$$2^{n-1} \geq 2^n - (2^{n/t} - 1)^t \tag{0.1}$$
$$(2^{n/t} - 1)^t \geq 2^{n-1} \tag{0.2}$$
$$2^{n/t} - 1 \geq 2^{n/t} \cdot 2^{-1/t} \tag{0.3}$$
$$2^{n/t}(1 - 2^{-1/t}) \geq 1 \tag{0.4}$$
$$2^{n/t} \geq \frac{1}{1 - 2^{-1/t}} \tag{0.5}$$
$$\frac{n}{t} \geq \log_2 \left(\frac{1}{1 - 2^{-1/t}} \right) \tag{0.6}$$
$$n \geq t \cdot \log_2 \left(\frac{1}{1 - 2^{-1/t}} \right) \tag{0.7}$$

Apply $e^x > x + 1$ for $x = -\frac{\ln 2}{t} \neq 0$:

$$2^{-1/t} > -\frac{\ln 2}{t} + 1 \tag{0.8}$$
$$\frac{\ln 2}{t} > 1 - 2^{-1/t} \tag{0.9}$$
$$\frac{1}{1 - 2^{-1/t}} > \frac{t}{\ln 2} \tag{0.10}$$

For $C = 1 + e^{-1}$ want to show

$$\frac{Cn}{\log_2 n} \cdot \log_2 \left(\frac{Cn}{\ln 2 \cdot \log_2 n} \right) \geq n.$$

This is equivalent to:

$$\log_n \left(\frac{Cn}{\ln 2 \cdot \log_2 n} \right) \geq \frac{1}{C} \tag{0.11}$$
$$\frac{Cn}{\ln 2 \cdot \log_2 n} \geq n^{1/C} \tag{0.12}$$
$$\frac{C}{\ln 2} \geq \log_2 n \cdot n^{1/C-1} \tag{0.13}$$
$$C \geq \ln n \cdot n^{1/C-1} \tag{0.14}$$

For $f(x) = \ln x \cdot x^{1/C-1}$ have

$$f'(x) = x^{1/C-2} + \left(\frac{1}{C} - 1 \right) \ln x \cdot x^{1/C-2} = \left(1 + \left(\frac{1}{C} - 1 \right) \ln x \right) x^{1/C-2},$$

which is zero iff

$$1 + \left(\frac{1}{C} - 1 \right) \ln x_0 = 0,$$

i.e.

$$x_0 = \exp \left(\frac{1}{1 - \frac{1}{C}} \right) = \exp \left(\frac{C}{C-1} \right).$$

There

$$f(x_0) = \frac{C}{C-1} \exp\left(\frac{C}{C-1} \cdot \frac{1-C}{C}\right) = \frac{1+e^{-1}}{e^{-1}} \cdot e^{-1} = C,$$

so $f(x_0) = C$ is a global maximum or minimum. But

$$f''(x) = \left(\left(\frac{1}{C} - 1\right) + \left(\frac{1}{C} - 2\right)\left(1 + \left(\frac{1}{C} - 1\right) \ln x\right)\right) x^{1/C-3},$$

plugging in $x = x_0$ gives

$$\frac{1-C}{C} + \frac{1-2C}{C}\left(1 + \frac{1-C}{C} \cdot \frac{C}{C-1}\right) = \frac{1-C}{C} < 0,$$

so $f''(x_0) < 0$ and thus

$$C = f(x_0) \geq f(x) = \ln x \cdot x^{1/C-1}$$

for all x.