Efficient recovery of non-periodic functions via samples

Felix Bartel, Kai Lüttgen, Nicolas Nagel, Tino Ullrich
TU Chemnitz

Based on the work in [BLNU]

Theory: cos-composition

- Periodization via cosine composition

$$(T_{\cos} f)(x_1, \dots, x_d) = f(\cos x_1, \dots, \cos x_d)$$

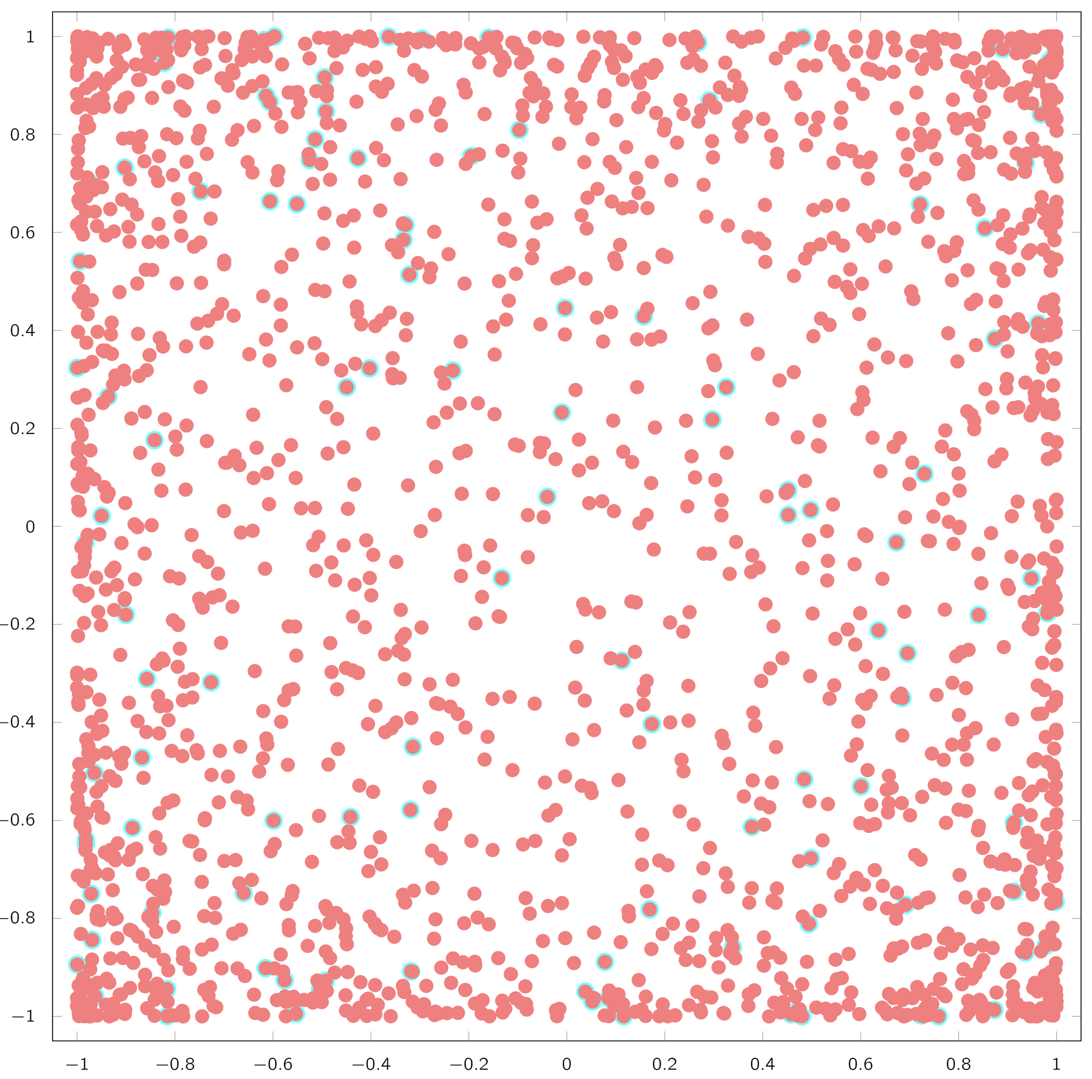
- Important: Preserves regularity, i.e. continuity of the operator

$$T_{\cos} : H^s_{\text{mix}}([-1,1]^d) \to H^s_{\text{mix}}(\mathbb{T}^d)$$

- Thus: Approximation of periodic functions via Fourier functions lifts to the non-periodic setting to Chebyshev polynomials

Algorithm: Subsampling

- Node set in $[-1,1]^d \to$ frame in $\mathbb{C}^d \to$ subsampling
- Existence: Via Kadison-Singer [MSS]
- Algorithm: Based on [BSS] and developed further in [BSU]
- Initial node set: random points, Chebyshev distributed $\prod_{i=1}^d \left(\pi\sqrt{1-x_i^2}\right)^{-1} \mathrm{d}\mathbf{x}$



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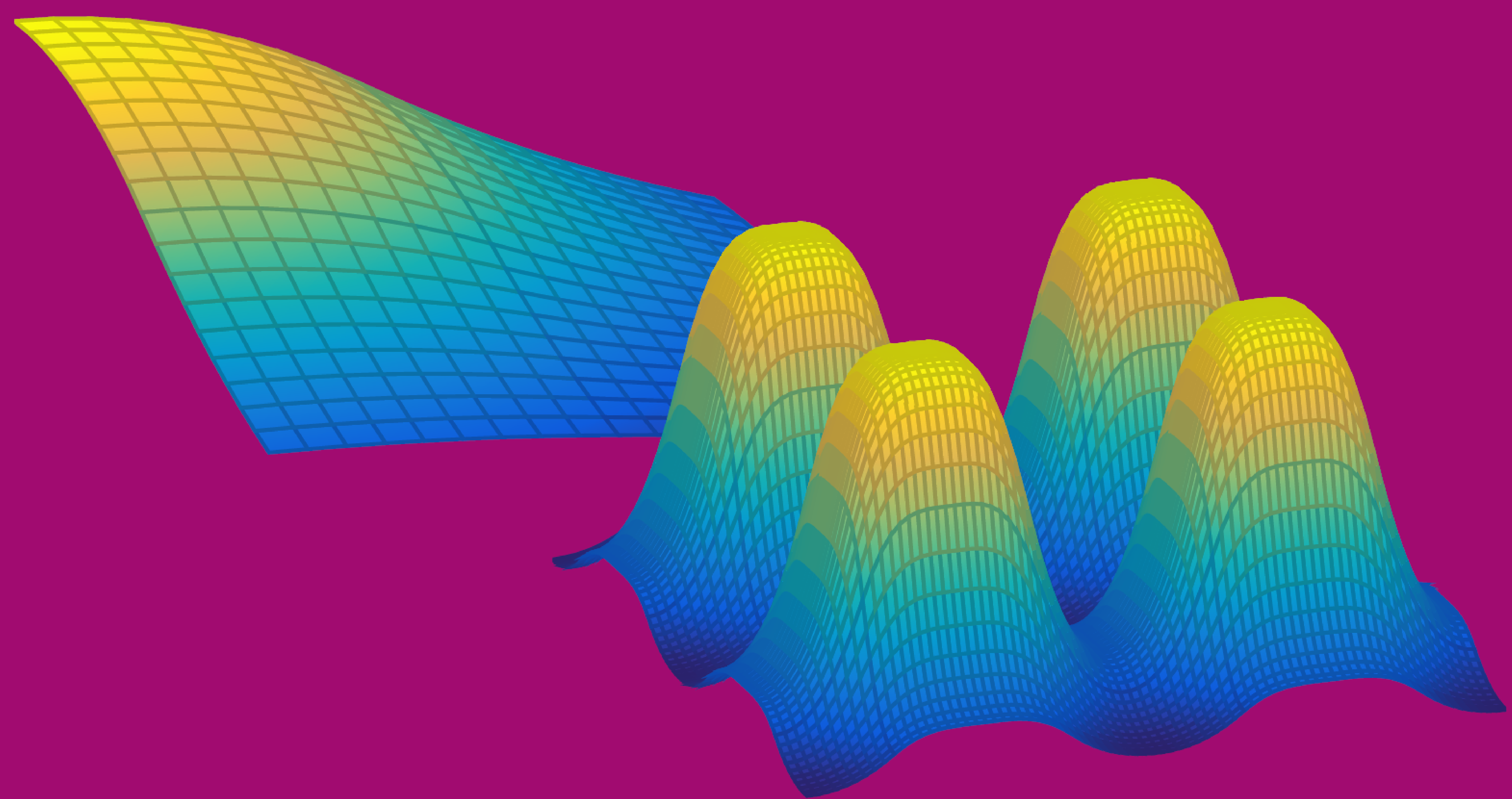
Chebyshev Basis Recovers Non-periodic Functions Efficiently



[BLNU] F. Bartel, K. Lüttgen, N. Nagel, T. Ullrich. *Efficient recovery of non-periodic multivariate functions from few scattered samples*. SampTA (2023)

[BSU] F. Bartel, M. Schäfer, T. Ullrich. *Constructive subsampling of finite frames with applications in optimal function recovery*. Appl. Comput. Harmon. Anal. (2023)

[BSS] J. D. Batson, D. A. Spielman, N. Srivastava. *Twice-Ramanujan sparsifiers*. SICOMP (2012)

[MSS] A. W. Marcus, D. A. Spielman, N. Srivastava. *Interlacing families II: Mixed characteristic polynomials and the Kadison–Singer problem*. Annals of Mathematics (2015)

{felix.bartel, kai.luettgen, nicolas.nagel, tino.ullrich}@math.tu-chemnitz.de

Numerical experiments

- (Tensored) test function

$$f(x) = \begin{cases} -\frac{1}{4}x^2 - \frac{1}{2}x + \frac{1}{2} & , -1 \le x \le 0 \\ \frac{1}{8}x^2 - \frac{1}{2}x + \frac{1}{2} & , 0 < x \le 1 \end{cases}$$

(smoothness $s = 2.5$)

- Hyperbolic cross

$$\Lambda := \{\mathbf{k} \in \mathbb{N}_0^d : \prod_{i=1}^d \max\{1, k_i\} \le R\},$$

$m := \#\Lambda$

- $M := \lceil 4m \log m \rceil$ random points $\{\mathbf{x}^i\}_{i=1}^M$ (Chebyshev measure)
- Subsample $\mathbf{u}_i = \left[\eta_{\mathbf{k}}(\mathbf{x}^i)\right]_{\mathbf{k}\in\Lambda}$ with $\eta_{\mathbf{k}}(\mathbf{x}^i) = \prod_{\ell=1}^d T_{k_\ell}(x_\ell^i)$, $T_k(x) = \sqrt{2}^{\min\{1,k\}} \cos(k \arccos(x))$

$\to \{\mathbf{x}^j\}_{j\in J}$ with $\#J := n = O(m)$

- $f(\mathbf{x}) \approx \sum_{\mathbf{k}\in\Lambda} c_{\mathbf{k}} \eta_{\mathbf{k}}(\mathbf{x})$ via least squares

$$\begin{bmatrix} \eta_{\mathbf{k}_1}(\mathbf{x}^{j_1}) & \cdots & \eta_{\mathbf{k}_m}(\mathbf{x}^{j_1}) \\ \vdots & & \vdots \\ \eta_{\mathbf{k}_1}(\mathbf{x}^{j_n}) & \cdots & \eta_{\mathbf{k}_m}(\mathbf{x}^{j_n}) \end{bmatrix} \begin{bmatrix} c_{\mathbf{k}_1} \\ \vdots \\ c_{\mathbf{k}_m} \end{bmatrix} \approx \begin{bmatrix} f(\mathbf{x}^{j_1}) \\ \vdots \\ f(\mathbf{x}^{j_n}) \end{bmatrix}$$

- (Chebyshev) L_2-error $\lesssim n^{-s}(\log n)^{(d-1)s+1/2}$
- For comparison: Uniform samples and half-period cosine basis gives (Lebesgue) L_2-error $\lesssim n^{-(1-\varepsilon)\min\{1.5,s\}}$