepsilon^4$.
- Optimal behaviour of $r(\varepsilon)$ not known, but have $r(\varepsilon) \ge \varepsilon^{-2}$ (*conference matrices*).

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- Kadison-Singer follows from an infinite dimensional analogue of Anderson's paving conjecture.
- Proof via compactness (Arzelà-Ascoli).

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Let T be a continuous, selfadjoint operator on $\mathcal{H} = \ell_2$ with zero diagonal. Then T can be (r, ε)-paved with $r \leq 136/\varepsilon^4$.

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The extension of a pure state $\varphi : \mathfrak{D} \to \mathbb{C}$ to a state $\psi : \mathfrak{B} \to \mathbb{C}$ is unique.

- Suffices to show: If Q is selfadjoint with zero diagonal then $\psi(Q) = 0$. Let $\varepsilon > 0$ arbitrary.
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Application of Weaver's Conjecture to Frames.

Corollary

Let $r \in \mathbb{N}$ *and* $u_1, ..., u_m \in \mathbb{C}^n$ *vectors fulfilling*

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and $\|u_i\|^2 \leq \delta$ *for all* i*. Then there exists a partition* $\{S_1, ..., S_r\}$ *of* $[m]$ *with*

$$\left\| \sum_{i \in S_j} u_i u_i^* \right\| \leq \left(\frac{1}{\sqrt{r}} + \sqrt{\delta} \right)^2 \quad \forall j = 1, ..., r.$$

Theorem (Weaver's KS_2-conjecture)

Let $u_1, ..., u_m \in \mathbb{C}^n$ *be vectors with* $\|u_i\| \leq 1$ *and suppose*

$$\sum_{i=1}^{m} |\langle w, u_i \rangle|^2 = 18\|w\|^2 \quad \forall w \in \mathbb{C}^n.$$

Then there is a partition $S_1 \dot{\cup} S_2 = [m]$ *with*

$$\sum_{i \in S_j} |\langle w, u_i \rangle|^2 \leq 16\|w\|^2 \quad \forall w \in \mathbb{C}^n \forall j = 1, 2.$$

Problems:

- In general we do not have a *tight* frame.
- Don't need a partition, just a good subframe.

First results by Nitzan, Olevskii, Ulanovskii (2016):

Theorem

Let $1 > \varepsilon > 0$ *and* $u_1, ..., u_m \in \mathbb{C}^n$ *be vectors with* $\|u_i\|^2 \leq \varepsilon$ *and suppose*

$$\sum_{i=1}^{m} |\langle w, u_i \rangle|^2 = \|w\|^2 \quad \forall w \in \mathbb{C}^n.$$

Then there is a partition $S_1 \dot{\cup} S_2 = [m]$ *with*

$$\frac{1 - 5\sqrt{\varepsilon}}{2} \|w\|^2 \leq \sum_{i \in S_j} |\langle w, u_i \rangle|^2 \leq \frac{1 + 5\sqrt{\varepsilon}}{2} \|w\|^2 \quad \forall w \in \mathbb{C}^n \forall j = 1, 2.$$

Corollary

Let $u_1, ..., u_m \in \mathbb{C}^n$ *with* $\|u_i\|^2 \leq \varepsilon$ *and*

$$\alpha\|w\|^2 \leq \sum_{i=1}^{m} |\langle w, u_i\rangle|^2 \leq \beta\|w\|^2 \quad \forall w \in \mathbb{C}^n,$$

where $0 < \varepsilon < \alpha \leq \beta$. *Then there is a partition* $S_1 \dot\cup S_2 = [m]$ *with*

$$\frac{1-5\sqrt{\varepsilon/\alpha}}{2} \cdot \alpha\|w\|^2 \leq \sum_{i \in S_j} |\langle w, v_i\rangle|^2 \leq \frac{1+5\sqrt{\varepsilon/\alpha}}{2} \cdot \beta\|w\|^2 \quad \forall w \in \mathbb{C}^n \forall j = 1, 2.$$

- Operator $Mw := \sum_{i=1}^{m} \langle w, u_i\rangle u_i$, then $M^{-1/2}u_i$ tight frame.

Corollary

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Iterative application of the previous Corollary:

Theorem (Nitzan, Olevskii, Ulanovskii 2016)

Let $u_1, ..., u_m \in \mathbb{C}^n$ *with* $\|u_i\|^2 = \frac{n}{m}$ *for all* i *and*

$$\sum_{i=1}^{m} |\langle w, u_i \rangle|^2 = \|w\|^2 \quad \forall w \in \mathbb{C}^n$$

Then there is a $J \subseteq [m]$ *with*

$$c \cdot \frac{n}{m}\|w\|^2 \leq \sum_{i \in J} |\langle w, u_i \rangle|^2 \leq C \cdot \frac{n}{m}\|w\|^2 \quad \forall w \in \mathbb{C}^n,$$

where $c \geq 25$ *and* $C \leq 3521$.

Theorem (Nitzan, Olevskii, Ulanovskii 2016)

There are constants $c, C > 0$ such that: For every set $S \subseteq \mathbb{R}$ of finite measure there is a discrete set $\Lambda \subseteq \mathbb{R}$ so that $E(\Lambda) = \{\exp(i\lambda\cdot)\}_{\lambda\in\Lambda}$ is a frame in $L_2(S)$ with frame bounds $c|S|$ and $C|S|$ (where $|S|$ is the measure of S), that is

$$c|S| \cdot \|h\|_2^2 \leq \sum_{u\in E(\Lambda)} |\langle h, u\rangle|^2 \leq C|S| \cdot \|h\|_2^2$$

for all $h \in L_2(S)$.

Note: $c \geq 6 \cdot 10^{-4}$ and $C \leq 6 \cdot 10^{8}$.

Theorem (Nitzan, Olevskii, Ulanovskii 2016)

Let $u_1, \dots, u_m \in \mathbb{C}^n$ *with* $\|u_i\|^2 = \frac{n}{m}$ *for all* i *and*

$$\sum_{i=1}^{m} |\langle w, u_i \rangle|^2 = \|w\|^2 \quad \forall w \in \mathbb{C}^n$$

Then there is a $J \subseteq [m]$ *with*

$$c \cdot \frac{n}{m}\|w\|^2 \leq \sum_{i \in J} |\langle w, u_i \rangle|^2 \leq C \cdot \frac{n}{m}\|w\|^2 \quad \forall w \in \mathbb{C}^n,$$

where $c \geq 25$ *and* $C \leq 3521$.

Problems:

- Do not necessarily have vectors of a precise norm, just bounded norm.
- Do not necessaily start out with a tight frame.
- How to bound $\#J$?

Theorem (N., Schäfer, T. Ullrich / Limonova, Temlyakov 2020)

Let $k_1, k_2, k_3 > 0$ and $u_1, \dots, u_m \in \mathbb{C}^n$ with $\|u_i\|^2 \le k_1 \frac{n}{m}$ for all i and

$$k_2\|w\|^2 \le \sum_{i=1}^{m} |\langle w, u_i\rangle|^2 \le k_3\|w\|^2 \quad \forall w \in \mathbb{C}^n.$$

Then there is a $J \subseteq [m]$ of size $\#J \le c_1 n$ with

$$c_2 \cdot \frac{n}{m}\|w\|^2 \le \sum_{i\in J} |\langle w, u_i\rangle|^2 \le c_3 \cdot \frac{n}{m}\|w\|^2 \quad \forall w \in \mathbb{C}^n,$$

where c_1, c_2, c_3 only depend on k_1, k_2, k_3.

- Constants c_1, c_2, c_3 may be huge: For $k_1 = k_3 = 2$ and $k_2 = 1/2$ have $c_1 = 6600, c_2 = 24, c_3 = 13200.$
- Better constants possible, see Martin Schäfer's talk.

Theorem (Limonova, Temlyakov 2020)

Let $\emptyset \neq \Omega \subseteq \mathbb{R}^d$ *and* μ *a probability measure on* Ω. *Let* $X \subseteq L_2(\Omega, \mu)$ *be an* n*-dimensional subspace. There are universal constants* C_1, C_2, C_3 *such that there are* $\{\xi^k\}_{k=1}^m \subseteq \Omega$ *with* $m \leq C_1 n$ *and nonnegative weights* $\{\lambda_k\}_{k=1}^m$ *with*

$$C_2 \|f\|_{L_2}^2 \leq \sum_{k=1}^m \lambda_k |f(\xi^k)|^2 \leq C_3 \|f\|_{L_2}^2 \quad \forall f \in X.$$

Corollary (Temlyakov 2020)

Sampling numbers $\lesssim$ *Kolmogorov numbers in* L_∞

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Theorem (N., Schäfer, T. Ullrich 2020)

Let $H(K)$ be a separable reproducing kernel Hilbert space on $D \subseteq \mathbb{R}^d$ with positive semidefinite kernel $K : D \times D \to \mathbb{C}$ satisfying

$$\int_D K(x,x) d\varrho(x) < \infty$$

for some measure ϱ on D. Then $\mathrm{Id}_{K,\varrho} : H(K) \to L_2(D,\varrho)$ is a Hilbert-Schmidt embedding, the corresponding sequence of singular numbers $(\sigma_k)_{k=1}^{\infty}$ square-summable. For the sequence of sampling numbers

$$g_n = \inf_{x^1,\dots,x^n \in D} \inf_{\varphi : \mathbb{C}^n \to L_2} \sup_{\|f\|_{H(K)} \le 1} \|f - \varphi(f(x^1),\dots,f(x^n))\|_{L_2(D,\varrho)}$$

we have the bound

$$g_n^2 \le C \frac{\log n}{n} \sum_{k \ge cn} \sigma_k^2 \quad \forall n \ge 2,$$

where $C, c > 0$ are universal constants.

Theorem (Dolbeault, Krieg, M. Ullrich 2022)

The setting as in the previous theorem, then even

$$g_n^2 \leq \frac{C}{n} \sum_{k \geq cn} \sigma_k^2,$$

matching the lower bound.

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