Bachelorarbeit

On the Kadison-Singer Problem and Weaver's Conjecture with Implications for Fourier Systems over Unbounded Sets

Nicolas Nagel

August 24, 2020

Contents

1 Introduction

1.1 Aim

The Kadison-Singer problem was for a long time one of the major open problems in functional analysis. Even though it started out as a problem concerning a certain infinite dimensional C*-algebra, it turned out that it has many equivalent finite dimensional formulations, many of which have a highly combinatorial nature to them.
We will give a thorough explanation of all the tools that were developed by Marcus, Spielman and Srivastava and many before them that finally lead to a solution of the Kadison-Singer problem. The first half of this thesis will therefore consist mostly of a retelling of [19] with some more detailed information if we feel the need for them.
Soon after their solution, it was realized that their techniques can be applied to show the existence of other structures that are of functional analytic and approximation theoretic interest. The second half of this thesis will present such an application of the subsequently developed theory by Nitzan, Olanevskii and Ulanovskii, showing the existence of exponential frames on sets of finite measure. The latter section will therefore follow closely their paper [20].
The aim of this thesis is to be, for the most part, a self contained reference of this topic without the major need of outside sources. We wish to give a deep explanation of all the involved concepts and with only a few exceptions we prove all their (for us) necessary properties. As for required preliminaries, the first chapters on the Kadison-Singer problem should be understandable only with knowledge from basic linear algebra and calculus (in particular complex analysis; the Kadison-Singer problem itself requiring some functional analysis). The chapter on exponential frames assumes some knowledge on general Hilbert spaces and the Fourier transform.

1.2 Notation

We will use $\mathbb{N}$ for the (positive) natural numbers and $\mathbb{N}_0$ when we want to include 0, $\mathbb{R}$ for the reals and $\mathbb{C}$ for the complex numbers. For a ring R, we denote by $R[x]$ and $R[x_1, ..., x_n]$ the ring of (finite) polynomials (univariate or multivariate respectively) with coefficients in R. Of course, we are mostly interested in the case $R \in \{\mathbb{R}, \mathbb{C}\}$. For real numbers $x_1, ..., x_k, y, z_1, ..., z_l$ we write

$$\begin{matrix} x_1 \\ \vdots \\ x_k \end{matrix} \leq y \leq \begin{matrix} z_1 \\ \vdots \\ z_l \end{matrix}$$

as a convenient way to express $x_p \le y \le z_q$ for all $p = 1, ..., k$ and all $q = 1, ..., l$. This is purely cosmetically and is just chosen to be easier to grasp than the equivalent formulation

$$\max\{x_p : p = 1, ..., k\} \le y \le \min\{z_q : q = 1, ..., l\}.$$

We will adopt the conventions $\mathbb{R}_{\ge 0} = [0, \infty)$ and $\mathbb{R}_{>0} = (0, \infty)$. We will sometimes denote the set $\{1, ..., n\}$ by $[n]$. For a square matrix A we will denote its characteristic polynomial as $\chi[A](t) = \det(tI - A)$. For partial derivatives we will write ∂_{x_i} or simply ∂_i for the derivative after the i-th variable. The expectation and probability will be denoted using the symbols $\mathbb{E}$ and $\mathbb{P}$ respectively. We will also use the Landau symbol O occasionally. By $\mathbf{1} \in \mathbb{R}^m$ we will denote the all ones vector, by $\mathbf{0}$ the all zeroes-vector and we will also use $\mathbb{O}$ for the all zeroes matrix. log will stand for the natural logarithm. To better distinguish between them, we will write $\|\cdot\|$ for the euclidean norm on finite dimensional spaces $\mathbb{C}^N$ and $\|\cdot\|_2$ for the usual norm in the Lebesgue spaces L^2. Occasionally we will write $K \subset\subset \Omega$ to denote compact subsets. As is often seen, we write $U_\varepsilon(x)$ for the open ε-ball around x (with respect to some metric, clear from context). Also the closure of a set Ω (with the topology clear from context) will be denoted by $\overline{\Omega}$.

2 Polynomials and Expectations

2.1 Interlacings and Families of Polynomials

Polynomials of the kind in the following definition are also often referred to as "hyperbolic", which originates from the theory of partial differential equations.

Definition 2.1. A polynomial $p \in \mathbb{C}[x]$ is said to be **real rooted** if all its roots are real numbers.

Of course, a real rooted polynomial which has at least one nonzero real coefficient (e.g. its leading coefficient) is a real polynomial, i.e. has only real coefficients. If we enumerate the roots $x_1, ..., x_n$ of a real rooted polynomial, we always do so in nondecreasing order, i.e. $x_1 \leq x_2 \leq ... \leq x_n$. For the major part, we are only interested in real rooted polynomials with real coefficients. Our main interest for now are certain families of real rooted polynomials.

Definition 2.2. For two real rooted polynomials, we say that $g(x) = \alpha_0 \prod_{i=1}^{n-1}(x - \alpha_i)$ **interlaces** $f(x) = \beta_0 \prod_{i=1}^{n}(x - \beta_i)$, if

$$\beta_1 \leq \alpha_1 \leq \beta_2 \leq \alpha_2 \leq ... \leq \beta_{n-1} \leq \alpha_{n-1} \leq \beta_n.$$

A family $\{f_j\}_{j=1}^k$ of real rooted polynomials with same degree has a **common interlacing**, if there is a real rooted polynomial g that interlaces all $f_j, j = 1, ..., k$ simultaneously, i.e. writing $f_j(x) = \beta_0^j \prod_{i=1}^{n}(x - \beta_i^j)$ and α_i as above we have

$$\begin{matrix} \beta_1^1 & & \beta_2^1 & & & \beta_{n-1}^1 & & \beta_n^1 \\ \vdots & \leq \alpha_1 \leq & \vdots & \leq \alpha_2 \leq & ... \leq & \vdots & \leq \alpha_{n-1} \leq & \vdots \\ \beta_1^k & & \beta_2^k & & & \beta_{n-1}^k & & \beta_n^k \end{matrix} .$$

If we have a family with common interlacing just like in the definition, we may think of the (multi-)sets $\{\beta_i^1, ..., \beta_i^k\}$ as "clusters" of roots, where two neighboring clusters can be "separated" by a corresponding root of g. Also note that therefore the interlacing property is more a property of the (multi-)sets of roots of the polynomials rather than of the polynomials themself. For some situations it might be better to think in terms of multisets, however the interpretation via polynomials will also turn out to be useful.

Remark 2.3. [11]

(i) If f has a root x_0 with multiplicity $d \geq 2$ and g interlaces f, then g also has x_0 as a root with multiplicity $\geq d-1$. Also, if $\{f_j\}_j$ has a common interlacing and one of the f_j has a root of order ≥ 3, then all of the f_j have that root (but not necessarily with the same multiplicities, even though they differ at most by two).

(ii) Assume $\{f_j\}_{j=1}^k$ is a family of real rooted polynomials all of degree n, such that β is a common root of the f_j, i.e. $f_j(\beta) = 0$ for all $j = 1, ..., k$. Write $\beta_1^j, ..., \beta_{n-1}^j$ for the roots of $f_j(x)/(x-\beta)$. Assume further that the family $\{f_j(x)/(x-\beta)\}_{j=1}^k$ has a common interlacing g with roots $\alpha_1, ..., \alpha_{n-2}$.

$$\begin{matrix} \beta_1^1 & & \beta_2^1 & & & & \beta_{n-2}^1 & & \beta_{n-1}^1 \\ \vdots & \leq \alpha_1 \leq & \vdots & \leq \alpha_2 \leq & \ldots & \leq & \vdots & \leq \alpha_{n-2} \leq & \vdots \\ \beta_1^k & & \beta_2^k & & & & \beta_{n-2}^k & & \beta_{n-1}^k \end{matrix}.$$

We want to reinsert β into this picture to show that the family $\{f_j\}_j$ has a common interlacing. If $\beta \leq \min\{\beta_1^j : 1 \leq j \leq k\} =: \underline{\beta}$ or $\overline{\beta} := \max\{\beta_{n-1}^j : 1 \leq j \leq k\} \leq \beta$ we have no difficulties with that, since we could simply add some $\alpha \in [\beta, \underline{\beta}]$ or $\alpha \in [\overline{\beta}, \beta]$ respectively to the roots of g to get a common interlacing $(x-\alpha)g(x)$ for $\{f_j\}_j$.
Otherwise we can order the polynomials f_j in such a way, that there is a $j_0 \in \{0, ..., k\}$ with

$$\begin{matrix} \beta_i^1 & & \beta_i^{j_0+1} \\ \vdots & \leq \beta \leq & \vdots \\ \beta_i^{j_0} & & \beta_i^k \end{matrix}$$

for some i. This means that the cluster $\{\beta_i^1, ..., \beta_i^k\}$ for the family $\{f_j(x)/(x-\beta)\}_j$ gets split up into $\{\beta_i^1, ..., \beta_i^{j_0}, \beta, ..., \beta\}$ and $\{\beta, ..., \beta, \beta_i^{j_0+1}, ..., \beta_i^k\}$ for the family $\{f_j\}_j$. But then the polynomial $(x-\beta)g(x)$ interlaces the family $\{f_j\}_j$.
Comparing this with the above case, $(x-\beta)g(x)$ is always a common interlacing for $\{f_j\}_j$, if g interlaces $\{f_j(x)/(x-\beta)\}_j$.

(iii) If a family $\{f_j\}_j$ of real rooted polynomials has the property, that any f_{j_1} and f_{j_2}, $j_1 \neq j_2$ have a common interlacing $g_{j_1 j_2}$, then $\{f_j\}_j$ already has a common interlacing. Indeed, again denoting the roots of f_j by β_i^j, by the assumption we know that the intervals $[\beta_i^{j_1}, \beta_{i+1}^{j_1}]$ and $[\beta_i^{j_2}, \beta_{i+1}^{j_2}]$ are not disjoint. Namely, the i-th root of $g_{j_1 j_2}$ lies in both by the interlacing property, see

$$\begin{matrix} \beta_i^{j_1} \\ \beta_i^{j_2} \end{matrix} \leq \alpha^{j_1 j_2} \leq \begin{matrix} \beta_{i+1}^{j_1} \\ \beta_{i+1}^{j_2} \end{matrix}. \tag{2.1}$$

Thus, for any given $i = 1, ..., n-1$ the system $\{[\beta_i^j, \beta_{i+1}^j] : j = 1, ..., k\}$ fulfills the so called (onedimensional) Helly property. A classical theorem by Helly and Radon [21, 15] then states, that also the total intersection

$$I_i := \bigcap_{j=1}^k [\beta_i^j, \beta_{i+1}^j]$$

is non-empty. It turns out that such a machinery however is not needed here and that there is even a short explanation for our situation. Simply choose $\beta_i^{j_0} := \max\{\beta_i^j : j = 1, ..., k\}$ which trivially fulfills $\beta_i^j \leq \beta_i^{j_0}$ for all j. But by (2.1) we also have $\beta_i^{j_0} \leq \alpha_i^{j_0 j} \leq \beta_{i+1}^j$ for all j. Thus $\beta_i^{j_0} \in [\beta_i^j, \beta_{i+1}^j]$ for all j, so that $\beta_i^{j_0} \in I_i$ is non-empty.
Choose now $\alpha_i \in I_i$ for $i = 1, ..., n-1$, these fulfill

$$\max\{\beta_i^j : j = 1, ..., k\} \leq \alpha_i \leq \min\{\beta_{i+1}^j : j = 1, ..., k\},$$

so that $g(x) = \prod_{i=1}^{n-1}(x - \alpha_i)$ is a common interlacing for $\{f_j\}_j$. This shows that "pairwise interlacable" and "totally interlacable" are indeed equivalent, the other direction being trivial.

The following result relies mainly on basic continuity arguments for polynomials.

Lemma 2.4. [18] Let $\{f_j\}_{j=1}^k$ be a family of real rooted polynomials all of the same degree having positive leading coefficients and define

$$f_\emptyset := \sum_{j=1}^{k} f_j.$$

If $\{f_j\}_{j=1}^k$ has a common interlacing, then there is a j such that the largest root of f_j is at most the largest root of $f_\emptyset$.

Proof. Let the f_j have common degree n and let g be a common interlacing with largest root α_{n-1}. Since all f_j have positive leading coefficient, $f_j(x)$ is positive for all j whenever x is sufficiently large. Also, by the fact that g is a common interlacing, every f_j has precisely one root $\geq \alpha_{n-1}$, so $f_j(\alpha_{n-1}) \leq 0$ for all $j = 1, ..., k$. Therefore $f_\emptyset(\alpha_{n-1}) \leq 0$ and, since $f_\emptyset$ has positive leading coefficient, we also have $f_\emptyset(x_1) > 0$ for a sufficiently large x_1, e.g. for all

$$x_1 > \max_{1 \leq j \leq k} \max\{x_0 : f_j(x_0) = 0\}.$$

This shows, that $f_\emptyset$ has a root in the interval $[\alpha_{n-1}, x_1)$ and is positive for all $x \geq x_1$, so the largest root of $f_\emptyset$ is larger than α_{n-1}, the largest root of g. Let β_n be this largest root of $f_\emptyset$.
Since $0 = f_\emptyset(\beta_n) = \sum_{j=1}^k f_j(\beta_n)$, there is a j with $f_j(\beta_n) \geq 0$. Collecting everything so far together, we get for this j:

$$f_j(\alpha_{n-1}) \leq 0, \quad f_j(\beta_n) \geq 0$$

and f_j has exactly one root $\geq \alpha_{n-1}$, so this root must be $\leq \beta_n$, which proves the claim. ∎

It will be convenient to define a similar concept for families of real rooted polynomials indexed by product sets.

Definition 2.5. Let $S_1, ..., S_m$ be finite sets and for all $(s_1, ..., s_m) \in S_1 \times ... \times S_m$ let $f_{s_1,...,s_m}$ be real rooted polynomials of common degree n with positive leading coefficients. For a partial assignment $(s_1, ..., s_k) \in S_1 \times ... \times S_k$ with $1 \leq k < m$ set

$$f_{s_1,...,s_k} := \sum_{s_{k+1} \in S_{k+1},...,s_m \in S_m} f_{s_1,...,s_k,s_{k+1},...,s_m},$$

as well as

$$f_\emptyset = \sum_{s_1 \in S_1,...,s_m \in S_m} f_{s_1,...,s_m}.$$

We say that the family $\{f_{s_1,...,s_m} : s_1 \in S_1, ..., s_m \in S_m\}$ forms an **interlacing family** if for all $k = 0, ..., m-1$ and all $(s_1, ..., s_k) \in S_1 \times ... \times S_k$, the polynomials

$$\{f_{s_1,...,s_k,t}\}_{t \in S_{k+1}}$$

have a common interlacing.

Theorem 2.6. [19] Let $S_1, ..., S_m$ be finite sets and $\{f_{s_1,...,s_m} : s_1 \in S_1, ..., s_m \in S_m\}$ an interlacing family of real rooted polynomials. Then there is an assignment $(s_1, ..., s_m) \in S_1 \times ... \times S_m$, such that the largest root of $f_{s_1,...,s_m}$ is at most the largest root of $f_\emptyset$.

Proof. By induction on m. The case $m = 1$ is precisely lemma 2.4 above. So assume the statement is true for $m - 1$. Using the induction hypothesis choose $(s_1, ..., s_{m-1}) \in S_1 \times ... \times S_{m-1}$ such that the largest root of $f_{s_1,...,s_{m-1}}$ is at most the largest root of $f_\emptyset$. The polynomials $\{f_{s_1,...,s_{m-1},t}\}_{t \in S_m}$ have a common interlacing by assumption. Since by definition we have

$$f_{s_1,...,s_{m-1}} = \sum_{t \in S_m} f_{s_1,...,s_{m-1},t},$$

again by lemma 2.4 there is an element $s_m \in S_m$, such that the largest root of $f_{s_1,...,s_m}$ is at most the largest root of $f_{s_1,...,s_{m-1}}$, which by choice of $(s_1, ..., s_{m-1})$ is at most the largest root of $f_\emptyset$ (with the induction hypothesis). ∎

Notice that this proof gives an iterative method for finding an assignment $(s_1, ..., s_m) \in S_1 \times ... \times S_m$ as in the statement. The following theorem gives a useful classification for families with a common interlacing.

Theorem 2.7. [11] Let $\{f_j\}_{j=1}^k$ be real polynomials (i.e. only real coefficients) of the same degree with positive leading coefficients. Then $\{f_j\}_{j=1}^k$ has a common interlacing, if and only if $\sum_{j=1}^k \lambda_j f_j$ is real rooted for all convex combinations $\lambda_j \geq 0, \sum_{j=1}^k \lambda_j = 1$.

Proof. We set $n := \deg f_1 = ... = \deg f_k$ their common degree and denote by $\beta_1^j \leq ... \leq \beta_n^j$ the roots of f_j.
($\Rightarrow$): By induction on n. Without loss of generality we may assume that the f_j have no common roots. If otherwise we had a β with $f_j(\beta) = 0$ for all $j = 1, ..., k$, we can apply the induction hypothesis on $\{f_j(x)/(x - \beta)\}_j$, where we use remark 2.3 (ii). Consider now a convex combination $f = \sum_{j=1}^k \lambda_j f_j$ where we may further assume $\lambda_j > 0$ for all

j, otherwise we can just restrict or considerations to the set of all indices j for which $\lambda_j > 0$.
Let g be a common interlacing for the f_j with roots α_i, i.e. we have the by now familiar situation

$$\begin{matrix} \beta_1^1 & & \beta_2^1 & & & & \beta_{n-1}^1 & & \beta_n^1 \\ \vdots & \leq \alpha_1 \leq & \vdots & \leq \alpha_2 \leq & \ldots & \leq & \vdots & \leq \alpha_{n-1} \leq & \vdots \\ \beta_1^k & & \beta_2^k & & & & \beta_{n-1}^k & & \beta_n^k \end{matrix}.$$

For $1 \leq j \leq k$, since f_j has positive leading coefficient and g interlaces f_j, we have

$$f_j(\alpha_i) \begin{cases} \leq 0 & n-i \text{ odd} \\ \geq 0 & n-i \text{ even} \end{cases}. \tag{2.2}$$

Since the f_j have no common root, $f_j(\alpha_i)$ cannot be zero for all i. With (2.2) we therefore have

$$f(\alpha_i) \begin{cases} < 0 & n-i \text{ odd} \\ > 0 & n-i \text{ even} \end{cases}$$

for the convex combination f. Thus in every open interval $(\alpha_1, \alpha_2), \ldots, (\alpha_{n-2}, \alpha_{n-1})$ the polynomial f admits a root. Since $f(\alpha_{n-1}) < 0$ and f has positive leading coefficient, it has another root $> \alpha_{n-1}$, so the polynomial f of degree n with real coefficients has at least $n-1$ real roots. But since complex roots of real polynomials come in pairs of two (via conjugates), f must have an n-th real root, thus f is real rooted.
($\Leftarrow$): By remark 2.3 (iii) it suffices to consider the case of $k = 2$ polynomials f_1, f_2. The proof of this side of the implication is somewhat lengthy and we proceed in multiple steps (each of them certainly worthy their own lemma, however we will only need these details here in this part of the proof). To shorten the following a bit, we will say that two real rooted polynomials f, g with common degree and positive leading coefficient are **compatible** if all their convex combinations $\lambda f + (1-\lambda)g, \lambda \in [0,1]$ are real rooted. So in what follows let f_1 and f_2 always be compatible of common degree n and let $h_\lambda = \lambda f_1 + (1-\lambda) f_2$ for $\lambda \in [0,1]$ be a convex combination. Furthermore, for $i = 1, 2$ and $x \in \mathbb{R}$ let $n_i(x)$ be the number of roots of f_i (counted with their multiplicities) that are $\geq x$. For real $a \in \mathbb{R}$ we also say that f_1 and f_2 **agree** in a, if $f_1(a) \neq 0 \neq f_2(a)$ and $f_1(a)$ and $f_2(a)$ have the same sign (so either both > 0 or both < 0).

(A) The derivatives f_1' and f_2' are also compatible:
All polynomials $h_\lambda = \lambda f_1 + (1-\lambda) f_2$ have real roots and by basic calculus the roots of the derivative h_λ' lie between the roots of h_λ (counted with their according multiplicities), so that $h_\lambda' = \lambda f_1' + (1-\lambda) f_2'$ has all $n-1$ real roots, thus f_1' and f_2' are compatible.

(B) If f_1 and f_2 agree in $a < b \in \mathbb{R}$, then $n_1(b) - n_1(a) = n_2(b) - n_2(a)$:
Since f_1 and f_2 agree in a and b, h_λ has no roots on the boundary of $[a, b]$. As $\lambda \in [0,1]$ varies continuously, so do the roots of h_λ (which are always real by compatibility of f_1 and f_2) between the roots of f_1 and f_2. By the first argument, now root of h_λ can cross a or b, so that the number of roots of h_λ in (a, b) stays constant. But the polynomials $f_1 = h_1$ and $f_2 = h_0$ have $n_1(b) - n_1(a)$ and $n_2(b) - n_2(a)$ roots respectively in this interval, so that they are the same, giving the claim.

(C) For $x \in \mathbb{R}$ we have $|n_1(x) - n_2(x)| \leq 1$:
We use induction on n, their common degree. The case $n = 1$ is clear. Since common roots of f_1 and f_2 contribute in the same way to n_1 as they do to n_2, we may assume f_1 and f_2 to have no common roots. Notice here that factoring out linear factors from both f_1 and f_2 preserve the compatibility property, as can be easily seen by considering the convex combinations h_λ.
Let now $n \geq 2$ and assume there is a $x_0 \in \mathbb{R}$ with $n_1(x_0) - n_2(x_0) \geq 2$ (if there only is a x_0 with $n_1(x_0) - n_2(x_0) \leq -2$ simply exchange the rolls of f_1 and f_2). Adding some small $\varepsilon > 0$ to x_0, by continuity we can assume that x_0 is a root of f_1. Furthermore, we may assume x_0 to be the largest root of f_1 with $n_1(x_0) - n_2(x_0) \geq 2$. By the assumptions so far (namely that f_1 and f_2 have no common roots) we also have $f_2(x_0) \neq 0$.
If $n_1(x_0) - n_2(x_0) > 2$ were true, consider the roots of the derivatives f_1' and f_2' and their in an analogous way defined quantities n_1' and n_2'. Since the roots of derivatives of real rooted polynomials lie between the roots of the original polynomial, we have $n_1'(x_0) = n_1(x_0) - 1$ and $n_2'(x_0) \leq n_2(x_0)$. But then $n_1'(x_0) - n_2'(x_0) \geq n_1(x_0) - n_2(x_0) + 1 \geq 2$, which is a contradiction to the induction hypothesis (with (A) giving the compatibility of f_1' and f_2'). This contradiction shows $n_1(x_0) - n_2(x_0) = 2$.
For a $y_2 \in \mathbb{R}$ strictly larger than all roots of f_1 and f_2, we have, since f_1 and f_2 have positive leading coefficients, that f_1 and f_2 agree in y_2 (both polynomials are positive there). By the fact that $n_1(x_0) - n_2(x_0) = 2$ is even, we can choose a $y_1 < x_0$, so that f_1 and f_2 also agree in y_1 and in such a way that f_1 and f_2 have no roots in $[y_1, x_0)$. But then counting the roots in $[y_1, y_2]$ we see that $n_1(y_2) - n_1(y_1) = n_2(y_2) - n_2(y_1) - 1 \neq n_2(y_2) - n_2(y_1)$, contradicting (B). Therefore such a x_0 cannot exist and we conclude the claim.

We are now able to show that f_1 and f_2 have a common interlacing. Denoting

$$f_1(x) = \beta_0^1 \prod_{i=1}^{n} (x - \beta_i^1), \quad f_2(x) = \beta_0^2 \prod_{i=1}^{n} (x - \beta_i^2)$$

with $\beta_0^j > 0, j = 1, 2$ and $\beta_1^j \leq ... \leq \beta_n^j, j = 1, 2$, we have to show that $[\beta_i^1, \beta_{i+1}^1] \cap [\beta_i^2, \beta_{i+1}^2]$ are nonempty for all $i = 1, ..., n-1$. As in remark 2.3 (iii) this implies the existence of a common interlacing.
Suppose otherwise and let i^* be the maximal $i = 1, ..., n-1$ with $[\beta_{i^*}^1, \beta_{i^*+1}^1] \cap [\beta_{i^*}^2, \beta_{i^*+1}^2] = \emptyset$. After possibly renaming f_1 and f_2, we may assume $\beta_{i^*+1}^1 < \beta_{i^*}^2$, so that the interval $[\beta_{i^*}^1, \beta_{i^*+1}^1]$ lies entirely to the left of $[\beta_{i^*}^2, \beta_{i^*+1}^2]$. But then $n_2(\beta_{i^*}^2) = n - i^*$ and $n_1(\beta_{i^*}^2) = n - i^* - 2$ (using maximality of i^*), contradicting (C). This finally proves that f_1 and f_2 have a common interlacing. ∎

2.2 Stable Polynomials

The following definition also originates from the theory of partial differential equations (as was already the case with the notion of real rootedness).

Definition 2.8. A polynomial $p \in \mathbb{C}[z_1, ..., z_m]$ is called **stable**, whenever one of the following (obviously) equivalent conditions hold

(i) $\forall i = 1, ..., m : \operatorname{Im} z_i > 0 \quad \Rightarrow \quad p(z_1, ..., z_m) \neq 0$;

(ii) $p(z_1, ..., z_m) = 0 \quad \Rightarrow \quad \exists i = 1, ..., m : \operatorname{Im} z_i \leq 0$.

A stable polynomial with real coefficients is called **real stable**.

Remark 2.9. [19] Notice that, by a similar argumentation as at the end of the proof of 2.7 ($\Rightarrow$), a real stable polynomial in one variable has only real roots. Thus, a univariate polynomial is real stable if and only if it is real rooted and has (at least one nonzero) real coefficients.

Our aim is to study certain polynomials which will turn out to be real stable. For a start we need a tool from complex analysis, known as Hurwitz's theorem. The proof of this statement (at least the one dimensional case) can be found in most text books on complex analysis, so we only sketch it here.

Theorem 2.10. [10] Let $D \subseteq \mathbb{C}^d$ be a domain (i.e. open and connected) and suppose that $(f_n)_n$ is a sequence of nonvanishing analytic functions on D, that converges uniformly on all compact subsets $K \subset\subset D$ to f. Then f is either nonvanishing on D or constant zero.

Proof sketch. Assume we already proved the one dimensional case $d = 1$, which can be done by basic complex analytical arguments. Assume for $c \in D$ we have $f(c) = 0$ and choose an open neighborhood $U := U_\varepsilon(c) \subseteq D$. By the one dimensional case, f is zero on the intersection $U \cap \{z \in \mathbb{C}^d : z_1 = c_1\}$. Repeat this argument for the other coordinates, so we have $f = 0$ on U, i.e. f vanishes on a (open) subdomain of D. By the (higher dimensional) identity theorem, f also vanishes on D. ■

Proposition 2.11. [3, 26] Let $A_1, ..., A_m \in \mathbb{C}^{n\times n}$ be hermitian positive semidefinite matrices and consider the polynomial

$$p(z_1, ..., z_m) = \det\left(\sum_{i=1}^{m} z_i A_i\right).$$

If p is not constant zero, then p is real stable.

Proof. Set $\mathbf{A} := \sum_{i=1}^{m} z_i A_i$. Since for real z_i's the matrix $\mathbf{A}$ is hermitian, we have then also that $p(z_1, ..., z_m)$ is real. Therefore p has real coefficients and it remains to show stability.
Using a limiting argument together with Hurwitz's theorem 2.10 above, it suffices to consider the case where all A_i are positive definite. In this case, the polynomial p is not the zero polynomial (not even if we fix $n - 1$ of the z_i's and consider it as a univariate polynomial in the remaining variable). With $\mathbf{z} \in \mathbb{R}^n$ and $\lambda \in \mathbb{R}^n_{>0}$ write

$z(t) := \mathbf{z} + t\lambda \in \mathbb{C}^n$ for $t \in \mathbb{C}$ and observe that $P := \sum_{i=1}^m \lambda_i A_i$ is hermitian positive definite, in particular it has an hermitian positive definite square root $P^{1/2}$. Then

$$\begin{aligned} p(z(t)) &= \det\left(\sum_{i=1}^m (z_i + t\lambda_i)A_i\right) = \det\left(\sum_{i=1}^m z_i A_i + t\sum_{i=1}^m A_i\right) \\ &= \det\left(\mathbf{A} + tP\right) = (\det P^{1/2})\det(tI + P^{-1/2}\mathbf{A}P^{-1/2})(\det P^{1/2}) \\ &= (-1)^n(\det P)\det(-tI - P^{-1/2}\mathbf{A}P^{-1/2}) \\ &= (-1)^n(\det P)\chi[P^{-1/2}\mathbf{A}P^{-1/2}](-t) \end{aligned}$$

is in essence the characteristic polynomial of an hermitian matrix $P^{-1/2}\mathbf{A}P^{-1/2}$ (remember, the z_i here are real) and thus has only real roots. This shows that $p(z_1, ..., z_n) = 0$ can only be if all z_i's are real, which proves the assertion. ■

In particular this show that the polynomial $\det(xI + \sum_{i=1}^m z_i A_i)$ is real stable for hermitian positive semidefinite $A_1, ..., A_m$, since it is clearly not constant zero by the xI term.

To generate real stable polynomials from known ones, there is a multitude of different approaches. One relatively simple one comes from the study of roots of complex polynomials. The following theorem may be compared to the Gauss-Lucas theorem and will be useful for our purposes.

Theorem 2.12. Let $q \in \mathbb{C}[z]$ be a complex polynomial of degree d and let $A \subseteq \mathbb{C}$ be a convex subset that contains all roots of q. For $\lambda \in \mathbb{C}$ the roots of the polynomial $q(z) - \lambda q'(z)$ lie in the region swept out by translating A in the direction and by the magnitude of $d\lambda$.

Proof. We will show that any root of $q(z) - \lambda q'(z)$ can be expressed as the sum of a convex combination of the roots of q plus $t\lambda$, where $t \in [0, d]$. Let $z_1, ..., z_d$ be the roots of q and choose a root z of $q - \lambda q'$. If z is also a root of q, then the above holds trivially, so let $q(z) \neq 0$. In this case we clearly have $\lambda \neq 0$. We can then write $q(z) - \lambda q'(z) = 0$ equivalently as

$$1 = \lambda\frac{q'(z)}{q(z)} = \lambda\sum_{i=1}^d \frac{1}{z - z_i} = \lambda\sum_{i=1}^d \frac{\overline{z} - \overline{z}_i}{|z - z_i|^2} \tag{2.3}$$

which gives

$$\frac{1}{\lambda} + \sum_{i=1}^d \frac{\overline{z}_i}{|z - z_i|^2} = \overline{z}\sum_{i=1}^d \frac{1}{|z - z_i|^2},$$

so, after taking complex conjugates,

$$\frac{1}{\overline{\lambda}\sum_{i=1}^d \frac{1}{|z-z_i|^2}} + \frac{1}{\sum_{i=1}^d \frac{1}{|z-z_i|^2}}\sum_{i=1}^d \frac{1}{|z - z_i|^2} z_i = z.$$

By setting

$$\mu_i = \frac{1/|z - z_i|^2}{\sum_{j=1}^d 1/|z - z_j|^2}$$

and

$$t\lambda = \frac{1}{\overline{\lambda} \sum_{i=1}^{d} \frac{1}{|z-z_i|^2}}$$

we get

$$t\lambda + \sum_{i=1}^{d} \mu_i z_i = z.$$

To finish the proof it remains to show that $t \in [0, d]$, i.e.

$$t = \frac{1}{|\lambda|^2 \sum_{i=1}^{d} \frac{1}{|z-z_i|^2}} \leq d$$

or equivalently

$$|\lambda|^{-2} \leq d \sum_{i=1}^{d} \frac{1}{|z-z_i|^2}. \tag{2.4}$$

Notice, by (2.3), that the left side can be expressed as

$$|\lambda|^{-2} = \left| \sum_{i=1}^{d} \frac{1}{z-z_i} \right|^2.$$

Using the triangle inequality, (2.4) gets implied by

$$\left(\sum_{i=1}^{d} \frac{1}{|z-z_i|} \right)^2 \leq d \sum_{i=1}^{d} \frac{1}{|z-z_i|^2},$$

but this follows from the (real) Cauchy-Schwarz inequality applied to the vectors $[1]_{i=1}^{d}$ and $[\frac{1}{|z-z_i|}]_{i=1}^{d}$. Reading the chain of inequalities backwards yields the assertion. ■

Corollary 2.13. [19] If $p \in \mathbb{R}[z_1, ..., z_m]$ is real stable, then so is

$$(1 - \partial_1)p = p - \partial_1 p.$$

Proof. Let $x_2, ..., x_n \in \mathbb{C}$ with positive imaginary part $\operatorname{Im} x_i > 0, i = 2, ..., m$. The univariate polynomial $q(z) = p(z, x_2, ..., x_m)$ (according to definition 2.8) can only have roots z_0 with $\operatorname{Im} z_0 \leq 0$, i.e. it is stable. The set $H := \{z \in \mathbb{C} : \operatorname{Im} z \leq 0\}$ is convex and invariant under translation by d (degree of q), while containing all the roots of q, so by theorem 2.12 the roots of $(1 - \partial_z)q$ also lie in H, i.e. $(1 - \partial_z)q$ is stable. Therefore, $(1 - \partial_1)p$ has no roots where all imaginary parts are positive. ■

Lemma 2.14. [26] Let $p \in \mathbb{R}[z_1, ..., z_m]$ be real stable and $a \in \mathbb{R}$. Then the polynomial $p|_{z_1=a} = p(a, z_2, ..., z_m)$ is constant zero or real stable.

Proof. It is obvious that $p|_{z_1=a}$ has real coefficients, so only need to show stability. As in the proof of corollary 2.13, we have that $p|_{z_1=\tilde{a}}$ for $\tilde{a} \in \mathbb{C}$ with $\operatorname{Im} \tilde{a} > 0$ is stable. A simple limiting argument by approximating a with such $\tilde{a}$, e.g. by $a + i/n, n \longrightarrow \infty$, together with Hurwitz's theorem 2.10 yields the claim. ■

It should also be clear that permuting the variables of $p(z_1, ..., z_m)$ as to obtain a polynomial $\tilde{p}(z_1, ..., z_m) = p(z_{\sigma(1)}, ..., z_{\sigma(m)})$ for a permutation σ of the set $\{1, ..., m\}$ also preserves real stability.

2.3 Trace Identities

This small section is a reminder from linear algebra and aims to derive a useful identity regarding the trace of matrices. For a vector $v \in \mathbb{C}^n$ we denote by v^* the transpose and componentwise complex conjugate of v. We will also use the trace functional tr on the space of (complex valued) square matrices, which induces an inner product $\langle A, B\rangle = \mathrm{tr}(AB^*)$, where A and B must be of the same format $m \times n$. For real matrices A, B, C and X, Y (with compatible formats) we have the straightforward identities

$$\mathrm{tr}(ABC) = \mathrm{tr}(CAB) = \mathrm{tr}(BCA) \tag{2.5}$$

and especially

$$\mathrm{tr}(XY) = \mathrm{tr}(YX).$$

Notice that, if $X \in \mathbb{C}^{m\times n}$ and $Y \in \mathbb{C}^{n\times m}$, then the trace in $\mathrm{tr}(XY)$ is a functional defined on $\mathbb{C}^{m\times m}$, while in the case of $\mathrm{tr}(YX)$ it is defined on $\mathbb{C}^{n\times n}$, so technically speaking these are two different functionals since they act (in general) on different ground sets. The same applies to the cyclic identities above. However, we will not distinguish between these notationwise and context will make it clear how these functionals are meant exactly. As an example, we will sometimes see chains of equalities looking like

$$\mathrm{tr}(Auv^*) = \mathrm{tr}(v^*Au) = v^*Au,$$

where A is a square matrix and u and v are (column) vectors of compatible dimensions. We will often simply skip the intermediate step, so we mention this point here to avoid possible confusion later on.

Lemma 2.15. If $A \in \mathbb{C}^{n\times n}$ is invertible and $u, v \in \mathbb{C}^n$, then

$$\det(A + uv^*) = (\det A)(1 + v^*A^{-1}u).$$

Proof. Observe

$$\begin{bmatrix} I & 0 \\ v^*A^{-1} & 1 \end{bmatrix} \begin{bmatrix} A + uv^* & u \\ 0 & 1 \end{bmatrix} \begin{bmatrix} I & 0 \\ -v^* & 1 \end{bmatrix} = \begin{bmatrix} A & u \\ 0 & v^*A^{-1}u + 1 \end{bmatrix},$$

so basic properties of the determinant then imply the claim. ■

Theorem 2.16. For $A, B \in \mathbb{C}^{n\times n}$ with A invertible we have

$$\partial_t \det(A + tB)|_{t=0} = \det A \cdot \mathrm{tr}(A^{-1}B).$$

Proof. Let $B = \sum_{i=1}^r u_i v_i^*$ be a rank-one decomposition of B. Applying lemma 2.15 iteratively yields

$$\begin{aligned}\det(A+tB) &= \det\left(A + t\sum_{i=1}^r u_i v_i^*\right) = (\det A)\prod_{i=1}^r (1 + t v_i^* A^{-1} u_i) \\ &= \det A + \det A \cdot t \sum_{i=1}^r v_i^* A^{-1} u_i + O(t^2) \text{ for } t \longrightarrow 0.\end{aligned}$$

Thus, $\partial_t \det(A+tB)|_{t=0}$ evaluates to $\det A \cdot \sum_{i=1}^r v_i^* A^{-1} u_i$, but

$$\sum_{i=1}^r v_i^* A^{-1} u_i = \sum_{i=1}^r \operatorname{tr}(A^{-1} u_i v_i^*) = \operatorname{tr}\left(A^{-1}\sum_{i=1}^r u_i v_i^*\right) = \operatorname{tr}(A^{-1}B),$$

proving the claim. ∎

2.4 Mixed Characteristic Polynomials

Definition 2.17. Let $A_1, ..., A_m \in \mathbb{C}^{d\times d}$ be (hermitian positive semidefinite) matrices, we call

$$\mu[A_1, ..., A_m](x) := \left(\prod_{i=1}^m 1 - \partial_i\right) \det\left(xI + \sum_{i=1}^m z_i A_i\right)\Bigg|_{z_1 = ... = z_m = 0}$$

the **mixed characteristic polynomial** of the matrices $A_1, ..., A_m$.

Such polynomials arise from the study of expected characteristic polynomials of random rank-one hermitian positive semidefinite matrices. This connection will be made precise in the following results.

Lemma 2.18. [19] For $A \in \mathbb{C}^{d\times d}$ and vector valued random variable $v \in \mathbb{C}^d$, such that the covariance matrix $\mathbb{E}vv^*$ exists, we have

$$\mathbb{E}\det(A - vv^*) = (1 - \partial_t)\det(A + t\mathbb{E}vv^*)|_{t=0}. \tag{2.6}$$

Proof. First assume A is invertible. By lemma 2.15 and linearity of the operators $\mathbb{E}$ and tr we have

$$\begin{aligned}\mathbb{E}\det(A - vv^*) &= \mathbb{E}[(\det A)(1 - v^* A^{-1} v)] \\ &= \det A \cdot \mathbb{E}[1 - \operatorname{tr}(A^{-1} vv^*)] \\ &= \det A - \det A \cdot \mathbb{E}\operatorname{tr}(A^{-1} vv^*) \\ &= \det A - \det A \cdot \operatorname{tr}(A^{-1}\mathbb{E}vv^*).\end{aligned}$$

On the other hand, by theorem 2.16 we have

$$\begin{aligned}(1-\partial_t)\det(A+t\mathbb{E}vv^*)|_{t=0} &= \det(A+t\mathbb{E}vv^*)|_{t=0} - \det A\cdot \operatorname{tr}(A^{-1}\mathbb{E}vv^*)\\ &= \det A - \det A\cdot \operatorname{tr}(A^{-1}\mathbb{E}vv^*).\end{aligned}$$

This yields (2.6) for invertible A. If A is not invertible notice that both sides of (2.6) are continuous (even polynomial) functions in the entries of A, therefore a limiting argument proves the claim for all $A \in \mathbb{C}^{d\times d}$. ■

Theorem 2.19. [19] Let $v_1, ..., v_m \in \mathbb{C}^d$ be independent random vectors with finite support (i.e. each v_i can only be one of the finitely many elements of $W_i = \{w_{i1}, ..., w_{il_i}\}, l_i \in \mathbb{N}$). For $A_i := \mathbb{E}v_iv_i^*$ the covariance matrices, we have

$$\mathbb{E}\chi\left[\sum_{i=1}^m v_iv_i^*\right](x) = \left(\prod_{i=1}^m 1-\partial_i\right)\det\left(xI+\sum_{i=1}^m z_iA_i\right)\Bigg|_{z_1=...=z_m=0}, \tag{2.7}$$

which is precisely the mixed characteristic polynomial of the A_i's.

Proof. First note that the covariance matrices really do exist, since the v_i's have finite support. We first write out the left hand side of (2.7) more clearly to get to

$$\mathbb{E}\left[\chi\left[\sum_{i=1}^m v_iv_i^*\right](x)\right] = \mathbb{E}\det\left(xI-\sum_{i=1}^m v_iv_i^*\right).$$

The claim then follows from the following general formula

$$\mathbb{E}\det\left(M-\sum_{i=1}^k v_iv_i^*\right) = \left(\prod_{i=1}^k 1-\partial_i\right)\det\left(M+\sum_{i=1}^k z_iA_i\right)\Bigg|_{z_1=...=z_k=0},$$

which we will prove with induction on k by applying lemma 2.18 repeatedly. The case $k=0$ is clear, even $k=1$ is just the statement of lemma 2.18. Otherwise, by using the independence of the v_i's, $A_i = \mathbb{E}v_iv_i^*$, linearity of $\mathbb{E}$ together with the finiteness of the v_i's and the aforementioned lemma 2.18 we get:

$$\begin{aligned}\mathbb{E}\det\left(M-\sum_{i=1}^k v_iv_i^*\right) &= \mathbb{E}_{v_1,...,v_{k-1}}\mathbb{E}_{v_k}\det\left(M-\sum_{i=1}^{k-1} v_iv_i^* - v_kv_k^*\right)\\ &= \mathbb{E}_{v_1,...,v_{k-1}}(1-\partial_k)\det\left(M-\sum_{i=1}^{k-1} v_iv_i^* + z_kA_k\right)\Bigg|_{z_k=0}\\ &= (1-\partial_k)\mathbb{E}_{v_1,...,v_{k-1}}\det\left(M-\sum_{i=1}^{k-1} v_iv_i^* + z_kA_k\right)\Bigg|_{z_k=0}\\ &= (1-\partial_k)\left(\prod_{i=1}^{k-1} 1-\partial_i\right)\det\left(M+z_kA_k+\sum_{i=1}^{k-1} z_iA_i\right)\Bigg|_{z_1=...=z_{k-1}=0}\Bigg|_{z_k=0}\\ &= \left(\prod_{i=1}^k 1-\partial_i\right)\det\left(M+\sum_{i=1}^k z_iA_i\right)\Bigg|_{z_1=...=z_k=0},\end{aligned}$$

as desired. Notice that we used the induction hypothesis on the matrix $M + z_k A_k$ instead of just M alone. ■

Corollary 2.20. The mixed characteristic polynomial

$$\mu[A_1, ..., A_m](x) := \left(\prod_{i=1}^{m} 1 - \partial_i\right) \det\left(xI + \sum_{i=1}^{m} z_i A_i\right)\Bigg|_{z_1=...=z_m=0}$$

of hermitian positive semidefinite matrices $A_1, ..., A_m$ is real rooted.

Proof. Observe that

$$p(x, z_1, ..., z_m) = \det\left(xI + \sum_{i=1}^{m} z_i A_i\right)$$

is nonconstant (because of the xI term), thus by proposition 2.11 we know that p is real stable. By corollary 2.13 we therefore also know that

$$\left(\prod_{i=1}^{m} 1 - \partial_i\right) \det\left(xI + \sum_{i=1}^{m} z_i A_i\right)$$

is real stable. Then with lemma 2.14 we get that

$$\left(\prod_{i=1}^{m} 1 - \partial_i\right) \det\left(xI + \sum_{i=1}^{m} z_i A_i\right)\Bigg|_{z_1=...=z_m=0}$$

is real stable. By remark 2.9 the mixed characteristic polynomial is therefore real rooted. ■

We will now proof that the random vectors v_i from theorem 2.19 lead naturally to a family of polynomials with a common interlacing. For that suppose that v_i takes on the value $w_{ij} \in W_i$ with probability p_{ij}. Define for $j_1 \in [l_1], ..., j_m \in [l_m]$ the univariate polynomials

$$q_{j_1 \dots j_m}(x) = \left(\prod_{i=1}^{m} p_{ij_i}\right) \cdot \chi\left[\sum_{i=1}^{m} w_{ij_i} w_{ij_i}^*\right](x).$$

Theorem 2.21. [19] The polynomials $\{q_{j_1 \dots j_m} : j_1 \in [l_1], ..., j_m \in [l_m]\}$ form an interlacing family (see definition 2.5).

Proof. For $1 \le k < m, j_1 \in [l_1], ..., j_k \in [l_k]$ let

$$q_{j_1 \dots j_k}(x) = \left(\prod_{i=1}^{k} p_{ij_i}\right) \cdot \mathbb{E}_{v_{k+1},...,v_m} \chi\left[\sum_{i=1}^{k} w_{ij_i} w_{ij_i}^* + \sum_{i=k+1}^{m} v_i v_i^*\right](x)$$

and also

$$q_\emptyset(x) = \mathbb{E}_{v_1,\dots,v_m}\chi\left[\sum_{i=1}^m v_i v_i^*\right](x).$$

We need to show that for a partial assignment $(j_1,\dots,j_k) \in [l_1]\times\dots\times[l_k]$ the polynomials $\{q_{j_1,\dots,j_k,t}\}_{t\in[l_{k+1}]}$ have a common interlacing. In view of theorem 2.7 we have to show that arbitrary convex combinations

$$\sum_{t=1}^{l_{k+1}} \lambda_t q_{j_1\dots j_k,t}(x)$$

are real rooted. We show this by denoting by u_{k+1} a random vector (independent from the v_i's) that takes on $w_{k+1,t}$ with probability

$$\frac{\lambda_t p_{k+1,t}}{\sum_{s=1}^{l_{k+1}} \lambda_s p_{k+1,s}} \in [0,1]$$

and observing that the convex combination $\sum_{t=1}^{l_{k+1}} \lambda_t q_{j_1\dots j_k,t}(x)$ then equals

$$\left(\sum_{s=1}^{l_{k+1}} \lambda_s p_{k+1,s}\right)\left(\prod_{i=1}^k p_{ij_i}\right)\cdot \mathbb{E}_{u_{k+1},v_{k+2},\dots,v_m}\chi\left[\sum_{i=1}^k w_{ij_i}w_{ij_i}^* + u_{k+1}u_{k+1}^* + \sum_{i=k+2}^m v_i v_i^*\right](x). \tag{2.8}$$

Since this is up to a real scalar multiple a mixed characteristic polynomial by theorem 2.19, it is real rooted by corollary 2.20, showing that the family $\{q_{j_1,\dots,j_k,t}\}_{t\in[l_{k+1}]}$ has a common interlacing by theorem 2.7. Indeed, to show the validity of (2.8) expand the expectation after u_{k+1} (remember all random vectors here are independent) to get

$$\begin{aligned}
&\mathbb{E}_{u_{k+1},v_{k+2},\dots,v_m}\chi\left[\sum_{i=1}^k w_{ij_i}w_{ij_i}^* + u_{k+1}u_{k+1}^* + \sum_{i=k+2}^m v_i v_i^*\right](x)\\
&=\sum_{t=1}^{l_{k+1}} \frac{\lambda_t p_{k+1,t}}{\sum_{s=1}^{l_{k+1}} \lambda_s p_{k+1,s}}\mathbb{E}_{v_{k+2},\dots,v_m}\chi\left[\sum_{i=1}^k w_{ij_i}w_{ij_i}^* + w_{k+1,t}w_{k+1,t}^* + \sum_{i=k+2}^m v_i v_i^*\right](x)
\end{aligned}$$

and compare this to

$$\begin{aligned}
&\sum_{t=1}^{l_{k+1}} \lambda_t q_{j_1\dots j_k,t}(x)\\
&=\sum_{t=1}^{l_{k+1}} \lambda_t \left(\prod_{i=1}^k p_{ij_i}\right) p_{k+1,t}\cdot \mathbb{E}_{v_{k+2},\dots,v_m}\chi\left[\sum_{i=1}^k w_{ij_i}w_{ij_i}^* + w_{k+1,t}w_{k+1,t}^* + \sum_{i=k+2}^m v_i v_i^*\right](x).
\end{aligned}$$

■

Notice that $q_\emptyset$ from the proof above is the mixed characteristic polynomial $\mu[A_1,\dots,A_m](x)$ by theorem 2.19, where again $A_i = \mathbb{E}v_i v_i^*$ are the covariance matrices.

3 Connections to the Kadison-Singer Problem

3.1 Root Estimations

We now want to give bounds on the roots of the mixed characteristic polynomial. For a start notice the following.

Remark 3.1. [19] Let $A_1, ..., A_m \in \mathbb{C}^{d\times d}$ be hermitian positive semidefinite matrices with $\sum_{i=1}^m A_i = I$. If we plug this into the mixed characteristic polynomial (see definition 2.17) we may write

$$\begin{aligned}\mu[A_1, ..., A_m](x) &= \left(\prod_{i=1}^m 1 - \partial_{z_i}\right) \det\left(xI + \sum_{i=1}^m z_i A_i\right)\Bigg|_{z_1=...=z_m=0} \\ &= \left(\prod_{i=1}^m 1 - \partial_{z_i}\right) \det\left(\sum_{i=1}^m (x + z_i) A_i\right)\Bigg|_{z_1=...=z_m=0} \\ &= \left(\prod_{i=1}^m 1 - \partial_{y_i}\right) \det\left(\sum_{i=1}^m y_i A_i\right)\Bigg|_{y_1=...=y_m=x},\end{aligned}$$

where we used the “linear substitution” $\partial_{y_i} f(y_i)|_{y_i=z_i+x} = \partial_{z_i} f(z_i + x)$. Writing

$$Q(y_1, ..., y_m) := \left(\prod_{i=1}^m 1 - \partial_{y_i}\right) \det\left(\sum_{i=1}^m y_i A_i\right)$$

this gives $\mu[A_1, ..., A_m](x) = Q(x, ..., x)$.

From now on we will always assume $\sum_{i=1}^m A_i = I$. We will come back to this function Q at a later point. The roots of Q will help in estimating the roots of the mixed characteristic polynomial.

Definition 3.2. For a multivariate polynomial $p \in \mathbb{R}[x_1, ..., x_m]$ we say that $x \in \mathbb{R}^m$ lies **above the roots** of p, if $p(x + t) > 0$ for all $t \in \mathbb{R}^m_{\geq 0}$, i.e. p is positive on the nonnegative orthant with origin in x. By Ab_p we denote the set of all points x, that lie above the roots of p, explicitly $\mathsf{Ab}_p = \{x \in \mathbb{R}^m : p(x + t) > 0 \text{ for all } t \in \mathbb{R}^m_{\geq 0}\}$.

Note that if we have a univariate polynomial $p(x)$ with positive leading coefficient, then x_0 lies above the roots of p if and only if x_0 is strictly larger then the largest root of p. If we consider Q as a real polynomial (which is possible since the matrices A_i defining it are hermitian, so Q has real coefficients) we can write

$$\begin{aligned}\mathsf{Ab}_Q &= \{y \in \mathbb{R}^m : Q(y+t) > 0 \forall t \in \mathbb{R}^m_{\geq 0}\} \\ &= \left\{ y \in \mathbb{R}^m : \left(\prod_{i=1}^m 1 - \partial_{y_i} \right) \det \left(\sum_{i=1}^m (y_i + t_i) A_i \right) > 0 \forall t \in \mathbb{R}^m_{\geq 0} \right\}.\end{aligned}$$

To study the roots of $\mu[A_1, ..., A_m](x) = Q(x, ..., x)$ we will investigate the elements in Ab_Q of the form $x\mathbf{1}, x \in \mathbb{R}$, i.e. the intersection $\mathsf{Ab}_Q \cap D^m$ with the "diagonal" $D^m := \{x\mathbf{1} : x \in \mathbb{R}\}$. In particular, we want to give an upper bound on $\inf\{x \in \mathbb{R} : x\mathbf{1} \in \mathsf{Ab}_Q\}$, which in turn gives an upper bound on the largest root of $\mu[A_1, ..., A_m](x) = Q(x, ..., x)$. This upper bound will depend on the matrices $A_1, ..., A_m$ (again, we will only be interested in the case $\sum_{i=1}^m A_i = I$), explicitly on the traces $\operatorname{tr} A_i$.
This will be achieved by iteratively applying the operators $1 - \partial_{y_i}$ (compare this with the definition of Q given in remark 3.1) and keeping track of the roots of the evolving polynomials. Let us now get into the details of the techniques involved

Definition 3.3. For $p \in \mathbb{R}[x_1, ..., x_m]$ a real stable polynomial and a point $x \in \mathsf{Ab}_p$ define the **barrier function** of p in direction j at x as

$$\Phi_p^j(x) := \frac{\partial_{x_j} p(x)}{p(x)} = \partial_{x_j} \log p(x).$$

If we set $q_{x,j}(t) := p(x_1, ..., x_{j-1}, t, x_{j+1}, ..., x_m) \in \mathbb{R}[t]$, we can give the equivalent definition

$$\Phi_p^j(x) = \Phi_p^j(x_1, ..., x_m) = \frac{q'_{x,j}(x_j)}{q_{x,j}(x_j)} = \sum_{i=1}^r \frac{1}{x_j - \lambda_i}.$$

where the univariate restriction $q_{x,j}(t)$ has the roots $\lambda_1, ..., \lambda_r$, which are real by lemma 2.14 (but in general they depend on the fixed variables $x_1, ..., x_{j-1}, x_{j+1}, ..., x_m$).

Theorem 3.4. [19, 23] Let $p \in \mathbb{R}[x_1, ..., x_m]$ be a real stable polynomial and let $x \in \mathsf{Ab}_p$ lie above the roots of p. Then for all $i, j \leq m$ and $\delta \geq 0$ we have

(i) $\Phi_p^j(x + \delta e_i) \leq \Phi_p^j(x)$;

(ii) $\Phi_p^j(x + \delta e_i) \leq \Phi_p^j(x) + \delta \cdot \partial_{x_i} \Phi_p^j(x + \delta e_i)$.

Point (i) describes the monotonicity (decreasing) of Φ_p^j and (ii) its convexity.

Proof. Notice that both claims follow once we showed that more generally we have $(-1)^k \partial_i^k \Phi_p^j(x) \geq 0$ for all $x \in \mathsf{Ab}_p, k \in \mathbb{N}_0$ ($k = 1$ is the monotonicity (i), $k = 2$ is the convexity (ii)).

First consider the univariate case $p = p(x) = \mu \prod_{i=1}^{n}(x - \lambda_i)$ with

$$\Phi_p(x) = \frac{p'(x)}{p(x)} = \sum_{i=1}^{n} \frac{1}{x - \lambda_i}.$$

We thus get

$$(-1)^k \Phi_p^{(k)}(x) = n! \sum_{i=1}^{n} \frac{1}{(x - \lambda_i)^{n+1}},$$

which is indeed positive, since x lies above the roots of p, i.e. $x > \lambda_i$ for all i.
As for the multivariate case, it suffices to fix $m-2$ of the variables to real numbers (preserving real stability by lemma 2.14), since we are only interested in the behavior of of the j-th variable under change of the i-th variable. Thus, without loss of generality, let $p = p(x_1, x_2)$ be real stable. We want to show that

$$(-1)^k \partial_2^k \Phi_p^1(x) \geq 0$$

whenever x lies above the roots of p. Rewriting

$$(-1)^k \partial_2^k \Phi_p^1(x) = (-1)^k \partial_2^k \partial_1 \log p(x) = \partial_1 \left((-1)^k \partial_2^k \log p(x) \right),$$

we will aim to show that $(-1)^k \partial_2^k \log p(x)$ is nondecreasing in the first variable. We will use basic continuity and smoothness properties of polynomials. In this circumstance, we will adopt the terminology of saying that a certain property holds for **generic** x, if it holds for all but finitely many x (under our consideration).
We can interpret $p_{x_1}(x_2) = p(x_1, x_2)$ as a univariate polynomial in x_2, which changes if we consider other (but fixed) x_1. For real x_1, the polynomial p_{x_1} is real stable (again by lemma 2.14). Denote its real roots by $\lambda_1(x_1), ..., \lambda_n(x_1)$ (with multiplicities). For generic x_1, the number n of roots does not depend on x_1, the roots $\lambda_i(x_1)$ can be labeled to vary smoothly with x_1 and the multiplicities of the $\lambda_i(x_1)$ are locally constant. We can then write

$$(-1)^k \partial_2^k \log p(x) = -(k-1)! \sum_{i=1}^{n} \frac{1}{(x_2 - \lambda_i(x_1))^k}$$

and aim to show that each term $\frac{1}{x_2 - \lambda_i(x_1)}$ is nonincreasing in x_1. Since $x = (x_1, x_2)$ lies over the roots of p, we have $\lambda_i(x_1) < x_2$, so it suffices to show that the $\lambda_i(x_1)$ are generically nonincreasing in x_1.
Assume otherwise, so that (as $\lambda_i(x_1)$ is smooth in x_1) it somewhere increases, so that it has positive derivative (differentiating after the variable x_1) on an open interval. In particular, there is an x_0 (from that interval), with positive derivative and constant multiplicity d in a (sufficiently small) neighborhood of x_0 (being entirely contained in the open interval from above). With help of the monotonicity and smoothness (ore concretely the fact, that p is a polynomial) of $\lambda_i(x_1)$ around x_0 and the inverse function theorem we can conclude that $\lambda_i(x_1)$ depends analytically on x_1 in this neighborhood around x_0. Continuing $\lambda_i(x_1)$ analytically to a small neighborhood of x_0 in the complex plane, we see that for complex z_1 in this extended neighborhood of x_0 the corresponding root $\lambda_i(z_1)$ of the polynomials $p_{z_1}(z_2) = p(z_1, z_2)$ (now in complex variables) lies in a neighborhood of $\lambda_i(x_0)$ (varying smoothly, i.e. analytically in z_1). Since the derivative of $\lambda_i(x_1)$ in x_0 is positive, we conclude that there is a complex root (z_1, z_2) of p (i.e. $z_2 = \lambda_i(z_1)$) in the corresponding neighborhood of $(x_0, \lambda_i(x_0))$ with $\operatorname{Im} z_1$ and $\operatorname{Im} z_2$ both positive (by applying the Cauchy-Riemann differential equations to $\lambda_i(z_1)$). This contradicts stability of p, so that $\lambda_i(x_1)$ are (generically) nonincreasing. This finally finishes the proof. ■

Taking once again a look at th definition of Q we are now interested in the relations between Ab_p and $\mathsf{Ab}_{(1-\partial_{x_j})p}$. The barrier function will help us for that.

Corollary 3.5. [19] Suppose that $p \in \mathbb{R}[x_1, ..., x_m]$ is real stable, $x \in \mathsf{Ab}_p$ lies above the roots of p and assume $\Phi_p^j(x) < 1$. Then x lies above the roots of $(1 - \partial_{x_j})p$, i.e. $x \in \mathsf{Ab}_{(1-\partial_{x_j})p}$.

Proof. For $t \in \mathbb{R}^m_{\geq 0}$ we have by the assumptions and theorem 3.4 (i) that $\Phi_p^j(x+t) \leq \Phi_p^j(x) < 1$. The definition of the barrier function then reads

$$\frac{\partial_{x_j} p(x+t)}{p(x+t)} < 1,$$

so

$$(p - \partial_{x_j} p)(x+t) > 0$$

which is precisely the fact that x lies above the roots of $(1 - \partial_{x_j})p$. ■

To apply this argument inductively we need stronger requirements on $\Phi_p^j(x)$, namely that it is bounded away from 1. The precise technicalities will be stated in the next result.

Lemma 3.6. [19] Let $p \in \mathbb{R}[x_1, ..., x_m]$ be a real stable polynomial, $x \in \mathsf{Ab}_p$ and $\delta > 0$ with

$$\Phi_p^j(x) \leq 1 - \frac{1}{\delta}. \tag{3.1}$$

Then for all $i = 1, ..., m$:

$$\Phi^i_{p - \partial_{x_j} p}(x + \delta e_j) \leq \Phi^i_p(x).$$

Proof. We start by taking a look at

$$(1 - \Phi_p^j)p = \left(1 - \frac{\partial_j p}{p}\right) p = p - \partial_j p,$$

and

$$\partial_i \Phi_p^j = \partial_i \left(\frac{\partial_j p}{p}\right) = \frac{\partial_i \partial_j p \cdot p - \partial_j p \cdot \partial_i p}{p^2} = \frac{\partial_j \partial_i p \cdot p - \partial_i p \cdot \partial_j p}{p^2} = \partial_j \left(\frac{\partial_i p}{p}\right) = \partial_j \Phi_p^i,$$

so $\partial_i \Phi_p^j = \partial_j \Phi_p^i$. Then we may write

$$\begin{aligned} \Phi^i_{p - \partial_{x_j} p} &= \frac{\partial_i (p - \partial_j p)}{p - \partial_j p} = \frac{\partial_i \left((1 - \Phi_p^j)p\right)}{(1 - \Phi_p^j)p} \\ &= \frac{(1 - \Phi_p^j) \cdot \partial_i p}{(1 - \Phi_p^j)p} + \frac{(\partial_i (1 - \Phi_p^j)) \cdot p}{(1 - \Phi_p^j)p} \\ &= \Phi_p^i - \frac{\partial_i \Phi_p^j}{1 - \Phi_p^j} = \Phi_p^i - \frac{\partial_j \Phi_p^i}{1 - \Phi_p^j}. \end{aligned}$$

Thus $\Phi^i_{p-\partial_{x_j}p}(x+\delta e_j) \leq \Phi^i_p(x)$ is equivalent to

$$\Phi^i_p(x+\delta e_j) - \frac{\partial_j \Phi^i_p(x+\delta e_j)}{1-\Phi^j_p(x+\delta e_j)} \leq \Phi^i_p(x)$$

or slightly rearranged

$$-\frac{\partial_j \Phi^i_p(x+\delta e_j)}{1-\Phi^j_p(x+\delta e_j)} \leq \Phi^i_p(x) - \Phi^i_p(x+\delta e_j).$$

By theorem 3.4 (ii) we get

$$-\delta \partial_j \Phi^i_p(x+\delta e_j) \leq \Phi^i_p(x) - \Phi^i_p(x+\delta e_j),$$

so to prove the claim it suffices to show

$$-\frac{\partial_j \Phi^i_p(x+\delta e_j)}{1-\Phi^j_p(x+\delta e_j)} \leq -\delta \partial_j \Phi^i_p(x+\delta e_j).$$

From theorem 3.4 (i) we get $-\partial_j \Phi^i_p(x+\delta e_j) \geq 0$ (of course the case when it is equal to 0 is clear), so dividing by it yields

$$\frac{1}{1-\Phi^j_p(x+\delta e_j)} \leq \delta.$$

Again by theorem 3.4 (i) we have $\Phi^j_p(x+\delta e_j) \leq \Phi^j_p(x)$, so this gets implied by

$$\frac{1}{1-\Phi^j_p(x)} \leq \delta,$$

which is just the assumption (3.1) and thus proves the claim. ■

To get back to a situation akin to remark 3.1 we can now prove the following:

Theorem 3.7. Suppose $A_1, ..., A_m \in \mathbb{C}^{d\times d}$ are hermitian positive semidefinite matrices with $\sum_{i=1}^m A_i = I$ and $\operatorname{tr} A_i \leq \varepsilon$ for all i. Then the largest root of the mixed characteristic polynomial

$$\mu[A_1, ..., A_m](x) = \left(\prod_{i=1}^m 1 - \partial_{y_i} \right) \det \left(\sum_{i=1}^m y_i A_i \right) \Bigg|_{y_1 = ... = y_m = x} = Q(x, ..., x)$$

is at most $(1+\sqrt{\varepsilon})^2$.

Compare this with remark 3.1 and the discussion after definition 3.2.

Proof. Let

$$P(y_1, ..., y_m) = \det \left(\sum_{i=1}^m y_i A_i \right)$$

and set $t = \sqrt{\varepsilon} + \varepsilon$. Since all A_i's are positive semidefinite and

$$\det\left(t\sum_{i=1}^{m} A_i\right) = \det(tI) > 0$$

we see that the vector $t\mathbf{1}$ lies above the roots of P. By theorem 2.16 we get

$$\Phi_P^j(y_1, ..., y_m) = \frac{\partial_j P(y_1, ..., y_m)}{P(y_1, ..., y_m)} = \operatorname{tr}\left(\left(\sum_{i=1}^{m} y_i A_i\right)^{-1} A_j\right),$$

so

$$\Phi_P^j(t\mathbf{1}) = \operatorname{tr}(t^{-1}A_j) = \frac{\operatorname{tr} A_j}{t} \leq \frac{\varepsilon}{t} = \frac{\varepsilon}{\sqrt{\varepsilon}+\varepsilon}. \tag{3.2}$$

We set

$$\phi := \frac{\varepsilon}{\varepsilon + \sqrt{\varepsilon}}, \quad \delta := \frac{1}{1-\phi} = 1 + \sqrt{\varepsilon}$$

and already note the similarities to lemma 3.6. For $k = 1, ..., m$ we also define

$$P_k(y_1, ..., y_m) := \left(\prod_{i=1}^{k} 1 - \partial_{y_i}\right) P(y_1, ..., y_m)$$

and observe that $P_m = Q$ (see remark 3.1).
Set $x^0 := t\mathbf{1}$ and $x^k := x^0 + \delta(e_1 + ... + e_k)$, i.e. x^k has $t + \delta$ in its first k components and t in the remaining. We therefore have $x^k = x^{k-1} + \delta e_k$ and applying corollary 3.5 and lemma 3.6 iteratively we see that $x^{k-1} + \delta e_k = x^k$ lies above the roots of $(1 - \partial_{y_k})P_{k-1} = P_k$ and

$$\Phi_{P_k}^j(x^k) = \Phi_{(1-\partial_{y_k})P_{k-1}}^j(x^{k-1} + \delta e_k) \leq \Phi_{P_{k-1}}^j(x^{k-1}) \leq \phi = 1 - \frac{1}{\delta}$$

by induction with (3.2) as the induction start. We therefore see that the largest root of $\mu[A_1, ..., A_m](x) = P_m(x\mathbf{1})$ (since $x^m = (t+\delta)\mathbf{1}$ lies above the roots of P_m) is at most

$$t + \delta = (\sqrt{\varepsilon} + \varepsilon) + (1 + \sqrt{\varepsilon}) = (1 + \sqrt{\varepsilon})^2,$$

as desired. ■

The choice of t (and thus of $\phi = \varepsilon/t$ and $\delta = 1/(1-\varphi)$) might seem a bit arbitrary at first. These values can be obtained by considering t as a unknown variable and then optimizing (minimizing) the final value

$$t + \delta = t + \frac{1}{1 - \frac{\varepsilon}{t}} = \frac{t^2 + (1-\varepsilon)t}{t - \varepsilon},$$

which exactly leads to $t = \varepsilon + \sqrt{\varepsilon}$.

3.2 Solutions to the Kadison-Singer Problem and Related Conjectures

The following results will build up to a solution of the Kadison-Singer problem.

Theorem 3.8. [19] If $\varepsilon > 0$ and $v_1, ..., v_m \in \mathbb{C}^d$ are independent random vectors with finite support (like in theorem 2.19) with

$$\sum_{i=1}^{m} \mathbb{E} v_i v_i^* = I$$

and $\mathbb{E}\|v_i\|^2 \leq \varepsilon$ for all i, then

$$\mathbb{P}\left(\left\|\sum_{i=1}^{m} v_i v_i^*\right\| \leq (1+\sqrt{\varepsilon})^2\right) > 0.$$

Proof. With $A_i := \mathbb{E} v_i v_i^*$ we see that

$$\operatorname{tr} A_i = \mathbb{E} \operatorname{tr} v_i v_i^* = \mathbb{E} v_i^* v_i = \mathbb{E}\|v_i\|^2 \leq \varepsilon$$

holds for all i. By theorems 3.7 and 2.19 the largest root of the mixed characteristic polynomial $\mu[A_1, ..., A_m](x)$ is at most $(1+\sqrt{\varepsilon})^2$. Using notation as in theorem 2.21 we know that the polynomials $\{q_{j_1 ... j_m} : j_1 \in [l_1], ..., j_m \in [l_m]\}$ form an interlacing family. By theorem 2.6 there is an assignment $(s_1, ..., s_m) \in [l_1] \times ... \times [l_m]$, such that the largest root of the characteristic polynomial

$$\chi\left[\sum_{i=1}^{m} w_{is_i} w_{is_i}^*\right](x)$$

is at most $(1+\sqrt{\varepsilon})^2$ (note that, with the small remark after the proof of theorem 2.21, we have $q_\emptyset(x) = \mu[A_1, ..., A_m](x)$). This shows that the largest eigenvalue of $\sum_{i=1}^{m} w_{is_i} w_{is_i}^*$ (and since the matrix is hermitian positive semidefinite this is the same as the largest singular value) is at most $(1+\sqrt{\varepsilon})^2$, so that the matrix norm is also at most $(1+\sqrt{\varepsilon})^2$ as well. The corresponding event has probability $\prod_{i=1}^{m} p_{is_i} > 0$, finishing the proof. ■

This result gives the existence of certain matrices (as described in the theorem) with relatively good controllable norms (i.e. singular values, i.e. eigenvalues). Note that this approach is nonconstructive. Even worse, unlike other probabilistic inequalities of similar type, we do not even get an arbitrarily high probability in our bound, just positive probability.
Now onto some more concrete applications of this result, which will lead to a solution of the Kadison-Singer problem.

Corollary 3.9. [19] Let $r \in \mathbb{N}$ and let $u_1, ..., u_m \in \mathbb{C}^d$ be vectors fulfilling

$$\sum_{i=1}^{m} u_i u_i^* = I$$

and $\|u_i\|^2 \leq \delta$ for all i. Then there exists a partition $\{S_1, ..., S_r\}$ of $[m]$, such that

$$\left\| \sum_{i \in S_j} u_i u_i^* \right\| \leq \left(\frac{1}{\sqrt{r}} + \sqrt{\delta} \right)^2$$

for all $j = 1, ..., r$.

Proof. Set

$$w_{i1} = \begin{bmatrix} u_i \\ \mathbf{0} \\ \vdots \\ \mathbf{0} \end{bmatrix}, w_{i2} = \begin{bmatrix} \mathbf{0} \\ u_i \\ \vdots \\ \mathbf{0} \end{bmatrix}, ..., w_{ir} = \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \\ \vdots \\ u_i \end{bmatrix},$$

which are all vectors from $\mathbb{C}^{rd}$. Define $v_1, ..., v_m$ to be independent random vectors such that v_i takes the values $W_i = \{\sqrt{r} w_{ik}\}_{k=1}^r$ each with probability $1/r$. We have

$$\mathbb{E} v_i v_i^* = \begin{bmatrix} u_i u_i^* & & & 0 \\ & u_i u_i^* & & \\ & & \ddots & \\ 0 & & & u_i u_i^* \end{bmatrix}$$

and $\|v_i\|^2 = r\|u_i\|^2 \leq r\delta$ (the norm of v_i does not depend on which vector w_{ik} it represents since all of them have norm $\|u_i\|$). So

$$\sum_{i=1}^m \mathbb{E} v_i v_i^* = I$$

(identity matrix over $\mathbb{C}^{rd}$) and thus we can apply theorem 3.8 with $\varepsilon = r\delta$. This shows that there is an assignment of the v_i that fulfills

$$\left(1 + \sqrt{r\delta}\right)^2 \geq \left\| \sum_{i=1}^m v_i v_i^* \right\| = \left\| \sum_{k=1}^r \sum_{i: v_i = w_{ik}} \left(\sqrt{r} w_{ik}\right) \left(\sqrt{r} w_{ik}\right)^* \right\|.$$

By setting $S_k = \{i : v_i = w_{ik}\}$ we get our desired partition:

$$\left\| \sum_{i \in S_k} u_i u_i^* \right\| = \left\| \sum_{i \in S_k} w_{ik} w_{ik}^* \right\| \leq \frac{1}{r} \left\| \sum_{k=1}^r \sum_{i: v_i = w_{ik}} \left(\sqrt{r} w_{ik}\right) \left(\sqrt{r} w_{ik}\right)^* \right\| \leq \left(\frac{1}{\sqrt{r}} + \sqrt{\delta} \right)^2.$$

To comment on the first "$\leq$": Notice that each partition class S_k corresponds to these vectors v_i (with the assignment given by theorem 3.8) which have their "weight" in the k-th spot, i.e. from the $((k-1)d+1)$-st to the kd-th component and zero everywhere else, so that

$$\sum_{i \in S_k} w_{ik} w_{ik}^* = \frac{1}{r} \sum_{i: v_i = w_{ik}} \left(\sqrt{r} w_{ik}\right) \left(\sqrt{r} w_{ik}\right)^* = \frac{1}{r} \sum_{i: v_i = w_{ik}} v_i v_i^*$$

is a block diagonal matrix of the form

$$\mathrm{Diag}(\mathbb{O}, ..., \mathbb{O}, \sum_{i \in S_k} u_i u_i^*, \mathbb{O}, ..., \mathbb{O}),$$

where the $\sum_{i\in S_k} u_i u_i^*$ stands in the k-th spot. The matrix

$$\frac{1}{r}\sum_{k=1}^{r}\sum_{i:v_i=w_{ik}} \left(\sqrt{r}w_{ik}\right)\left(\sqrt{r}w_{ik}\right)^*$$

however is precisely

$$\mathrm{Diag}\left(\sum_{i\in S_1} u_i u_i^*, ..., \sum_{i\in S_r} u_i u_i^*\right).$$

It thus "contains" the matrix $\mathrm{Diag}(\mathbb{O}, ..., \sum_{i\in S_k} u_i u_i^*, ..., \mathbb{O})$ inside its block diagonal structure from which it can be easily seen that this will only increase the matrix norm. ■

Theorem 3.10 (Weaver's KS_2-conjecure)**.** [19, 27] There exist universal constants $\eta \geq 2$ and $\theta > 0$ such that the following holds: Let $v_1, ..., v_m \in \mathbb{C}^d$ be unit vectors (i.e. $\|v_i\| = 1$ in the euclidean norm for all i) and suppose

$$\sum_{i=1}^{m} |\langle w, v_i\rangle|^2 = \eta \tag{3.3}$$

for all unit vectors $w \in \mathbb{C}^d$. Then we can find a partition $[m] = S_1 \dot\cup S_2$ with

$$\sum_{i\in S_j} |\langle w, v_i\rangle|^2 \leq \eta - \theta \tag{3.4}$$

for all unit vectors $w \in \mathbb{C}^d$ and each $j = 1, 2$.

By "universal constants" we mean that η and θ do not depend on m or d. The set of (normalized) v_i's from above form a tight frame with frame bound η.

Proof. Let the v_i be unit vectors as in the assumptions. We want to apply corollary 3.9 with $r = 2$. How to choose δ now and what does this mean for η and θ? For this set $u_i = v_i/\sqrt{\eta}$ and consider the hermitian positive semidefinite matrix

$$H := \sum_{i=1}^{m} u_i u_i^*,$$

which has the property

$$w^* H w = \frac{1}{\eta}\sum_{i=1}^{m}(w^* v_i)\cdot(v_i^* w) = \frac{1}{\eta}\sum_{i=1}^{m} |\langle w, v_i\rangle|^2 = 1$$

for all unit vectors $w \in \mathbb{C}^d$ by (3.3). This shows that $H = I$ is the identity, since

$$w^*(H - I)w = w^* H w - w^* w = 1 - 1 = 0$$

for all unit vectors w, so that the u_i fulfill the requirements of corollary 3.9 with

$$\|u_i\|^2 = \frac{1}{\eta}\|v_i\|^2 = \frac{1}{\eta} =: \delta.$$

This shows the existence of a partition $S_1 \dot\cup S_2 = [m]$ with

$$\frac{1}{\eta}\left\|\sum_{i\in S_j} v_i v_i^*\right\| = \left\|\sum_{i\in S_j} u_i u_i^*\right\| \leq \left(\frac{1}{\sqrt{r}} + \sqrt{\delta}\right)^2 = \left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{\eta}}\right)^2$$

for $j = 1, 2$, so that

$$\left\|\sum_{i\in S_j} v_i v_i^*\right\| \leq \left(\frac{\sqrt{\eta}}{\sqrt{2}} + 1\right)^2 = \frac{1}{2}\eta + \sqrt{2\eta} + 1.$$

Notice that for $\eta > (2+\sqrt{2})^2$ we have $\frac{1}{2}\eta + \sqrt{2\eta} + 1 < \eta$, so for such η we have

$$\left\|\sum_{i\in S_j} v_i v_i^*\right\| \leq \left(\frac{\sqrt{\eta}}{\sqrt{2}} + 1\right)^2 = \eta - \theta$$

for $\theta = \frac{1}{2}\eta - \sqrt{2\eta} - 1 > 0$. Then

$$\sum_{i\in S_j} |\langle w, v_i\rangle|^2 = w^* \left(\sum_{i\in S_j} v_i v_i^*\right) w \leq \left\|\sum_{i\in S_j} v_i v_i^*\right\| w^* w \leq \eta - \theta$$

for each $= 1, 2$ and all unit vectors $w \in \mathbb{C}^n$, so choosing for example $\eta = 18 > (2+\sqrt{2})^2$ (so $\delta = 1/18$) and calculating $\theta = 2$ with the above relations we get our desired universal constants, proving (3.4). ■

Remark 3.11. Notice that the choice of $\eta = 18$ and $\theta = 2$ is somewhat arbitrary, even though they are particularly nice numerical values. This is just to stress that these constants can really be chosen universally, so they are applicable in any situation as described in the statement of theorem 3.10 independent of any other parameters (i.e. of m, d or the choice of the v_i's).
We will reiterate however that we can choose any $\eta > (2+\sqrt{2})^2$ with the corresponding $\theta = \frac{1}{2}\eta - \sqrt{2\eta} - 1$, so that $\eta - \theta = (1 + \sqrt{\eta/2})^2 > (2+\sqrt{2})^2$. In [7], it was shown that the constraint on η can be loosened under certain extra assumptions, yielding better constants in that case. Their new approach uses a more careful analysis for theorem 3.7. For us, the theoretic implications of the existence of such η and θ will be enough.

Notably, Weaver [27] showed that theorem 3.10 is equivalent to the Kadison-Singer problem, which will be stated precisely at the end of this section. The proof that theorem 3.10 is indeed equivalent to the Kadison-Singer problem goes by showing that Weaver's conjecture is equivalent to another classical conjecture which is in turn equivalent to Kadison-Singer, namely Anderson's paving conjecture. The methods from [19] can even show this other conjecture directly, which we also want to demonstrate here. In particular, we can give a self contained proof of the Kadison-Singer problem, without needing to show the equivalence between Weaver's conjecture and the paving conjecture (both of course are now theorems). We will need theorem 3.10 later on however, so it is worthwhile proving it as well. To get to the paving conjecture, we start with the following central notion.

Definition 3.12. A matrix $T \in \mathbb{C}^{d\times d}$ can be (r,ε)-**paved** for $r \in \mathbb{N}$ and $\varepsilon > 0$, if there are coordinate projections $P_1, ..., P_r$ with $\sum_{i=1}^r P_i = I$ and $\|P_iTP_i\| \leq \varepsilon\|T\|$ for all i.

The coordinate projections $P_1, ..., P_r$ from the definition can also be seen as a partition $S_1, ..., S_r$ of $[d]$ (reflected by the property $\sum_{i=1}^r P_i = I$), so that the matrix norms of the principle submatrices of T corresponding the the S_i's are small in comparison to the norm of T itself. It should also be clear that only $\varepsilon \in (0,1)$ are interesting choices. We also see that the property of being (r,ε)-pavable can be extended to infinite matrices acting continuously on the infinite dimensional, separable Hilbert space $l^2(\mathbb{N}) = l^2$.

Lemma 3.13. [8] Suppose there is a function $r : \mathbb{R}_{>0} \to \mathbb{N}$ such that every hermitian projection $Q \in \mathbb{C}^{2n\times 2n}$ (i.e. $Q^* = Q$ and $Q^2 = Q$) with diagonal entries equal to 1/2 can be $(r(\varepsilon), \frac{1+\varepsilon}{2})$-paved for all $\varepsilon > 0$. Then every hermitian $T \in \mathbb{C}^{n\times n}$ with zeroes in its main diagonal can be $(r(\varepsilon)^2, \varepsilon)$-paved for all $\varepsilon > 0$.

Proof. As a note: Throughout the proof we will denote the identity matrices on $\mathbb{C}^{n\times n}$ and $\mathbb{C}^{2n\times 2n}$ both by I.
Let $T \in \mathbb{C}^{n\times n}$ be hermitian with zero diagonal and assume, without loss of generality, that $\|T\| \leq 1$ (otherwise scale accordingly by a real scalar). Consider the auxiliary matrices

$$R = \begin{bmatrix} T & (I-T^2)^{1/2} \\ (I-T^2)^{1/2} & -T \end{bmatrix} \in \mathbb{C}^{2n\times 2n}$$

and

$$Q = \frac{1}{2}(I+R) \in \mathbb{C}^{2n\times 2n}.$$

Notice, by the conditions we set on T, that R (i.e. $(I-T^2)^{1/2}$) is well defined and hermitian. In particular, R and then also Q are hermitian. Since R has zero diagonal, the matrix Q has only 1/2-entries on its main diagonal. Lastly, we note that

$$R^2 = \begin{bmatrix} T^2 + (I-T^2) & T(I-T^2)^{1/2} - (I-T^2)^{1/2}T \\ T(I-T^2)^{1/2} - (I-T^2)^{1/2}T & (I-T^2) + T^2 \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} = I$$

is the identity, where we used the fact that T and $(I-T^2)^{1/2}$ commute. Therefore, we see that

$$Q^2 = \frac{1}{4}(I + 2R + R^2) = \frac{1}{4}(2I + 2R) = \frac{1}{2}(I+R) = Q,$$

so that Q is as in the statement of the lemma and thus $(r(\varepsilon), \frac{1+\varepsilon}{2})$-pavable by assumption.
Let $r = r(\varepsilon)$ and $P_1, ..., P_r$ be coordinate projections of $\mathbb{C}^{2n}$ with $\|P_iQP_i\| \leq \frac{1+\varepsilon}{2}\|Q\| = \frac{1+\varepsilon}{2}$ (Q projection, so $\|Q\| = 1$) for all $i = 1, ..., r$. We then have

$$0 \preceq P_iQP_i \preceq \frac{1+\varepsilon}{2}P_iIP_i = \frac{1+\varepsilon}{2}P_i,$$

where $\preceq$ is the Loewner order on the set of hermitian matrices. By $R = 2Q - I$ we get

$$-P_i \preceq P_iRP_i \preceq \varepsilon P_i.$$

Applying the same reasoning to $-R$ with the corresponding matrix $\tilde{Q} = \frac{1}{2}(I - R)$ we get a (possibly) different collection of coordinate projections $\tilde{P}_1, ..., \tilde{P}_r$ with $\|\tilde{P}_i \tilde{Q} \tilde{P}_i\| \leq \frac{1+\varepsilon}{2}$, but by the universality of r and ε as in the statement we still have the same parameters. Analogously we therefore get

$$-\varepsilon \tilde{P}_i \preceq \tilde{P}_i R \tilde{P}_i \preceq \tilde{P}_i.$$

The set $\{P_i\tilde{P}_j : i, j = 1, ..., r\}$ (with $P_i\tilde{P}_j = \tilde{P}_j P_i$) of coordinate projections therefore fulfills

$$\sum_{i,j=1}^{r} P_i\tilde{P}_j = \left(\sum_{i=1}^{r} P_i\right)\left(\sum_{j=1}^{r} \tilde{P}_j\right) = I^2 = I$$

and by the above

$$-\varepsilon(P_i\tilde{P}_j) \preceq (P_i\tilde{P}_j)R(P_i\tilde{P}_j) \preceq \varepsilon(P_i\tilde{P}_j),$$

so $\|(P_i\tilde{P}_j)R(P_i\tilde{P}_j)\| \leq \varepsilon\|P_i\tilde{P}_j\| \leq \varepsilon$ for all $i, j = 1, ..., r$. Also have $\|R\| = 1$ by

$$\|Rz\|^2 = \langle R^2 z, z\rangle = \|z\|^2$$

for all $z \in \mathbb{C}^{2n}$ (using selfadjointness of R and $R^2 = I$), so we see that R is (r^2, ε)-pavable.
By restricting the paving of R above to the first n coordinates, we therefore see that T is also (r^2, ε)-pavable. ■

The following result is known as Anderson's paving conjecture [1], where we can even give an explicit bound on r (originally, one only asked for the existence of such r).

Theorem 3.14 (Paving Conjecture)**.** [19] For $0 < \varepsilon < 1$ any hermitian $T \in \mathbb{C}^{n\times n}$ with zero diagonal can be (r, ε)-paved with $r \leq 136/\varepsilon^4$.

Note that r depends only on ε, but not on n or T itself. This will be crucial in the proof of Kadison-Singer, where we need an infinite dimensional analogue.

Proof. In sight of lemma 3.13, let $Q \in \mathbb{C}^{2n\times 2n}$ be an hermitian projection with diagonal entries all 1/2. The claim follows, if we have shown that Q is $(\tilde{r}, \frac{1+\varepsilon}{2})$-pavable with a suitable bound on $\tilde{r}$. To do so, write $Q = [u_i^* u_j]_{i,j=1}^{2n}$ as a Gram matrix (remember $Q^* = Q = Q^2$, so Q is hermitian positive semidefinite which allows us to write Q in that form) for $u_1, ..., u_{2n} \in \mathbb{C}^{2n}$. By the structure of Q we have $\|u_i\|^2 = u_i^* u_i = 1/2 =: \delta$ for all i. Applying corollary 3.9 to these vectors and an $\tilde{r} \in \mathbb{N}$ to be specified later we get a partition $S_1, ..., S_{\tilde{r}}$ of $[2n]$ with corresponding coordinate projections $P_1, ..., P_{\tilde{r}}$, such that

$$\|P_k Q P_k\| = \left\|[u_i^* u_j]_{i,j\in S_k}\right\| = \left\|\sum_{i\in S_k} u_i u_i^*\right\| \leq \left(\frac{1}{\sqrt{\tilde{r}}} + \sqrt{\delta}\right)^2 = \left(\frac{1}{\sqrt{\tilde{r}}} + \frac{1}{\sqrt{2}}\right)^2$$

for all $k = 1, ..., \tilde{r}$, where we used the fact that for arbitrary rectangular matrices U we have the equality $\|U^*U\| = \|UU^*\|$. By $\|Q\| = 1$ and

$$\left(\frac{1}{\sqrt{\tilde{r}}} + \frac{1}{\sqrt{2}}\right)^2 = \frac{1}{2} + \frac{\sqrt{2}}{\sqrt{\tilde{r}}} + \frac{1}{\tilde{r}},$$

we see that Q can be $(\tilde{r}, \frac{1+\varepsilon}{2})$-paved for

$$\frac{\varepsilon}{2} = \frac{\sqrt{2}}{\sqrt{\tilde{r}}} + \frac{1}{\tilde{r}} \quad \Rightarrow \quad \tilde{r} = \left(\frac{\sqrt{2}+\sqrt{2+2\varepsilon}}{\varepsilon}\right)^2 \leq \frac{(2+\sqrt{2})^2}{\varepsilon^2},$$

so lemma 3.13 shows that any hermitian $T \in \mathbb{C}^{n\times n}$ with zero diagonal can be (r, ε)-paved for

$$r = \tilde{r}^2 \leq \frac{(2+\sqrt{2})^4}{\varepsilon^4} \leq \frac{136}{\varepsilon^4}.$$

■

Of course, the above bound may be optimized further. However, it is known, see [8], that we have an dependency between r and ε of at least $r \geq \varepsilon^{-2}$ (with so called conference matrices from combinatorial matrix theory, see [2] as a starting point on those, almost attaining that bound) and that matrices Q as in lemma 3.13 cannot be $(r, \frac{1+\varepsilon}{2})$-paved for $r = 2$ and $\frac{1+\varepsilon}{2} < 1$ (in fact, more generally we have the relation $\frac{r}{2(r-1)} \leq \frac{1+\varepsilon}{2}$).
For the proof of the Kadison-Singer problem we will need an infinite dimensional extension of theorem 3.14, which will require the compactness theorem of Arzela and Ascoli.

Corollary 3.15. [23] Let T be a continuous, selfadjoint operator on the infinite dimensional, separable Hilbert space l^2 and further assume that T only has zeroes on its main diagonal. Then T can be (r, ε)-paved with $r \leq 136/\varepsilon^4$.

Proof. Let $\varepsilon > 0$ arbitrary, T as in the statement and consider the matrices (operators) $T^{(m)}$ to be the upper-left principal $(m \times m)$-submatrix of T extended by zeroes to an infinite matrix. Of course we then have $\|T^{(m)}\| \leq \|T\|$ for all $m \in \mathbb{N}$. Applying theorem 3.14 to these $T^{(m)}$ we get the existence of coordinate projections $P_1^{(m)}, ..., P_r^{(m)}$ (all of them are infinite matrices) with $\sum_{i=1}^r P_i^{(m)} = I$ (identity operator on l^2) and $\|P_i^{(m)} T^{(m)} P_i^{(m)}\| \leq \varepsilon \|T^{(m)}\| \leq \varepsilon\|T\|$ for all $i = 1, ..., r$, where $r \leq 136/\varepsilon^4$ does not depend on m, so we may assume that there are r such projections $P_i^{(m)}$ for all m (some of these projections then may be the zero operator).
Applying the theorem of Arzela-Ascoli to the set $\{P_i^{(m)} : m \in \mathbb{N}, 1 \leq i \leq r\}$ (closedness and boundedness are clear, equicontinuity is also fulfilled by the fact that these are linear operators whos norms are all bounded uniformly by 1), we can choose a sequence of increasing natural numbers $(m(l))_l$, such that $P_i^{(m(l))} \longrightarrow P_i, l \longrightarrow \infty$ pointwise (i.e. componentwise) to some P_i for all $i = 1, ..., r$ simultaneously. By the properties of the $P_i^{(m)}$ we see that the P_i are again coordinate projections and that

$$I = \sum_{i=1}^r P_i^{(m(l))} \longrightarrow \sum_{i=1}^r P_i \quad , \quad l \longrightarrow \infty,$$

so that $\sum_{i=1}^r P_i = I$.
For unite vectors $u, v \in l^2$ which have finite support (note that the vectors of finite support form a dense subset of l^2), we see that $P_i^{(m(l))} u = P_i u$ and $P_i^{(m(l))} v = P_i v$ for all $i = 1, ..., r$ and all sufficiently large l. Since $P_i u$ and $P_i v$ have finite support, we also

have $T^{(m(l))}P_i u = T_i u$ and $T^{(m(l))}P_i v = T_i v$ for sufficiently large l. We therefore see, for l big enough, that

$$\langle T^{(m(l))}P_i^{(m(l))}u, P_i^{(m(l))}v\rangle = \langle TP_i u, TP_i v\rangle = \langle P_i TP_i u, v\rangle$$

for all $i = 1, ..., r$. Since $\|P_i^{(m)}T^{(m)}P_i^{(m)}\| \leq \varepsilon\|T\|$ for all $i = 1, ..., r$, the Cauchy-Schwarz inequality yields

$$\langle P_i TP_i u, v\rangle = \langle P_i^{(m(l))}T^{(m(l))}P_i^{(m(l))}u, v\rangle \leq \varepsilon\|T\|,$$

so taking suprema over all unit vectors u and v we get

$$\|P_i TP_i\| \leq \varepsilon\|T\|$$

for all $i = 1, ..., r$. This finishes the proof. ■

To conclude this chapter we will now discuss the Kadison-Singer problem: Let again l^2 be the (complex) Hilbert space of square summable sequences. By the Fischer-Riesz theorem from basic functional analysis we known that every complex separable Hilbert space is isometrically isomorphic to l^2, so for our considerations it suffices to investigate this case. We set $\mathfrak{B} := \mathcal{L}(l^2)$ to be the space of linear, bounded operators $l^2 \to l^2$, which becomes a C*-algebra with the involution * being defined as taking adjoint operators. Let $\mathfrak{A} \subseteq \mathfrak{B}$ be a (closed) unital subalgebra (i.e. $I \in \mathfrak{A}$ for I the identity operator) closed under the *-operation (i.e. $A \in \mathfrak{A} \Rightarrow A^* \in \mathfrak{A}$). A notable example of such an $\mathfrak{A}$ would be the set $\mathfrak{D}$ of diagonal operators on l^2. A **state** of $\mathfrak{A}$ is a continuous linear functional $\varphi : \mathfrak{A} \to \mathbb{C}$, that fulfills

(i) $\varphi(I) = 1$;

(ii) $\varphi(P) \geq 0$ for all positive operators $P \in \mathfrak{A}$ (i.e. $\langle Px, x\rangle \geq 0$ for all $x \in l^2$).

The set $\mathcal{S}(\mathfrak{A})$ of all states on $\mathfrak{A}$ is a convex subset in the dual space $\mathfrak{A}'$, which is also weak*-compact ($\mathfrak{A}'$ equipped with the weak*-topology induced by $\mathfrak{A}$), as can be shown using the Banach-Alaoglu theorem. By the Krein-Milman theorem $\mathcal{S}(\mathfrak{A}) \subseteq \mathfrak{A}'$ is the closed convex hull of its extreme points. These extreme points are called the **pure states** of $\mathfrak{A}$, so that pure states are precisely those states that cannot be expressed as a proper convex combination (i.e. at least two coefficients of the convex combination are not zero) of other states. The following should help to give some intuition on how pure states work.

Lemma 3.16. [23] Let φ be a pure state on $\mathfrak{D}$ and $P \in \mathfrak{D}$ a diagonal projection, i.e. P has zeroes on its off-diagonal entries and only zeroes or ones on its main diagonal. Then $\varphi(P) \in \{0, 1\}$.

Proof. Let P be as above. Both P and $I - P$ are positive operators, thus by assumption $\varphi(P) \geq 0$ and $\varphi(I - P) \geq 0$, so that we have $0 \leq \varphi(P) \leq 1$. Assume, for contradiction, that $\varphi(P) = \theta \in (0, 1)$. We can then define new states φ_1, φ_2 on $\mathfrak{D}$ by $\varphi_1(Q) = \frac{1}{\theta}\varphi(PQ)$

and $\varphi_2(Q) = \frac{1}{1-\theta}\varphi((I-P)Q)$, which are easily seen to satisfy the properties we require for a state. But then we have

$$\theta\varphi_1(Q) + (1-\theta)\varphi_2(Q) = \varphi(PQ) + \varphi((I-P)Q) = \varphi(Q),$$

contradicting the fact that φ is a pure state. We therefore see that $\varphi(P) \in \{0,1\}$ for φ and P as above. ■

By the Hahn-Banach theorem, any state φ on $\mathfrak{A}$ can be extended to a continuous linear functional on $\mathfrak{B}$. In the case that $\mathfrak{A} = \mathfrak{D}$ are the diagonal operators, the existence of such an extension can be seen even simpler, by setting all off-diagonal entries of an $H \in \mathfrak{B}$ to zero and applying φ on this restricted operator. The Kadison-Singer problem states further:

Theorem 3.17 (Kadison-Singer)**.** [16, 27, 23] The extension of a pure state on $\mathfrak{D}$ to a state on $\mathfrak{B}$ is unique.

Proof. Let φ be a pure state on $\mathfrak{D}$ and ψ an extension of φ to a state on $\mathfrak{B}$. It can be seen directly that the claim follows once we showed that ψ is zero on all $Q \in \mathfrak{B}$ whos main diagonal only consists of zero entries. Furthermore, it suffices to consider selfadjoint operators, since we can split up

$$Q = \frac{Q + Q*}{2} + \frac{1}{i} \cdot \frac{i(Q - Q^*)}{2},$$

so we can always write Q as a linear combination of selfadjoint operators (still having only zeroes on their main diagonal). Once we showed the case for selfadjoint Q, we therefore get by linearity of ψ the wanted statement for general Q.
Let now $\varepsilon > 0$ be arbitrary. Using corollary 3.15, for all selfadjoint $Q \in \mathfrak{B}$ with zero diagonal we can find coordinate projections $P_1, ..., P_r$ with $I = \sum_{i=1}^r P_i$ and $\|P_i Q P_i\| \leq \varepsilon\|Q\|$ for all $i = 1, ..., r$. As seen multiple times by now, $r \leq 136/\varepsilon^4$ only depends on ε (while the actual projections $P_1, ..., P_r$ depend on Q).
By lemma 3.16 we have that $\psi(P_i) = \varphi(P_i) \in \{0,1\}$, so with

$$1 = \varphi(I) = \sum_{i=1}^{r} \varphi(P_i)$$

we see that exactly one of the P_i, say P_{i_0}, has $\varphi(P_{i_0}) = 1$, while the others fulfill $\varphi(P_i) = 0, i \neq i_0$. For what follows take an arbitrary $i \neq i_0$.
We now verify that $\langle A, B\rangle := \psi(AB^*)$ is a semiscalar product on $A, B \in \mathfrak{B}$. Linearity in the first component is clear, sesquilinearity in the second component follows in the same way using $\langle A, iB\rangle = \psi(A(iB)^*) = -i\psi(AB^*) = -i\langle A, B\rangle$, positive semidefiniteness is a consequence of property (ii) of the state ψ (since XX^* is a positive operator for $X \in \mathfrak{B}$, we have $\langle X, X\rangle = \psi(XX^*) \geq 0$) and conjugate symmetry is settled by the already proven properties together with

$$\mathbb{R} \ni \langle A+B, A+B\rangle - \langle A-B, A-B\rangle = 2(\langle A, B\rangle + \langle B, A\rangle)$$

$$\Rightarrow \operatorname{Im}\langle A, B\rangle = -\operatorname{Im}\langle B, A\rangle$$

and

$$\mathbb{R} \ni \langle iA + B, iA + B\rangle - \langle A + iB, A + iB\rangle = 2i(\langle A, B\rangle - \langle B, A\rangle)$$
$$\Rightarrow \mathrm{Re}\langle A, B\rangle = \mathrm{Re}\langle B, A\rangle,$$

so that indeed $\langle A, B\rangle = \overline{\langle B, A\rangle}$. We can therefore apply the Cauchy-Schwarz inequality for the semiscalar product on $\mathfrak{B}$ above, so that

$$|\psi(P_iR^*)|^2 \leq \psi(P_iP_i^*)\psi(RR^*)$$

for all $R \in \mathfrak{B}$. Since $\psi(P_iP_i^*) = \psi(P_i) = 0$ (remember $i \neq i_0$), we conclude $\psi(P_iR^*) = 0$ (and consequently $\psi(RP_i) = 0$) for all $R \in \mathfrak{B}$, $i \neq i_0$. Thus

$$\psi(Q) = \sum_{i=1}^{r} \psi(P_iQP_i) = \psi(P_{i_0}QP_{i_0})$$

and by $\|P_{i_0}QP_{i_0}\| \leq \varepsilon\|Q\|$ we therefore get

$$|\psi(Q)| = |\psi(P_{i_0}QP_{i_0})| \leq \|\psi\| \cdot \|P_{i_0}QP_{i_0}\| \leq \|\psi\| \cdot \varepsilon\|Q\|$$

($\|\psi\|$ the norm of ψ as a functional, $\|Q\|$ the norm of Q as an operator). Since $\varepsilon > 0$ was chosen arbitrarily, we see that indeed $\psi(Q) = 0$. ■

As stated right before theorem 3.17, we also know how this extension has to look. In particular, the extension is again a pure state (now on $\mathfrak{B}$). There are more general formulations of the Kadison-Singer problem (generalizations to more C*-algebras as described above, not just $\mathfrak{D}$) and deeper theoretical connection (correspondence of pure states on $\mathfrak{D}$ with ultrafilters on $\mathbb{N}$), but we will not go deeper into these topics.

4 Exponential Frames

We now want to give a concrete application of the above results for functional analytic and approximation theoretic purposes. But first a reminder on the notions.

Definition 4.1. For a Hilbert space $H = (H, \langle \cdot, \cdot \rangle, \| \cdot \|)$, a subset $E = \{u_i\}_{i \in I} \subseteq H$ is called a **frame**, if there are constants $0 < a \leq A$ (then called the **frame bounds**), so that

$$a\|x\|^2 \leq \sum_{i \in I} |\langle x, u_i \rangle|^2 \leq A\|x\|^2$$

for all $x \in H$.

We will consider the Hilbert space $L^2(S)$ of (equivalence classes of) complex valued square integrable functions on $S \subseteq \mathbb{R}$ (always with respect to the Lebesgue measure). We want to construct a frame of $L^2(S)$ of a certain form, namely consisting of exponential functions $\{\exp(i\lambda \cdot) : \lambda \in \Lambda\}$ for a "discrete" set of frequencies Λ. For these exponential functions to be in $L^2(S)$, we have to require S to be of finite measure. The problem under consideration thus states:
For a measurable $S \subseteq \mathbb{R}$ of finite measure, find a (in some sense) discrete set of frequencies $\Lambda \subseteq \mathbb{R}$, such that $E(\Lambda) := \{\exp(i\lambda \cdot) : \lambda \in \Lambda\} \subseteq L^2(S)$ forms a frame and give estimates for the corresponding frame bounds.
We will make everything precise further down. Of course, from the get go it is not even clear that such Λ exist at all. For example it is not always possible to find frequencies, such that $E(\Lambda)$ forms an orthogonal basis. Since in our situation an exponential orthogonal basis results in a frame with equal frame bounds $a = A = |S|$ (the measure of S), frames are the natural generalization for this problem.

4.1 Consequences of Weaver's Conjecture

In order to get such Λ, we will use the partition given by theorem 3.10. For convenience, we will restate theorem 3.10 a bit more explicitly.

Theorem 4.2. [20] Let $0 < \varepsilon$ and $u_1, ..., u_m \in \mathbb{C}^n$ with $\|u_i\|^2 \leq \varepsilon$ for all $i = 1, ..., m$ and

$$\sum_{i=1}^{m} |\langle w, u_i \rangle|^2 = \|w\|^2$$

for all $w \in \mathbb{C}^n$. Then there is a partition $S_1 \dot\cup S_2 = [m]$ with

$$\sum_{i \in S_j} |\langle w, u_i \rangle|^2 \leq \frac{(1+\sqrt{2\varepsilon})^2}{2} \|w\|^2$$

for each $j = 1, 2$ and all $w \in \mathbb{C}^n$. In the case of $\varepsilon < 1$ we therefore have

$$\sum_{i \in S_j} |\langle w, u_i \rangle|^2 \leq \frac{1+5\sqrt{\varepsilon}}{2} \|w\|^2.$$

Proof. Without loss of generality let $\varepsilon < (2+\sqrt{2})^{-2}$ and let the u_i fulfill the assumptions. The rescaled vectors $v_i := u_i/\sqrt{\varepsilon}$ fulfill the assumptions of theorem 3.10 with $\eta = 1/\varepsilon > (2+\sqrt{2})^2$ (compare with remark 3.11). We conclude that for a suitable partition $S_1 \dot\cup S_2 = [m]$ we have

$$\sum_{i \in S_j} |\langle w, v_i \rangle|^2 \leq (\eta - \theta) \cdot \|w\|^2$$

for all $j = 1, 2$ and $w \in \mathbb{C}^n$. Scaling back gives

$$\sum_{i \in S_j} |\langle w, u_i \rangle|^2 \leq \frac{\eta - \theta}{\eta} \cdot \|w\|^2,$$

so plugging in η and $\eta - \theta$ (see theorem 3.10 and remark 3.11) gives the desired

$$\sum_{i \in S_j} |\langle w, v_i \rangle|^2 \leq \frac{(1+1/\sqrt{2\varepsilon})^2}{1/\varepsilon} \cdot \|w\|^2 = \frac{(1+\sqrt{2\varepsilon})^2}{2} \|w\|^2.$$

For the last part in the statement simply observe $(1+\sqrt{2\varepsilon})^2 \leq 1 + 5\sqrt{\varepsilon}$ for $\varepsilon < 1$. ■

Indeed, the constant 5 here could even be slightly improved to $2 + 2\sqrt{2}$, or even if we only want this type of inequality to hold for $0 < \varepsilon < (2+\sqrt{2})^{-2}$ we can further improve the constant to $2 + \sqrt{2} < 3.42$. We will continue as in [20], however one may keep this in mind if one is interested in better constants at the end (for this, a look in [7] can also be useful).

We will give some first insight into why this might be related to frames.

Remark 4.3. [20] With notation as in theorem 4.2 we have

$$\sum_{i \in S_1} |\langle w, u_i \rangle|^2 = \|w\|^2 - \sum_{i \in S_2} |\langle w, u_i \rangle|^2,$$

so for $\varepsilon < 1$ theorem 4.2 even gives the two sided estimate

$$\frac{1-5\sqrt{\varepsilon}}{2} \|w\|^2 \leq \sum_{i \in S_j} |\langle w, u_i \rangle|^2 \leq \frac{1+5\sqrt{\varepsilon}}{2} \|w\|^2$$

for each $j = 1, 2$ and $w \in \mathbb{C}^n$.

We will continue with more concrete application of theorem 4.2 to frames.

Corollary 4.4. [14] Let $v_1, ..., v_k \in \mathbb{C}^n$ with $\|v_i\|^2 \leq \delta$ for all $i = 1, ..., k$ and

$$\alpha\|w\|^2 \leq \sum_{i=1}^{k} |\langle w, v_i\rangle|^2 \leq \beta\|w\|^2$$

for all $w \in \mathbb{C}^n$, where $\beta \geq \alpha > \delta > 0$ are some constants. Then there is a partition $S_1\dot{\cup}S_2 = [k]$, such that

$$\frac{1 - 5\sqrt{\delta/\alpha}}{2} \cdot \alpha\|w\|^2 \leq \sum_{i\in S_j} |\langle w, u_i\rangle|^2 \leq \frac{1 + 5\sqrt{\delta/\alpha}}{2} \cdot \beta\|w\|^2$$

for each $j = 1, 2$ and all $w \in \mathbb{C}^n$.

Proof. Consider the operator $M : \mathbb{C}^n \to \mathbb{C}^n, w \mapsto \sum_{i=1}^{k}\langle w, v_i\rangle v_i$. Observe $\langle Mw, w\rangle = \sum_{i=1}^{k} |\langle w, v_i\rangle|^2 \geq 0$, so that M is a positive operator which is also hermitian by

$$\langle Mu, w\rangle = \sum_{i=1}^{k}\langle u, v_i\rangle\langle v_i, w\rangle = \overline{\sum_{i=1}^{k}\langle w, v_i\rangle\langle v_i, u\rangle} = \overline{\langle Mw, u\rangle} = \langle u, Mw\rangle.$$

We can therefore form $M^{1/2}$ which has the property

$$\|M^{1/2}w\|^2 = w^*Mw = w^*\sum_{i=1}^{k}\langle w, v_i\rangle v_i = \sum_{i=1}^{k} |\langle w, v_i\rangle|^2,$$

so that by the assumptions

$$\alpha\|w\|^2 \leq \|M^{1/2}w\|^2 \leq \beta\|w\|^2$$

for all $w \in \mathbb{C}^n$.
Setting now $u_i = M^{-1/2}v_i$, we have $\|u_i\|^2 \leq \|v_i\|^2/\alpha \leq \delta/\alpha$, but also

$$\sum_{i=1}^{k}\langle w, u_i\rangle u_i = M^{-1/2}\sum_{i=1}^{k}\langle M^{-1/2}w, v_i\rangle v_i = M^{-1/2}M(M^{-1/2}w) = w$$

for all $w \in \mathbb{C}^n$, since $M^{-1/2}$ is hermitian. This shows, that the u_i fulfill the assumptions of theorem 4.2 with $m = k$ and $\varepsilon = \delta/\alpha < 1$. Let $S_1\dot{\cup}S_2 = [k]$ be a partition as guaranteed by theorem 4.2. We thus get (with the above bound on the operator norm of $M^{1/2}$)

$$\begin{aligned}\sum_{i\in S_j} |\langle w, v_i\rangle|^2 = \sum_{i\in S_j} |\langle M^{1/2}w, u_i\rangle|^2 &\leq \frac{1 + 5\sqrt{\varepsilon}}{2}\|M^{1/2}w\|^2 \\ &\leq \frac{1 + 5\sqrt{\delta/\alpha}}{2} \cdot \beta\|w\|^2.\end{aligned}$$

For the lower bound we use remark 4.3, to get analogously

$$\begin{aligned}\sum_{i\in S_j} |\langle w, v_i\rangle|^2 = \sum_{i\in S_j} |\langle M^{1/2}w, u_i\rangle|^2 &\geq \frac{1 - 5\sqrt{\varepsilon}}{2}\|M^{1/2}w\|^2 \\ &\geq \frac{1 - 5\sqrt{\delta/\alpha}}{2} \cdot \alpha\|w\|^2.\end{aligned}$$

■

The formulation given in corollary 4.4 is particularly useful for inductive purposes, as will be seen for the next result.

Theorem 4.5. [20] Let $u_1, ..., u_m \in \mathbb{C}^n$ with $\|u_i\|^2 = \frac{n}{m}$ for all $i = 1, ..., m$ and

$$\sum_{i=1}^{m} |\langle w, u_i \rangle|^2 = \|w\|^2 \tag{4.1}$$

for all $w \in \mathbb{C}^n$. Then there is a $J \subseteq [m]$ with

$$c_0 \cdot \frac{n}{m}\|w\|^2 \leq \sum_{i \in J} |\langle w, u_i \rangle|^2 \leq C_0 \cdot \frac{n}{m}\|w\|^2 \tag{4.2}$$

for all $w \in \mathbb{C}^n$. Here, c_0 and C_0 are universal positive constants.

Proof. As a first step consider for $0 < \delta < 1/100$ the inductively defined numbers $\alpha_0 = \beta_0 = 1$ and

$$\alpha_{j+1} := \alpha_j \cdot \frac{1 - 5\sqrt{\delta/\alpha_j}}{2}, \quad \beta_{j+1} := \beta_j \cdot \frac{1 + 5\sqrt{\delta/\alpha_j}}{2}.$$

We want to show that then there is a universal constant $C > 0$ and an $L \in \mathbb{N}$, such that $\alpha_j \geq 100\delta$ for all $j \leq L$, as well as $25\delta \leq \alpha_{L+1} < 100\delta$ and $\beta_{L+1} < C\alpha_{L+1} \leq 100C\delta$. Notice that

$$\frac{1 - 5\sqrt{\delta/\alpha_j}}{2}$$

is strictly monotonically increasing in $\alpha_j > 0$. For $\alpha_j \geq 100\delta$ we thus have

$$\frac{1}{4} = \frac{1 - 5\sqrt{\frac{\delta}{100\delta}}}{2} \leq \frac{1 - 5\sqrt{\delta/\alpha_j}}{2} < \lim_{x \longrightarrow \infty} \frac{1 - 5\sqrt{\delta/x}}{2} = \frac{1}{2},$$

therefore

$$\frac{\alpha_j}{4} \leq \alpha_j \cdot \frac{1 - 5\sqrt{\delta/\alpha_j}}{2} = \alpha_{j+1} < \frac{\alpha_j}{2}.$$

Set $L := \max\{j \in \mathbb{N} : \alpha_j \geq 100\delta\}$ (L depends on δ), so that $\alpha_j \geq 100\delta$ for all $j \leq L$. Note that $\alpha_j \searrow 0$ decreasingly as $j \longrightarrow \infty$, so that $L < \infty$. By the definition of L we have $\alpha_{L+1} < 100\delta$, but by the above also

$$\alpha_{L+1} \geq \frac{\alpha_L}{4} \geq 25\delta.$$

It remains to find a $C > 0$ not depending on δ, such that $\beta_{L+1} < C\alpha_{L+1}$. This will be obtained by iteratively inserting the definition of the α_j and β_j from above:

$$\begin{aligned} \frac{\beta_{L+1}}{\alpha_{L+1}} &= \frac{\beta_L}{\alpha_L} \cdot \frac{1 + 5\sqrt{\delta/\alpha_L}}{1 - 5\sqrt{\delta/\alpha_L}} = \frac{\beta_{L-1}}{\alpha_{L-1}} \cdot \frac{1 + 5\sqrt{\delta/\alpha_{L-1}}}{1 - 5\sqrt{\delta/\alpha_{L-1}}} \cdot \frac{1 + 5\sqrt{\delta/\alpha_L}}{1 - 5\sqrt{\delta/\alpha_L}} = ... \\ &= \underbrace{\frac{\beta_0}{\alpha_0}}_{=1} \cdot \prod_{j=0}^{L} \frac{1 + 5\sqrt{\delta/\alpha_j}}{1 - 5\sqrt{\delta/\alpha_j}}. \end{aligned}$$

Have $\alpha_L \geq 100\delta$, so that $5\sqrt{\delta/\alpha_L} \leq 1/2$ and using

$$\alpha_{j+1} < \frac{\alpha_j}{2} \quad \Rightarrow \quad 5\sqrt{\delta/\alpha_j} < \frac{1}{\sqrt{2}} \cdot 5\sqrt{\delta/\alpha_{j+1}}$$

inductively, we get $5\sqrt{\delta/\alpha_{L-j}} < 2^{-1-j/2}$ for $j = 1, ..., L$ (with "$\leq$" for $j = 0$). Thus

$$\frac{\beta_{L+1}}{\alpha_{L+1}} = \prod_{j=0}^{L} \frac{1+5\sqrt{\delta/\alpha_j}}{1-5\sqrt{\delta/\alpha_j}} < \prod_{j=0}^{\infty} \frac{1+2^{-1-j/2}}{1-2^{-1-j/2}} =: C < 35.21,$$

which yields the final claim.
With this at hand, consider now the situation as in the statement of the theorem. First notice that $n \leq m$, as otherwise there is a $w \in \mathbb{C}^n \setminus \{0\}$, which is orthogonal to all u_i, $i = 1, ..., m$. Plugging this into (4.1), the left hand side would yield 0, while the right hand side would be strictly positive. This contradiction shows $n \leq m$, so $n/m \leq 1$.
If $1 \geq n/m \geq 1/100$, then (4.2) holds with $J = [m]$, $C_0 = 100$ and $c_0 = 1$. Assume for the rest $\delta := n/m < 1/100$ and let α_j and β_j be as in the first part of the proof. The vectors $v_i = u_i$ satisfy the assumption of corollary 4.4 with $\alpha = \alpha_0 = 1$ and $\beta = \beta_0 = 1$. Hence, there is a set $J_1 \subseteq [m]$ with

$$\alpha_1 \|w\|^2 \leq \sum_{i \in J_1} |\langle w, u_i \rangle|^2 \leq \beta_1 \|w\|^2$$

for all $w \in \mathbb{C}^n$. We may now apply corollary 4.4 again (restricted on the indices in the set J_1). Since $\alpha_1 \geq \alpha_L \geq 100\delta > \delta$, corollary 4.4 gives the existence of a $J_2 \subseteq J_1$ with

$$\alpha_2 \|w\|^2 \leq \sum_{i \in J_2} |\langle w, u_i \rangle|^2 \leq \beta_2 \|w\|^2.$$

If we keep on going like this, after $L+1$ applications of corollary 4.4 (which is doable by $\alpha_L \geq 100\delta > \delta$) we get

$$\alpha_{L+1} \|w\|^2 \leq \sum_{i \in J_{L+1}} |\langle w, u_i \rangle|^2 \leq \beta_{L+1} \|w\|^2.$$

By what was proven in the first part, we therefore get

$$25\delta \|w\|^2 \leq \sum_{i \in J_{L+1}} |\langle w, u_i \rangle|^2 < 100C\delta \|w\|^2.$$

The claim now follows (once we remembered $\delta = n/m$) for $J = J_{L+1}$, $c_0 = 25$ and $C_0 = 100C < 3521$. ■

Remark 4.6. We have seen that we have to split the set $[m]$ a total of $L+1$ times. If at every step we choose to take the smaller of the two sets we can estimate $\#J \leq 2^{-(L+1)}m$. By $a_0 = 1$ and $a_j/2 \geq a_{j+1}$ we easily see that $a_j \leq 2^{-j}$, so that we can guarantee $a_{L+1} < 100\delta$ if we have $2^{-(L+1)} \leq 100\delta$. Since $\delta = n/m$ we therefore get an estimate of the form $\#J \leq 100n$. All this of course only makes sense if $m > 100n$.
If we take instead in every step the larger of the two sets, we get that there is also a J as above with $\#J \geq 100n$. Depending on our purposes, we might want J to be small or J to be large, but in general $\#J \approx 100n$ might be a good guess loosely speaking.

We give a further reformulation of theorem 4.5, which is more akin to the concept of matrices with the so called restricted isometry property (RIP).

Corollary 4.7. There exist universal constants $c_0, C_0 > 0$ with the following property: Whenever $A \in \mathbb{C}^{m\times n}$ is a matrix such that the columns of A form an orthonormal system and every row of A has equal euclidean norm (for every $i = 1, ..., m$, the norm $\|A_i\|$ of the i-th row of A is independent of i), then we can find a set $J \subseteq [m]$, such that

$$c_0 \cdot \frac{n}{m}\|w\|^2 \leq \|A_J w\|^2 \leq C_0 \cdot \frac{n}{m}\|w\|^2$$

for all $w \in \mathbb{C}^n$, where A_J denotes the submatrix of A corresponding to the row indices from J.

Note that the norm in $\|w\|$ is the euclidean norm on $\mathbb{C}^n$, while $\|A_J w\|$ is the euclidean norm on $\mathbb{C}^J$.

Proof. As above, let A_i for $i = 1, ..., m$ be the rows of A (we will consider A_i as column vectors) and denote by $\kappa = \|A_i\|^2$ their common norm. Then have

$$m \cdot \kappa = \sum_{i=1}^m \|A_i\|^2 = \sum_{i=1}^m \sum_{j=1}^n |a_{ij}|^2 = \sum_{j=1}^n \underbrace{\sum_{i=1}^m |a_{ij}|^2}_{=1,\text{ since ONS}} = n,$$

so $\kappa = n/m$, thus $\|A_i\|^2 = n/m$ for all $i = 1, ..., m$. But for arbitrary $w \in \mathbb{C}^n$ we have

$$\sum_{i=1}^m |\langle w, A_i\rangle|^2 = \sum_{i=1}^m \overline{A_i^* w} \cdot A_i^* w = w^* \left(\sum_{i=1}^m A_i A_i^*\right) w,$$

where $\sum_{i=1}^m A_i A_i^* = \overline{A^* A} = I$ (columns are orthonormal), so $\sum_{i=1}^m |\langle w, A_i\rangle|^2 = \|w\|^2$. Thus the $A_i = u_i$ fulfill the assumptions (4.2) of theorem 4.5, giving us a $J \subseteq [m]$ with

$$c_0 \cdot \frac{n}{m}\|w\|^2 \leq \sum_{i\in J} |\langle w, A_i\rangle|^2 \leq C_0 \cdot \frac{n}{m}\|w\|^2$$

for all $w \in \mathbb{C}^n$. But

$$\sum_{i\in J} |\langle A_i, w\rangle|^2 = \|A_J w\|^2,$$

which finishes the proof. ■

4.2 Analysis of the Paley-Wiener Space

We first fix some notation. For f a complex valued, integrable function on $\mathbb{R}$ (with respect to the Lebesgue measure), we write

$$\hat{f}(\xi) = (2\pi)^{-1/2} \int_{\mathbb{R}} f(x) e^{-ix\xi} dx$$

for its Fourier transform, where we continue F in the usual way to all L^2-functions. Before we get to the analysis itself, we first need some auxiliary notions.

Definition 4.8. For a set $\Lambda \subseteq \mathbb{R}$ we define its **separation constant**

$$d(\Lambda) := \inf_{\lambda \neq \lambda' \in \Lambda} |\lambda - \lambda'|.$$

We call such a $\Lambda \subseteq \mathbb{R}$ **discrete**, if $d(\Lambda) > 0$. If $(\Lambda_n)_n$ is a sequence of discrete sets with $d(\Lambda_n) \geq d$ for all $n \in \mathbb{N}$, where $d > 0$ is a fixed constant, we say that $(\Lambda_n)_n$ **converges weakly** to a countable $\Lambda \subseteq \mathbb{R}$, and call Λ its **weak limit**, if

$$\forall \varepsilon > 0 \forall \Omega = (a,b), a,b \in (\mathbb{R} \setminus \Lambda) \exists N \in \mathbb{N} \forall n \geq N : \Lambda_n \cap \Omega \subseteq (\Lambda \cap \Omega) + (-\varepsilon, \varepsilon)$$
$$\text{and } \Lambda \cap \Omega \subseteq (\Lambda_n \cap \Omega) + (-\varepsilon, \varepsilon).$$

(Notice that we require the above statement only to hold for all intervals Ω, which have end points *not* in Λ, i.e. for which $\overline{\Omega} \subseteq \mathbb{R} \setminus \Lambda$ hold) We then write $\Lambda_n \xrightarrow{w} \Lambda$.
Furthermore, we define $\Lambda \subseteq \mathbb{R}$ to be **quasidiscrete**, if for all $\lambda \in \Lambda$ there is an $\varepsilon = \varepsilon(\lambda) > 0$, such that $\Lambda \cap U_\varepsilon(\lambda) = \{\lambda\}$.

Some remarks: For technical reasons we require the weak limit Λ to be countable. Otherwise every sequence $(\Lambda_n)_n$ as described in the definition would trivially converge weakly towards $\mathbb{R}$ for example, since then we will never find a suitable Ω, so the condition is trivially fulfilled. Notice also the importance of the restriction for the intervals Ω. As an example consider the sequence defined by $\Lambda_n = \{1/n\}$ for $n \in \mathbb{N}$. It certainly sounds plausible for this sequence to converge towards $\Lambda = \{0\}$ and it indeed does (as one checks easily) in the weak sense as above. However, if we would try to apply the above definition for the interval $\Omega = (0,2)$ (which has its left boundary point in Λ), then for no $\varepsilon > 0$ the first condition on N will be met, as the left hand side is always nonempty, while the right hand side is always empty.
Note that it is possible for the weak limit of discrete sets to be empty, for example $\{n\} \xrightarrow{w} \emptyset$ in the above sense. In the case that $\#\Lambda \leq 1$, we have $d(\Lambda) = \infty$. Obviously, every discrete set is quasidiscrete. Also observe that in general, the union of two discrete sets is not discrete again (according to the above definition), as can be seen with $\Lambda_1 = \mathbb{N}$ and $\Lambda_2 = \{n - \frac{1}{n} : n \in \mathbb{N}\}$. However, it is easily seen that the union of finitely many (quasi-)discrete sets is quasidiscrete, but also keep in mind that not ever quasidiscrete set is the union of finitely many discrete sets, as can be seen with $\Lambda = \{\sqrt{n} : n \in \mathbb{N}\}$ for example.
We will next prove some technical properties of this type of convergence.

Proposition 4.9. Let $(\Lambda_n)_n$ be a sequence of nonempty discrete sets with $d(\Lambda_n) \geq d > 0$ for all $n \in \mathbb{N}$.

(i) Let $\Lambda_n \xrightarrow{w} \Lambda$. For any $\lambda \in \Lambda$ there is a sequence $(\lambda_n)_n$ with $\lambda_n \in \Lambda_n$ for all $n \in \mathbb{N}$ and $\lambda_n \longrightarrow \lambda$. If on the other hand $(\lambda_n)_n$ is a convergent sequence with $\lambda_n \in \Lambda_n$ for all $n \in \mathbb{N}$, then the limit $\lim\limits_{n \longrightarrow \infty} \lambda_n$ lies in Λ.

(ii) If the weak limit exists, it is unique.

(iii) If $\Lambda_n \xrightarrow{w} \Lambda$, then $d(\Lambda) \geq d$, so Λ is discrete.

(iv) There is a subsequence $(\Lambda_{n_k})_k$, that converges weakly to a discrete $\Lambda \subseteq \mathbb{R}$ with $d(\Lambda) \geq d$.

Proof. (i): Let $\lambda \in \Lambda$ arbitrary and fix $n > 1/d$. Then, for every $k \in \mathbb{N}$ there is a $N_k \in \mathbb{N}$, such that for every $m \geq N_k$ we have

$$\lambda \in \Lambda \cap \underbrace{\left(\lambda - \frac{\delta_k}{2n+k}, \lambda + \frac{\delta_k}{2n+k}\right)}_{=\Omega} \subseteq \left(\Lambda_m \cap \underbrace{\left(\lambda - \frac{\delta_k}{2n+k}, \lambda + \frac{\delta_k}{2n+k}\right)}_{=\Omega}\right) + \underbrace{\left(-\frac{1}{n}, \frac{1}{n}\right)}_{=(-\varepsilon,\varepsilon)},$$

where $\delta_k \in (0, 1]$ ensures that the boundary of Ω does not contain any points from Λ, which is possible since Λ is countable.
Notice that the interval Ω has length $\frac{2\delta_k}{2n+k} < d$, so that $\Lambda_m \cap \Omega$ contains at most one point, and since the far right hand side is not empty (it contains λ for example) there is a $\lambda_{k,m} \in \Lambda_m \cap \Omega$. It should be clear that we can choose these N_k to be strictly increasing, i.e. $N_k < N_{k+1}$. We now select

$$\lambda_1 \in \Lambda_1 \quad , \quad \dots \quad , \quad \lambda_{N_1-1} \in \Lambda_{N_1-1}$$

arbitrary, then

$$\lambda_{N_1} = \lambda_{1,N_1} \quad , \quad \lambda_{N_1+1} = \lambda_{1,N_1+1} \quad , \quad \dots \quad , \quad \lambda_{N_2-1} = \lambda_{1,N_2-1},$$

then

$$\lambda_{N_2} = \lambda_{2,N_2} \quad , \quad \lambda_{N_2+1} = \lambda_{2,N_2+1} \quad , \quad \dots \quad , \quad \lambda_{N_3-1} = \lambda_{2,N_3-1},$$

and so on (all Λ_n are nonempty). The sequence $(\lambda_n)_n$ is now defined in a way according to the claim, where convergence is ensured by the ever shrinking "$(-\varepsilon, \varepsilon)$" term at the end.
On the other hand let now $(\lambda_n)_n$ be a sequence as in the second part of the claim and set $\lambda = \lim_{n \longrightarrow \infty} \lambda_n$. For an arbitrary neighborhood $\Omega = U_\varepsilon(\lambda)$, only finitely many of the sets $\Lambda_m \cap \Omega$ can be empty. Then have

$$\Lambda_m \cap \Omega \subseteq (\Lambda \cap \Omega) + (-\varepsilon, \varepsilon)$$

for all but finitely many m. Since we can choose ε arbitrarily small (with similar conclusions from the countability of Λ as were already used in the first part), the principle of nested intervals yields $\lambda \in \Lambda$.
(ii): By (i) we get a complete description of Λ, showing that the weak limit must be unique. To write it out for once:

$$\Lambda = \left\{ \lambda \in \mathbb{R} : \exists \lambda_n \in \Lambda_n, n \in \mathbb{N} \text{ with } \lim_{n \longrightarrow \infty} \lambda_n = \lambda \right\}.$$

(iii): Again by (i), choosing according sequences $\lambda_n \longrightarrow \lambda$ and $\lambda'_n \longrightarrow \lambda'$ for $\lambda \neq \lambda' \in \Lambda$, then by uniqueness of the limit for the convergence in $(\mathbb{R}, |\cdot|)$, the sequences $(\lambda_n)_n$ and $(\lambda'_n)_n$ can only be equal in at most finitely many places. Therefore, they are different from a point onward, say for all $n \geq N$. Since the Λ_n's all have a separation constant $\geq d$, we have $|\lambda_n - \lambda'_n| \geq d$ for all $n \geq N$. Taking limits yields $|\lambda - \lambda'| \geq d$, so $d(\Lambda) \geq d$ is discrete.
(iv): The basic idea is to use the already proven characterization of the weak limit from point (i) together with a diagonalization argument and the Bolzano-Weierstrass theorem. For this, consider for $l \in \mathbb{Z}$ the intervals $I_l := [ld, (l+1)d)$ (d as in the statement of the proposition). We see that $\mathbb{R} = \bigcup_{l \in \mathbb{Z}} I_l$ disjointly and since $d(\Lambda_n) \geq d$ for all $n \in \mathbb{N}$, we also have $\#(I_l \cap \Lambda_n) \leq 1$ for all $l \in \mathbb{Z}, n \in \mathbb{N}$. Furthermore set

$J_l^0 := \{n \in \mathbb{N} : I_l \cap \Lambda_n \neq \emptyset\}$ for $l \in \mathbb{Z}$, so J_l^0 is the set of indices n, such that Λ_n has an element from I_l for given l.
If we have $\#J_l^0 < \infty$ for all $l \in \mathbb{Z}$, then we easily see that $\Lambda_n \xrightarrow{w} \emptyset$, in which case we are done with $\Lambda = \emptyset$. So assume there is an $l_1 \in \mathbb{Z}$, so that $\#J_{l_1}^0 = \infty$. Then the sequence $(\lambda_j^1)_{j \in J_{l_1}^0}$, where λ_j^1 is the unique element in $I_{l_1} \cap \Lambda_j$ for $j \in J_{l_1}^0$ (notice that this intersection is nonempty by definition of $J_{l_1}^0$), consists of infinitely many values from $I_{l_1} \subseteq [l_1 d, (l_1+1)d]$, so by the Bolzano-Weierstrass theorem it has a convergent subsequence. Denote the indices of this convergent subsequence by $J^1 = \{j_1^1 < j_2^1 < \ldots\}$, so that $\lambda_{j_p^1}^1 \longrightarrow \lambda_1$ as $p \longrightarrow \infty$ for some $\lambda_1 \in \mathbb{R}$.
Set now $J_l^1 := J_l^1 \cap J^1$ and repeat the above process to get either that $\#J_l^1 < \infty$ for all $l \in \mathbb{Z} \setminus \{l_1\}$, in which case we have $\Lambda_{j_p^1} \xrightarrow{w} \{\lambda_1\}$ so the claim follows for $\Lambda = \{\lambda_1\}$, or the existence of an $l_2 \in \mathbb{Z} \setminus \{l_1\}$ with $\#J_{l_2}^1 = \infty$. In the latter case we again choose indices $J_{l_2}^1 \supseteq J^2 = \{j_1^2 < j_2^2 < \ldots\}$ with $I_{l_2} \cap \Lambda_{j_p^2} \ni \lambda_{j_p^2}^2 \longrightarrow \lambda_2$ as $p \longrightarrow \infty$.
If this process can only be repeated finitely many times, until it stops at an $m \in \mathbb{N}$, then $\Lambda_{j_p^m} \xrightarrow{w} \{\lambda_1, \ldots, \lambda_{m-1}\}$, so the weak limit is the desired Λ. If on the other hand the above process continues indefinitely, then by the way the subsequences and subsequences of subsequences and so on are constructed, a typical diagonalization argument shows that $\Lambda_{j_p^p} \xrightarrow{w} \{\lambda_1, \lambda_2, \ldots\} =: \Lambda$.
With (iii) we have $d(\Lambda) \geq d$ in either case, which finishes the prove. ■

Consider for another example the discrete sets $\Lambda_n = \{(-1)^n\}$. This sequence does not converge, as can be seen directly by the definition or easier through proposition 4.9 (i). However, one can easily identify $(\Lambda_{2n})_n$ or $(\Lambda_{2n-1})_n$ as convergent subsequences. Moreover, if we choose an arbitrary bijection $q : \mathbb{N} \to \mathbb{Q}$, the sequence of sets defined by $\Lambda_n = \{q(n)\}$ is again divergent, but for any $\lambda \in \mathbb{R}$ there is a corresponding subsequence converging to $\Lambda = \{\lambda\}$. Notice also that the fact, that the $d(\Lambda_n)$ are uniformly bounded away from 0 is important, as can be seen by the sequence of discrete sets $(\frac{1}{n}\mathbb{Z})_n$, for which weak convergence towards a countable Λ does not make sense in the above definition 4.8 and indeed is unsuitable for our purposes below, as will be seen later.
We now come to the central concept for this section.

Definition 4.10. For $S \subseteq \mathbb{R}$, the **Paley-Wiener space** PW_S is defined to be the space of all functions $f \in L^2(\mathbb{R})$, such that $\hat{f}$ vanishes almost everywhere outside of S, i.e. its essential support fulfills $\operatorname{supp} \hat{f} \subseteq S$.

Remark 4.11. [20]

(i) If $S \subseteq \mathbb{R}$ has finite measure $|S| < \infty$, then it is a well known consequence of Hölder's inequality that

$$\int_S |F(\xi)| d\xi \leq \|F\|_2 \cdot |S|^{1/2}$$

for all $F \in L^2(S)$, so $L^2(S) \subseteq L^1(S)$. Hence, if $f \in \mathsf{PW}_S$, then $\hat{f} \in L^2(\mathbb{R})$, but also $\operatorname{supp} \hat{f} \subseteq S$, so that we can identify $\hat{f}$ with $\hat{f}|_S$ (restricting onto S) and we therefore get $\hat{f} \in L^2(S) \subseteq L^1(S)$. By well known properties of the Fourier transform we get that then f is continuous, i.e. PW_S consists of continuous

functions (if $|S| < \infty$). In particular, point evaluation $f(x), x \in \mathbb{R}$ makes sense for $f \in \mathsf{PW}_S$.
To better understand the Paley-Wiener space, it might be useful to consider the following: Let $S \subseteq \mathbb{R}$ of finite measure $|S| < \infty$ and let $(f_n)_n$ be a sequence in PW_S (the sequence elements being equivalence classes, however we can identify them with there unique continuous representative, see above), such that $\hat{f}_n \longrightarrow \hat{f}, n \longrightarrow \infty$ in $L^1(S)$. Then

$$\begin{aligned} |f_n(x) - f(x)| &= \left|(2\pi)^{-1/2} \int_{\mathbb{R}} (\hat{f}_n(\xi) - \hat{f}(\xi)) e^{ix\xi} d\xi\right| \\ &\leq (2\pi)^{-1/2} \int_S |\hat{f}_n(\xi) - \hat{f}(\xi)| d\xi \\ &= (2\pi)^{-1/2} \|\hat{f}_n - \hat{f}\|_1 \longrightarrow 0, \end{aligned}$$

so that $f_n \longrightarrow f, n \longrightarrow \infty$ uniformly. Thus, for $S \subseteq \mathbb{R}$ of finite measure one easily sees (using Riemann-Lebesgue), that PW_S is (or more formally can be identified with) a closed subspace of $(C_0(\mathbb{R}), \|\cdot\|_\infty)$, where $C_0(\mathbb{R})$ is the set of all continuous functions $\mathbb{R} \to \mathbb{C}$ which decay towards 0 for $|x| \longrightarrow \infty$.

(ii) The following easy observation is crucial in some of the coming results: For $S \subseteq T \subseteq \mathbb{R}$ definition 4.10 immediately implies $\mathsf{PW}_S \subseteq \mathsf{PW}_T$. Also note that if S and T differ only in a null set, i.e $|S \Delta T| = 0$ for the symmetric difference, then $\mathsf{PW}_S = \mathsf{PW}_T$, as the corresponding null set cannot be "seen" by the integral of the inverse Fourier transform (the equality of spaces can also be seen by the definition directly).
We further have some transformation properties of the Paley-Wiener spaces: Translating S by $-t$ to get $-t + S$, the map $\tau_t : \mathsf{PW}_{-t+S} \to \mathsf{PW}_S, (\tau_t f)(x) = e^{itx} f(x)$ is well defined by

$$\begin{aligned} (\tau_t f)^\wedge(\xi) &= (2\pi)^{-1/2} \int_{\mathbb{R}} e^{itx} f(x) e^{-ix\xi} dx \\ &= (2\pi)^{-1/2} \int_{\mathbb{R}} f(x) e^{-ix(\xi - t)} dx = \hat{f}(\xi - t) \end{aligned}$$

for $f \in \mathsf{PW}_{-t+S}$ and easily seen to be an isometric isomorphism (for $|S| < \infty$). Similarly scaling S by $\alpha > 0$ to get αS, the map $\mu_\alpha : \mathsf{PW}_S \to \mathsf{PW}_{\alpha S}, (\mu_\alpha f)(x) = f(\alpha x)$ is also well defined by

$$\begin{aligned} (\mu_\alpha f)^\wedge(\xi) &= (2\pi)^{-1/2} \int_{\mathbb{R}} f(\alpha x) e^{-ix\xi} dx \\ &= \frac{(2\pi)^{-1/2}}{\alpha} \int_{\mathbb{R}} f(y) e^{-iy\frac{\xi}{\alpha}} dy = \frac{1}{\alpha} \hat{f}\left(\frac{\xi}{\alpha}\right) \end{aligned}$$

for $f \in \mathsf{PW}_S$ and further $\|\mu_\alpha f\|_2 = \alpha^{-1/2} \|f\|_2$ (for $|S| < \infty$).

We first establish some more general auxiliary results, before we come to the main theorem of this chapter. The proof of the following statement relies heavily on abstract Hilbert space theory and complex analysis, so we will only state the result here and refer to [28] for the proof.

Theorem 4.12. [20, 28] Given a discrete set $\Lambda \subseteq \mathbb{R}$, i.e. $d(\Lambda) > 0$, and a bounded set $S \subseteq \mathbb{R}$, there is a constant $K = K(d(\Lambda), \operatorname{diam}(S))$, where $\operatorname{diam}(S) = \sup\{|x-y| : x, y \in S\}$ is the diameter of S, such that

$$\sum_{\lambda \in \Lambda} |f(\lambda)|^2 \leq K \|f\|_2^2$$

for all $f \in \mathsf{PW}_S$.

This will be used for the following two limiting results.

Lemma 4.13. [20] Let $S \subseteq \mathbb{R}$ be bounded and of positive measure $|S| > 0$. Let further $(\Lambda_k)_k$ be a sequence of discrete sets in $\mathbb{R}$ with $d(\Lambda_k) \geq \delta > 0$ for all $k \in \mathbb{N}$ and $\Lambda_k \xrightarrow{w} \Lambda$ for some countable $\Lambda \subseteq \mathbb{R}$. Then

$$\lim_{k \longrightarrow \infty} \sum_{\lambda \in \Lambda_k} |f(\lambda)|^2 = \sum_{\lambda \in \Lambda} |f(\lambda)|^2$$

holds for all $f \in \mathsf{PW}_S$.

Note that the set Λ may be empty, so that the empty sum on the right is 0. In that case the lemma may be seen as a Riemann-Lebesgue type result.

Proof. First note that Λ is countable by definition 4.8, so that the sum over Λ makes sense in principle. Let $f \in \mathsf{PW}_S$, by remark 4.11 (i) we know that f is continuous, so that for every $l \in \mathbb{Z}$ it is possible to choose a $x_l \in [l\delta - \frac{\delta}{2}, l\delta + \frac{\delta}{2}]$ with

$$|f(x_l)| = \max\{|f(x)| : |x - l\delta| \leq \delta/2\}.$$

By definition it is clear that $x_{l+2} - x_l \geq \delta$, so that the sets $\{x_l : l \in \mathbb{Z} \text{ even}\}$ and $\{x_l : l \in \mathbb{Z} \text{ odd}\}$ are discrete sets with separation constants $\geq \delta$ each. Applying theorem 4.12 to both sets separately yields

$$\sum_{l \in \mathbb{Z}} |f(x_l)|^2 \leq 2K \|f\|_2^2 < \infty. \tag{4.3}$$

For $R > 0$ with $-R, R \notin \Lambda$ but otherwise arbitrary, we now have by splitting up the sums

$$\begin{aligned}
\left| \sum_{\lambda \in \Lambda_k} |f(\lambda)|^2 - \sum_{\lambda \in \Lambda} |f(\lambda)|^2 \right| &\leq \left| \sum_{\substack{\lambda \in \Lambda_k \\ |\lambda| < R}} |f(\lambda)|^2 - \sum_{\substack{\lambda \in \Lambda \\ |\lambda| < R}} |f(\lambda)|^2 \right| + \left| \sum_{\substack{\lambda \in \Lambda_k \\ |\lambda| \geq R}} |f(\lambda)|^2 - \sum_{\substack{\lambda \in \Lambda \\ |\lambda| \geq R}} |f(\lambda)|^2 \right| \\
&\leq \left| \sum_{\substack{\lambda \in \Lambda_k \\ |\lambda| < R}} |f(\lambda)|^2 - \sum_{\substack{\lambda \in \Lambda \\ |\lambda| < R}} |f(\lambda)|^2 \right| + \sum_{\substack{\lambda \in \Lambda_k \\ |\lambda| \geq R}} |f(\lambda)|^2 + \sum_{\substack{\lambda \in \Lambda \\ |\lambda| \geq R}} |f(\lambda)|^2.
\end{aligned}$$

Since the Λ_k have separation constant $\geq \delta$ and so does Λ by proposition 4.9 (iii), the way the x_l's are defined yields

$$\left|\sum_{\lambda\in\Lambda_k}|f(\lambda)|^2-\sum_{\lambda\in\Lambda}|f(\lambda)|^2\right|\leq\left|\sum_{\substack{\lambda\in\Lambda_k\\|\lambda|<R}}|f(\lambda)|^2-\sum_{\substack{\lambda\in\Lambda\\|\lambda|<R}}|f(\lambda)|^2\right|+2\sum_{|l|\geq R/\delta}|f(x_l)|^2.$$

By continuity of f, the discreteness of the Λ_k as well as of Λ and proposition 4.9 (i) and (ii), the first term of the right hand side goes to zero as $k\longrightarrow\infty$. By (4.3), so does the second term for $R\longrightarrow\infty$ on the right hand side. This establishes the claim. ■

Lemma 4.14. [20] Let $S_1\subseteq S_2\subseteq\ldots$ be an infinite, increasing sequence of bounded subsets of $\mathbb{R}$, such that $S:=\bigcup_{k\in\mathbb{N}}S_k$ has finite measure. Let $\Lambda\subseteq\mathbb{R}$ be a discrete set and $k,K>0$ constants, such that for all $j\in\mathbb{N}$ there holds

$$k\|f_j\|_2^2\leq\sum_{\lambda\in\Lambda}|f_j(\lambda)|^2\leq K\|f_j\|_2^2$$

for all $f_j\in\mathsf{PW}_{S_j}$. Then we have

$$k\|f\|_2^2\leq\sum_{\lambda\in\Lambda}|f(\lambda)|^2\leq K\|f\|_2^2$$

for all $f\in\mathsf{PW}_S$.

Proof. Given an $f\in\mathsf{PW}_S$, let $f_j\in\mathsf{PW}_{S_j}$ be the inverse Fourier transform of $\hat{f}\cdot\mathbf{1}_{S_j}$. By dominated convergence then have $\|\hat{f}-\hat{f}_j\|_1\longrightarrow 0, j\longrightarrow\infty$, so as in remark 4.11 (i) we get that $f_j\longrightarrow f$ uniformly for $j\longrightarrow\infty$. Also notice that

$$\|f-f_j\|_2^2=\|\hat{f}-\hat{f}_j\|_2^2\longrightarrow 0, j\longrightarrow\infty,$$

again by dominated convergence.
For given $R>0$ we have

$$\sum_{\substack{\lambda\in\Lambda\\|\lambda|<R}}|f_j(\lambda)|^2\leq K\|f_j\|_2^2.$$

First taking $j\longrightarrow\infty$ gives (together with the above convergence properties for the f_j)

$$\sum_{\substack{\lambda\in\Lambda\\|\lambda|<R}}|f(\lambda)|^2\leq K\|f\|_2^2,$$

letting then $R\longrightarrow\infty$ yields

$$\sum_{\lambda\in\Lambda}|f(\lambda)|^2\leq K\|f\|_2^2.$$

To get the other side of the claimed inequality, we will use the already proven inequality (and the triangle inequality) to get

$$\begin{aligned}\left(\sum_{\lambda\in\Lambda}|f(\lambda)|^2\right)^{1/2}&\geq\left(\sum_{\lambda\in\Lambda}|f_j(\lambda)|^2\right)^{1/2}-\left(\sum_{\lambda\in\Lambda}|(f-f_j)(\lambda)|^2\right)^{1/2}\\&\geq k^{1/2}\|f_j\|_2-K^{1/2}\|f-f_j\|_2\end{aligned}$$

($\mathsf{PW}_S \supseteq \mathsf{PW}_{S_j}$, see remark 4.11, are vector spaces, so $f - f_j \in \mathsf{PW}_S$). Again letting $j \longrightarrow \infty$ gives the assertion

$$k\|f\|_2^2 \leq \sum_{\lambda\in\Lambda} |f(\lambda)|^2.$$

■

Remark 4.15. A similar assertion to that of the above lemma holds more general for Λ, which are the finite union of discrete sets. Writing $\Lambda = \bigcup_{l=1}^r \Lambda_l$ in such a way, we have

$$\sum_{\lambda\in\Lambda_l} |f_j(\lambda)|^2 \leq \sum_{\lambda\in\Lambda} |f_j(\lambda)|^2 \leq K\|f_j\|_2^2,$$

for all $f_j \in \mathsf{PW}_{S_j}$ and $l = 1, ..., r$. Then lemma 4.14 yields

$$\sum_{\lambda\in\Lambda_l} |f(\lambda)|^2 \leq K\|f\|_2^2$$

for all $f \in \mathsf{PW}_S$ and $l = 1, ..., r$. Thus

$$\sum_{\lambda\in\Lambda} |f(\lambda)|^2 \leq \sum_{l=1}^{r} \sum_{\lambda\in\Lambda_l} |f(\lambda)|^2 \leq rK\|f\|_2^2$$

for all $f \in \mathsf{PW}_S$. Similar to the proof above, then get

$$\begin{aligned}\left(\sum_{\lambda\in\Lambda} |f(\lambda)|^2\right)^{1/2} &\geq \left(\sum_{\lambda\in\Lambda} |f_j(\lambda)|^2\right)^{1/2} - \left(\sum_{\lambda\in\Lambda} |(f - f_j)(\lambda)|^2\right)^{1/2} \\ &\geq k^{1/2}\|f_j\|_2 - (rK)^{1/2}\|f - f_j\|_2,\end{aligned}$$

thus we conclude as above that

$$k\|f\|_2^2 \leq \sum_{\lambda\in\Lambda} |f(\lambda)|^2 \leq rK\|f\|_2^2$$

for all $f \in \mathsf{PW}_S$. Therefore, we only have to pay an additional factor in the upper bound.

The following lemma helps in controlling the discrete set Λ of sample points when taking limiting processes.

Lemma 4.16. [20] Assume that there is a constant $C > 0$, an $S \subseteq \mathbb{R}$ of finite measure $|S| < \infty$ and a countable $\Lambda \subseteq \mathbb{R}$, such that

$$\sum_{\lambda\in\Lambda} |f(\lambda)|^2 \leq C|S| \cdot \|f\|_2^2$$

holds for all $f \in \mathsf{PW}_S$. Then there is a constant $\eta = \eta(S) > 0$, so that

$$\#(\Lambda \cap \Omega) \leq 9C$$

for every interval $\Omega \subseteq \mathbb{R}$ with $|\Omega| = \eta$.

Proof. Let $h \in \mathsf{PW}_S$ be the inverse Fourier transform of the indicator function $\mathbf{1}_S$ ($\mathbf{1}_S$ is L^1 by finiteness of $|S|$). Thus h is continuous and we further have

$$h(0) = (2\pi)^{-1/2} \int_{\mathbb{R}} \mathbf{1}_S(\xi) e^{i\xi\cdot 0} d\xi = (2\pi)^{-1/2}|S| > |S|/3$$

and

$$\|h\|_2^2 = \|\mathbf{1}_S\|_2^2 = |S|.$$

By continuity it is now possible to choose an $\eta > 0$ small enough, so that $|h(x)| \geq |S|/3$ whenever $|x| \leq \eta/2$ (notice that η only depends on S). By the assumption from the lemma applied to $f = h$ we get for $\Omega = [-\eta/2, \eta/2]$ then

$$\#(\Lambda \cap \Omega) \cdot \left(\frac{|S|}{3}\right)^2 \leq \sum_{\lambda \in \Lambda \cap \Omega} |h(\lambda)|^2 \leq \sum_{\lambda \in \Lambda} |h(\lambda)|^2 \leq C|S| \cdot \|h\|_2^2 = C|S|^2,$$

so that $\#(\Lambda \cap \Omega) \leq 9C$ for this particular Ω. Since translating h yields a modulation of $\hat{h}$, we have that $h(\cdot - x_0) \in \mathsf{PW}_S$ for all $x_0 \in \mathbb{R}$. This corresponds to translating the set Ω, so that the assertion holds for general intervals of length η.

■

Obviously, if the assertion of the above lemma holds for all intervals of length η, then it holds for all intervals of length $\leq \eta$.

Remark 4.17. We will go into some more detail on the dependence between S and the constant η. To reiterate, for $S \subseteq \mathbb{R}, 0 < |S| < \infty$ the constant η is chosen in such a way that

$$|h(x)| = (2\pi)^{-1/2} \left| \int_S e^{ix\xi} d\xi \right| \geq \frac{1}{3}|S|$$

for all $|x| \leq \eta/2$. We formally define

$$\eta(S) := \max\left\{\eta > 0 : |x| \leq \frac{1}{2}\eta \Rightarrow |h(x)| \geq \frac{1}{3}|S|\right\},$$

where the max is justified (instead of sup) by continuity of h since $|S| < \infty$ (see remark 4.11 (i)). Notice first, that there is no universal lower bound on the $\eta(S)$, not even for $|S| \leq M$ uniformly bounded by some $M > 0$, since for $n \in \mathbb{N}$ we can consider the sets $S_n = [-n-1/2, -n] \cup [n, n+1/2]$, all of measure $|S_n| = 1$. A straightforward calculation leads to

$$h(x) = (2\pi)^{-1/2} \cdot 2 \int_n^{n+1/2} \cos(x\xi) d\xi = \sqrt{\frac{2}{\pi}} \cdot \frac{\sin\big((n+1/2)x\big) - \sin(nx)}{x},$$

for which one can show, using the Taylor expansion of the sinc-function for example, that $\eta(S_n) = O(n^{-1/2})$, so in particular $\eta(S_n) \longrightarrow 0$ as $n \longrightarrow \infty$. Since this is only for demonstrative purposes, we will not go into further details.
Let $S \subseteq \mathbb{R}$ with $0 < |S| < \infty$. The above example seems to suggest that a general bound may not be straightforward to find, so we only consider the special case that additionally

$$\int_S |\xi| d\xi < \infty$$

holds (in particular, this holds for all bounded S). Using the mean value theorem on $\operatorname{Re} h$ and $\operatorname{Im} h$ (which are continuously differentiable by the additional assumption) for all real $x < y$ there are $\sigma, \tau \in (x, y)$ with

$$\begin{aligned}
\left|\frac{h(y)-h(x)}{y-x}\right| &= \left|\frac{\operatorname{Re} h(y)-\operatorname{Re} h(x)}{y-x} + i\cdot\frac{\operatorname{Im} h(y)-\operatorname{Im} h(x)}{y-x}\right| \\
&= |\operatorname{Re} h'(\sigma) + i\cdot \operatorname{Im} h'(\tau)| \\
&= (2\pi)^{-1/2}\left|\int_S \xi\cos(\sigma\xi)d\xi + i\int_S \xi\sin(\tau\xi)d\xi\right| \\
&\leq (2\pi)^{-1/2}\cdot 2\int_S |\xi| d\xi,
\end{aligned}$$

so that h is Lipschitz-continuous with Lipschitz-constant $L = \sqrt{2/\pi}\int_S |\xi|d\xi$. Therefore, using $h(0) = (2\pi)^{-1/2}|S|$, we get the chain of implications

$$\begin{aligned}
|x| \leq \frac{\left((2\pi)^{-1/2}-\frac{1}{3}\right)|S|}{\sqrt{2/\pi}\int_S|\xi|d\xi} &\Rightarrow |h(x)-h(0)| \leq \sqrt{2/\pi}\int_S |\xi|d\xi\cdot|x| \leq \left((2\pi)^{-1/2}-\frac{1}{3}\right)|S| \\
&\Rightarrow |h(x)| \geq |h(0)| - |h(x)-h(0)| \geq \frac{1}{3}|S|
\end{aligned}$$

and conclude from that

$$\eta(S) \geq \frac{1}{2}\cdot\frac{\left((2\pi)^{-1/2}-\frac{1}{3}\right)|S|}{\sqrt{2/\pi}\int_S|\xi|d\xi} = \underbrace{\left(\frac{1}{4}-\frac{1}{12}\sqrt{2\pi}\right)}_{>0.04}\cdot\frac{|S|}{\int_S|\xi|d\xi} > 0.$$

4.3 Exponential Frames on General Sets of Finite Measure

The following few results will culminate in the main result of this chapter.

Lemma 4.18. [20] Let $n < m \in \mathbb{N}$ and let $S \subseteq \mathbb{R}$ of the type

$$S = \bigcup_{r\in I}\left[\frac{2\pi r}{m}, \frac{2\pi(r+1)}{m}\right],$$

where $I \subseteq \{0, ..., m-1\}, \#I = n$. Then there is a $\Lambda \subseteq \mathbb{Z}$ with

$$c_0'|S|\cdot\|f\|_2^2 \leq \sum_{\lambda\in\Lambda}|f(\lambda)|^2 \leq C_0'|S|\cdot\|f\|_2^2$$

for all $f \in \mathsf{PW}_S$, where $c_0', C_0' > 0$ are universal constants.

Proof. Writing $I = \{r_1, ..., r_n\}$, let

$$\begin{aligned}\mathcal{F}(I) &= \left[\exp\left(2\pi i \frac{kr}{m}\right)\right]_{k=0,...,m-1; r\in I} \\ &= \begin{bmatrix} \exp\left(2\pi i \cdot \frac{0\cdot r_1}{m}\right) & \cdots & \exp\left(2\pi i \cdot \frac{0\cdot r_n}{m}\right) \\ \vdots & & \vdots \\ \exp\left(2\pi i \cdot \frac{(m-1)\cdot r_1}{m}\right) & \cdots & \exp\left(2\pi i \cdot \frac{(m-1)\cdot r_n}{m}\right) \end{bmatrix}\end{aligned}$$

be the (nonnormalized) Fourier matrix with columns indexed by I. Notice that $m^{-1/2}\mathcal{F}(I)$ fulfills the requirements of corollary 4.7 (with common squared euclidean row norm $\kappa = n/m$), so that there is a row selection $J \subseteq \{0, ..., m-1\}$ with

$$c_0 \cdot \frac{n}{m}\|w\|^2 \leq \|m^{-1/2}\mathcal{F}(I)_J \cdot w\|^2 \leq C_0 \cdot \frac{n}{m}\|w\|^2$$

for all $w \in \mathbb{C}^I$, where the norms are the euclidean norms in the spaces $\mathbb{C}^I$ and $\mathbb{C}^J$ respectively. The constants c_0 and C_0 are universal constants and are taken from corollary 4.7. We therefore have

$$c_0 n\|w\|^2 \leq \|\mathcal{F}(I)_J \cdot w\|^2 \leq C_0 n\|w\|^2 \tag{4.4}$$

for all $w \in \mathbb{C}^I$.

Observe that every "function" $F \in L^2(S)$ can be written as

$$F(t) = \sum_{r\in I} F_r\left(\xi - \frac{2\pi r}{m}\right),$$

where the $F_r \in L^2[0, 2\pi/m] \subseteq L^2(\mathbb{R})$ (extend by 0) are defined by

$$F_r(\xi) := F\left(\xi + \frac{2\pi r}{m}\right)\mathbf{1}_{[0,2\pi/m]}(\xi),$$

so we simply split up F according to the intervals defining S. Taking inverse Fourier transforms we therefore see that every $f \in \mathsf{PW}_S$ can be represented as

$$f(x) = \sum_{r\in I} e^{2\pi i\cdot\frac{r}{m}x} f_r(x)$$

for some $f_r \in \mathsf{PW}_{[0,2\pi/m]}, r \in I$ (the inverse Fourier transforms of the F_r). Using the unitarity of the Fourier transform (and its inverse) and the structure of the F_r above we see that the $e^{2\pi i\cdot\frac{r}{m}x} f_r(x)$ are pairwise orthogonal in $L^2(\mathbb{R})$. For general $h \in \mathsf{PW}_{[0,2\pi/m]}$ we also have the fact that

$$\frac{1}{m}\|h\|_2^2 = \sum_{\lambda\in m\mathbb{Z}} |h(\lambda)|^2 \tag{4.5}$$

at hand. Indeed, since $\{e_k^m(\cdot) = (m/2\pi)^{1/2}\exp(imk\cdot) : k \in \mathbb{Z}\}$ is an orthonormal basis in $L^2[0, 2\pi/m]$, we have

$$\begin{aligned}\|h\|_2^2 = \|\hat{h}\|_2^2 = \sum_{k\in\mathbb{Z}} |\langle\hat{h}, e_k^m\rangle|^2 &= \sum_{k\in\mathbb{Z}} \left|\sqrt{\frac{m}{2\pi}}\int_0^{2\pi/m} \hat{h}(\xi)e^{-imk\xi}d\xi\right|^2 \\ &= \sum_{k\in\mathbb{Z}} m\left|(2\pi)^{-1/2}\int_{\mathbb{R}} \hat{h}(\xi)e^{i\cdot(-mk)\xi}d\xi\right|^2 \\ &= m\sum_{k\in\mathbb{Z}} |h(-mk)|^2 = m\sum_{\lambda\in m\mathbb{Z}} |h(\lambda)|^2.\end{aligned}$$

We now claim that for $\Lambda := \{j + km : j \in J, k \in \mathbb{Z}\}$ we get the desired estimate. To verify this, take an arbitrary $f \in \mathsf{PW}_S$ (which also gives f_r as described above). We then have

$$\sum_{\lambda\in\Lambda} |f(\lambda)|^2 = \sum_{j\in J}\sum_{k\in\mathbb{Z}} |f(j+km)|^2 = \sum_{j\in J}\sum_{k\in\mathbb{Z}} \left|\sum_{r\in I} e^{2\pi i\cdot\frac{jr}{m}} f_r(j+km)\right|^2,$$

so applying (4.5) for every $j \in J$ to $h_j(x) = \sum_{r\in I} e^{2\pi i\cdot\frac{jr}{m}} f_r(j+x)$ we further get

$$\begin{aligned}\sum_{\lambda\in\Lambda} |f(\lambda)|^2 &= \frac{1}{m}\sum_{j\in J}\int_{\mathbb{R}} \left|\sum_{r\in I} e^{2\pi i\cdot\frac{jr}{m}} f_r(j+x)\right|^2 dx\\ &= \frac{1}{m}\int_{\mathbb{R}}\sum_{j\in J} \left|\sum_{r\in I} e^{2\pi i\cdot\frac{jr}{m}} f_r(x)\right|^2 dx\\ &= \frac{1}{m}\int_{\mathbb{R}} \|\mathcal{F}(I)_J[f_r(x)]_{r\in I}\|^2 dx,\end{aligned}$$

where $[f_r(x)]_{r\in I}$ is the vector consisting of the $f_r, r \in I$ all evaluated at x. By (4.4) we get

$$c_0 \cdot \frac{n}{m}\int_{\mathbb{R}}\sum_{r\in I}|f_r(x)|^2 dx \le \sum_{\lambda\in\Lambda}|f(\lambda)|^2 \le C_0\cdot\frac{n}{m}\int_{\mathbb{R}}\sum_{r\in I}|f_r(x)|^2 dx.$$

The claim now follows from the calculation (using the orthogonality described above)

$$\int_{\mathbb{R}}\sum_{r\in I}|f_r(x)|^2 dx = \int_{\mathbb{R}}\sum_{r\in I}|e^{2\pi i\cdot\frac{r}{m}x} f_r(x)|^2 dx = \int_{\mathbb{R}}\left|\sum_{r\in I} e^{2\pi i\cdot\frac{r}{m}x} f_r(x)\right|^2 dx = \int_{\mathbb{R}} |f(x)|^2 dx$$

and the fact that $|S| = \frac{2\pi n}{m}$, so that $c_0' = c_0/2\pi$ and $C_0' = C_0/2\pi$. ■

We now refine this result.

Corollary 4.19. [20] Let $S \subseteq [0, 2\pi]$ be a set of positive measure. Then there is a $\Lambda \subseteq \mathbb{Z}$ with

$$c_2'|S|\cdot\|f\|_2^2 \le \sum_{\lambda\in\Lambda}|f(\lambda)|^2 \le C_2'|S|\cdot\|f\|_2^2$$

for all $f \in \mathsf{PW}_S$. (c_2', C_2' positive universal constants)

Proof. We start with compact $K \subseteq [0, 2\pi]$. Let $|K| > \varepsilon > 0$, then K can be covered by a set $K_\varepsilon \subseteq [0, 2\pi]$, so that K_ε is of the form described in lemma 4.18 with $K \subseteq K_\varepsilon$ and $|K_\varepsilon \setminus K| < \varepsilon$ (in particular $|K| \le |K_\varepsilon| \le 2|K|$). It is easily seen that this can indeed be done, since compact sets in $\mathbb{R}$ are the finite union of closed and bounded intervals. By the denseness of the rational numbers in $\mathbb{R}$ (to be more specific here of $2\pi\mathbb{Q} \subseteq \mathbb{R}$) all of those closed and bounded intervals can be covered arbitrarily well by sets of the above form, which can then be extended to a cover of the original compact set of the desired

type.
By lemma 4.18, this gives a $\Lambda_\varepsilon \subseteq \mathbb{Z}$ with

$$c_0'|K_\varepsilon| \cdot \|f\|_2^2 \le \sum_{\lambda \in \Lambda_\varepsilon} |f(\lambda)|^2 \le C_0'|K_\varepsilon| \cdot \|f\|_2^2$$

for all $f \in \mathsf{PW}_{K_\varepsilon}$. Using remark 4.11 (ii) and $K \subseteq K_\varepsilon$, we have $\mathsf{PW}_K \subseteq \mathsf{PW}_{K_\varepsilon}$, so that the aforementioned bound holds in particular for all $f \in \mathsf{PW}_K$. This gives

$$c_1'|K| \cdot \|f\|_2^2 \le c_0'|K_\varepsilon| \cdot \|f\|_2^2 \le \sum_{\lambda \in \Lambda_\varepsilon} |f(\lambda)|^2 \le C_0'|K_\varepsilon| \cdot \|f\|_2^2 \le C_1'|K| \cdot \|f\|_2^2,$$

for all $f \in \mathsf{PW}_K$, which yields the statement for the compact case for $c_1' = c_0'$ and $C_1' = 2C_0'$ with $\Lambda = \Lambda_\varepsilon$.
More generally, for open $U \subseteq [0, 2\pi]$ (with respect to the subspace topology), U consists of a disjoint union of at most countably many open intervals. Thus, if we go over to the closure $\overline{U} \subseteq [0, 2\pi]$, we add at most countably many points to U, so that $\overline{U} \setminus U$ is a null set and by that, using remark 4.11 (ii), have $\mathsf{PW}_{\overline{U}} = \mathsf{PW}_U$. But $\overline{U}$ is compact, by the first part of the proof we therefore get

$$c_1'|U| \cdot \|f\|_2^2 \le \sum_{\lambda \in \Lambda} |f(\lambda)|^2 \le C_1'|U| \cdot \|f\|_2^2$$

for all $f \in \mathsf{PW}_U$, where $c_1' C_1'$ and Λ are just as above.
Using now the outer regularity of the Lebesgue measure, for measurable $S \subseteq [0, 2\pi]$ with $|S| > 0$ there is an open $U \subseteq [0, 2\pi]$ with $S \subseteq U$ and $|U \setminus S| < |S|$, in particular $|U| \le |S| + |U \setminus S| \le 2|S|$. By the open case, we have

$$c_1'|U| \cdot \|f\|_2^2 \le \sum_{\lambda \in \Lambda} |f(\lambda)|^2 \le C_1'|U| \cdot \|f\|_2^2$$

for all $f \in \mathsf{PW}_U$, but again by remark 4.11 (ii) it holds $\mathsf{PW}_S \subseteq \mathsf{PW}_U$, so that as in the first part of the proof we get

$$c_2'|S| \cdot \|f\|_2^2 \le c_1'|U| \cdot \|f\|_2^2 \le \sum_{\lambda \in \Lambda} |f(\lambda)|^2 \le C_1'|U| \cdot \|f\|_2^2 \le C_2'|S| \cdot \|f\|_2^2$$

for all $f \in \mathsf{PW}_S$, where Λ gets taken over from the open case and $c_2' = c_1'$ and $C_2' = 2C_1'$, similar to above. This concludes the proof. ■

We note that for the above purposes, it is easier to get estimates for the Paley-Wiener space by approximating the defining sets from the outside. By the properties of the Lebesgue measure, this can be done to an arbitrarily small error, which allows us to get the desired bounds. In what comes below however we are required to make an approximation from the inside, which is why we introduced the machinery developed for the Paley-Wiener space in the previous section.
It is also noteworthy that the additional factor 2 (which was used twice in the above proof) can be replaced by $1 + \varepsilon$ for an arbitrary $\varepsilon > 0$. However, if we do not want to introduce an additional factor, i.e. $C_0' = C_1' = C_2'$, we certainly have to use more sophisticated techniques. For simplicity, we will be satisfied with the above bounds for now.
We can easily extend the result even further.

Corollary 4.20. [20] For every bounded $S \subseteq \mathbb{R}$ of positive measure, which is contained in an interval of length $2\pi d$, there is a a set $\Lambda \subseteq \frac{1}{d}\mathbb{Z}$ with

$$c_2'|S| \cdot \|f\|_2^2 \leq \sum_{\lambda \in \Lambda} |f(\lambda)|^2 \leq C_2'|S| \cdot \|f\|_2^2 \tag{4.6}$$

for all $f \in \mathsf{PW}_S$. (c_2', C_2' positive universal constants)

Proof. First note that applying the translation operator τ_t form remark 4.11 (ii) does not change the quantities in the above bound, so we may assume $S \subseteq [0, 2\pi d]$. Using the bijective scaling operator $\mu_{1/d}$ from remark 4.11 (ii) for $f \in \mathsf{PW}_S$, apply corollary 4.19 to $\mu_{1/d} f \in \mathsf{PW}_{\frac{1}{d}S}$ where $\frac{1}{d}S \subseteq [0, 2\pi]$ to get

$$c_2' \left| \frac{1}{d} S \right| \cdot \|\mu_{1/d} f\|_2^2 \leq \sum_{\lambda \in \Lambda'} |(\mu_{1/d} f)(\lambda)|^2 \leq C_2' \left| \frac{1}{d} S \right| \cdot \|\mu_{1/d} f\|_2^2.$$

for some $\Lambda' \subseteq \mathbb{Z}$. Since $(\mu_{1/d} f)(\lambda) = f(\lambda/d)$ and $\|\mu_{1/d} f\|_2^2 = d\|f\|_2^2$, we get (4.6) for $\Lambda = \frac{1}{d}\Lambda' \subseteq \frac{1}{d}\mathbb{Z}$. ■

We answer now the question from the start of this chapter.

Theorem 4.21. [20] There are universal constants $c, C > 0$ such that the following holds: For every set $S \subseteq \mathbb{R}$ of finite measure there is a quasidiscrete set $\Lambda \subseteq \mathbb{R}$, so that $E(\Lambda) = \{\exp(i\lambda \cdot)\}_{\lambda \in \Lambda}$ is a frame in $L^2(S)$ with frame bounds $c|S|$ and $C|S|$, i.e.

$$c|S| \cdot \|h\|_2^2 \leq \sum_{u \in E(\Lambda)} |\langle h, u \rangle|^2 \leq C|S| \cdot \|h\|_2^2$$

for all $h \in L^2(S)$.

Proof. The claimed bound somewhat more explicitly written out reads

$$c|S| \cdot \|h\|_2^2 \leq \sum_{u \in E(\Lambda)} \Bigg| \underbrace{\int_S h(t) e^{-i\lambda t} dt}_{=\sqrt{2\pi}\hat{h}(\lambda)} \Bigg|^2 \leq C|S| \cdot \|h\|_2^2,$$

so going over to the Fourier transform we need to show the existence of universal constants $c', C' > 0$ ($c' = c/2\pi$, $C' = C/2\pi$) with

$$c'|S| \cdot \|f\|_2^2 \leq \sum_{\lambda \in \Lambda} |f(\lambda)|^2 \leq C'|S| \cdot \|f\|_2^2$$

for all $f \in \mathsf{PW}_{-S}$ (the minus coming from the fact that h is the inverse Fourier transform of f but in what follows we will replace $-S$ by S which does not change the validity of the statement by a transformation argument just like that in remark 4.11 (ii)). The case for bounded $S \subseteq \mathbb{R}$ is clear by corollary 4.20, so consider unbounded $S \subseteq \mathbb{R}$ of finite measure.
Take a sequence $S_1 \subseteq S_2 \subseteq ...$ of bounded sets S_j with $S = \bigcup_j S_j$ (e.g. $S_j = [-j, j] \cap S$),

on which we may impose further properties to or liking, if they do not conflict with generality. Using corollary 4.20 we get discrete sets Λ_j with (4.6), i.e

$$c_2'|S_j| \cdot \|f_j\|_2^2 \le \sum_{\lambda \in \Lambda_j} |f_j(\lambda)|^2 \le C_2'|S_j| \cdot \|f_j\|_2^2$$

for all $f_j \in \mathsf{PW}_{S_j}$ and all $j \in \mathbb{N}$. Since $\mathsf{PW}_{S_j} \subseteq \mathsf{PW}_{S_k}$ for $j \le k$ (with remark 4.11 (ii)) we even have

$$c_2'|S_k| \cdot \|f_j\|_2^2 \le \sum_{\lambda \in \Lambda_k} |f_j(\lambda)|^2 \le C_2'|S_k| \cdot \|f_j\|_2^2 \tag{4.7}$$

for all $f_j \in \mathsf{PW}_{S_j}$ and $j \le k$.
Denote by

$$h(x) := (2\pi)^{-1/2} \int_S e^{ix\xi} d\xi, \quad h_j(x) := (2\pi)^{-1/2} \int_{S_j} e^{ix\xi} d\xi,$$

so that $\hat{h} = \mathbf{1}_S$ and $\hat{h}_j = \mathbf{1}_{S_j}$. By remark 4.11 (i) we know that h and the h_j are continuous. Therefore, there are $\eta(S)$ and $\eta(S_j)$ as in remark 4.17. Our first aim will be to show that the $\eta(S_j)$ are uniformly bounded from below by some $\rho > 0$. Since $S_j \nearrow S$ with S of finite measure, dominated convergence gives that $\mathbf{1}_{S_j} \longrightarrow \mathbf{1}_S$ in L^1. Again by remark 4.11 (i), we conclude that $h_j \longrightarrow h$ uniformly, so in particular $|h_j| \longrightarrow |h|$ uniformly (by the reverse triangle inequality). Thus, we can choose an index $j_0 \in \mathbb{N}$, such that

$$|h_j(x)| \ge \frac{1}{3}|S|$$

for all $j \ge j_0$ and $|x| \le \frac{1}{4}\eta(S)$. By disregarding the first few sequence elements of $(h_j)_j$, we may assume without loss of generality that $j_0 = 1$. We therefore have that $\eta(S_j) \ge \frac{1}{2}\eta(S) =: \rho$ is uniformly bounded from below.
Applying lemma 4.16 (together with the small remark under its proof) to the S_j, for which $\sum_{\lambda \in \Lambda_j} |f_j(\lambda)|^2 \le C_2'|S_j| \cdot \|f_j\|_2^2$ holds for all $f_j \in \mathsf{PW}_{S_j}$, we have that

$$\#(\Lambda_j \cap \Omega) \le 9C_2'$$

for all intervals Ω of length $|\Omega| \le \rho$. We can therefore partition the sets Λ_j into subsets $\Lambda_j^{(l)}, l = 1, ..., r$, by considering the intersections $\Lambda_j \cap [n\rho, (n+1)\rho]$ for $n \in \mathbb{Z}$, such that $d(\Lambda_j^{(l)}) \ge \rho$ (see definition 4.8), where $r \le 18C_2'$ is universal for all $j \in \mathbb{N}$ (the additional factor of 2 comes from the same sort of even-odd-argument as was already used in the proof of lemma 4.13). By taking appropriate subsequences (and subsequences of subsequences etc.) we can assume by proposition 4.9 (iv) that $\Lambda_j^{(l)} \xrightarrow{w} \Lambda^{(l)}, j \longrightarrow \infty$ (which a priori are not known to be nonempty) with $d(\Lambda^{(l)}) \ge \rho, l = 1, ..., r$. Then the set $\Lambda := \bigcup_{l=1}^r \Lambda^{(l)}$ is quasidiscrete, or more specifically the union of finitely many discrete sets.
Writing $\sum_{\lambda \in \Lambda_k} |f_j(\lambda)|^2 = \sum_{l=1}^r \sum_{\lambda \in \Lambda_k^{(l)}} |f_j(\lambda)|^2$, and taking limits $k \longrightarrow \infty$ as in lemma 4.13, (4.7) becomes

$$c_2'|S| \cdot \|f_j\|_2^2 \le \sum_{l=1}^r \sum_{\lambda \in \Lambda^{(l)}} |f_j(\lambda)|^2 \le C_2'|S| \cdot \|f_j\|_2^2$$

for all $f_j \in \mathsf{PW}_{S_j}$. In particular, the lower bound implies that at least one of the $\Lambda^{(l)}$ (and by extension also Λ) is nonempty. We also have

$$\sum_{\lambda\in\Lambda} |f_j(\lambda)|^2 \leq \sum_{l=1}^{r} \sum_{\lambda\in\Lambda^{(l)}} |f_j(\lambda)|^2 \leq r \sum_{\lambda\in\Lambda} |f_j(\lambda)|^2$$

(in general, the first "$\leq$" might not be replaceable by an "$=$", since the $\Lambda^{(l)}$ are not guaranteed to be pairwise disjoint), so that we have

$$\frac{c_2'}{r}|S| \cdot \|f_j\|_2^2 \leq \sum_{\lambda\in\Lambda} |f_j(\lambda)|^2 \leq C_2'|S| \cdot \|f_j\|_2^2$$

for all $f_j \in \mathsf{PW}_{S_j}$. A final application of lemma 4.14 together with remark 4.15 for $j \longrightarrow \infty$ yields the assertion for unbounded S with $c' = \frac{c_2'}{r}$ and $C' = rC_2'$. ■

Notice that, after going through all the arguments again, we can even get numerical values for c and C. However, at this point $c \geq 6 \cdot 10^{-4}$ may be very small while $C \leq 6 \cdot 10^{8}$ may be very large (the biggest contributers being the large values of C_0 in theorem 4.5 and of r from the proof of theorem 4.21), which certainly makes the above result very interesting in theory, but not very good for practical applications, disregarding the whole fact that it is not even easy to get Λ concretely. As was already mentioned at various points throughout the arguments, there is still much room for improvements.

5 Final Remarks

There are some open ends to these topics that, as far as the author is aware of, have not been further studied so far. It is known that theorem 3.8 can be improved (under certain extra assumptions), as was done in [7]. This also leads to improved constants in an accordingly adapted version of theorem 3.10. However, the most important open problem here seems to be how large the probability in theorem 3.8 can be, in particular how many partitions fulfilling corollary 3.9 and theorem 3.10 there are and maybe even how they can be found in an efficient way. To this extend, the formulation of corollary 4.7 may be improved to get more knowledge on RIP matrices (and related concepts from the theory of compressed sensing, see [22]), especially there existence (in a somewhat deterministic way) for a small number of rows. As was noted in the small remark after theorem 3.14, there also seems to be connections to the theory of combinatorial matrices regarding optimal bounds for such situations.

It is also useful to take a look at [9] or [29] (in German), which give an overview of all the problems, which are related to Weaver's conjectures, Anderson's paving conjecture and the Kadison-Singer problem in general. In particular, the Feichtinger conjecture also might be useful in the context of frames, as was discussed above, and which is also dealt with in [7]. As was already discussed after theorem 4.21, the constants there can probably be improved by a lot.

It is also natural to ask, whether theorem 4.21 holds for the higher dimensional case of $S \subseteq \mathbb{R}^d$ of finite but positive measure. To this end and other related topics, [24] gives more general results. In principle, all but maybe theorem 4.12 are easily seen to be generalizable to the multidimensional setting. It might also be not to far fetched to expect the multidimensional analogue of theorem 4.21 to also give new insights into discrepancy theory or related concepts. There are also more abstract results regarding frames over Lebesgue spaces or even more generally on Hilbert spaces, see for example [6] or [7]. All this theory has its roots in [19].

As a final note, this manuscript is, for a complete solution of the Kadison-Singer problem, not the first of its kind, see among others [5] or [25]. The author of this work hopes however, that this presentation is not only useful for understanding the Kadison-Singer problem and its solution, but it also aims to demonstrate its further applications to (but certainly not just) functional analysis and approximation theory.

Acknowledgment

I want to thank Prof. Dr. Tino Ullrich for suggesting this fascinating topic and always keeping me motivated. I also want to thank Dr. Martin Schäfer for giving extra resources and helping ironing out some problems from the original papers. Last but not least, I want to thank everyone who attended my talk on this topic shortly before this manuscript was finished, which seemed to give a good overlook on the mentioned material and helped flesh out some of the extra details explained here.

Bibliography

[1] C. A. Akemann, J. Anderson, Lyapunov Theorems for Operator Algebras, Memoirs of the American Mathematical Society, 1991.

[2] N. A. Balonin, J. Seberry, A review and new symmetric conference matrices, Faculty of Engineering and Information Sciences - Papers: Part A. 2748, 2014.

[3] J. Borcea, P. Brändén, Applications of stable polynomials to mixed determinants: Johnson's conjectures, unimodality, and symmetrized Fischer products, arXiv:math/0607755 [math.SP], 2008.

[4] J. Borcea, P. Brändén, Multivariate Polya-Schur classification problems in the Weyl algebra, Proceedings in the London Mathematical Society, Volume 101, Issue 1, pp. 73-104, 2016.

[5] M. Bownik, The Kadison-Singer Problem, arXiv:1702.04578 [math.FA], 2017.

[6] M. Bownik, Continuous frames and the Kadison-Singer problem, Springer Proceedings in Physics, Volume 205, 2018.

[7] M. Bownik, P. G. Casazza, A. W. Marcus, D. Speegle, Improved bounds in Weaver and Feichtinger Conjectures, Journal für die reine und angewandte Mathematik, Volume 749, pp. 267–293, 2019.

[8] P. Casazza, D. Edidin, D. Kalra, V. I. Paulsen, Projections and the Kadison-Singer problem, Operators and Matrices, Volume 1, Number 3, pp. 391-408, 2007.

[9] P. Casazza, M. Fickus, J. C. Tremain, E. Weber, The Kadison-Singer problem in mathematics and engeneering: a detailed account, arXiv:math/0510024 [math.FA], 2006.

[10] Y.-B. Choe, J. G. Oxley, A. D. Sokal, D. G. Wagner, Homogeneous multivariate polynomials with the half-plane property, Advances in Applied Mathematics, Volume 32, Issues 1-2, pp. 88-187, 2002.

[11] M. Chudnovsky, P. Seymour, The roots of the independence polynomial of a clawfree graph, Journal of Combinatorial Theory, Series B 97, pp. 350–357, 2007.

[12] J. P. Dedieu, Obreschkoff's theorem revisited: What convex sets are contained

in the set of hyperbolic polynomials?, Journal of Pure and Applied Algebra, Volume 93, Issue 1, pp. 111-112, 1992.

[13] L. GÅDING, An Inequality for Hyperbolic Polynomials, Journal of Mathematics and Mechanics, Volume 8, 1959.

[14] N. J. A. HARVEY, N. OLVER, Pipage Rounding, Pessimistic Estimators and Matrix Concentration, Proceedings of the 2014 Annual ACM-SIAM Symposium on Discrete Algorithms, pp. 926-945, 2014.

[15] E. HELLY, Über Mengen konvexer Körper mit gemeinschaftlichen Punkten, Jahresbericht der Deutschen Mathematiker-Vereinigung, Volume 32, pp. 175-176, 1923.

[16] R. V. KADISON, I. M. SINGER, Extensions of Pure States, American Journal of Mathematics, Volume 81, No. 2, pp. 383-400, 1959.

[17] A. S. LEWIS, P. A. PARRILO, M. V. RAMANA, The Lax conjecture is true, Proceedings of the American Mathematical Society, Volume 133, No. 9, pp. 2495-2499, 2005.

[18] A. W. MARCUS, D. A. SPIELMAN, N. SRIVASTAVA, Interlacing Families I: Bipartite Ramanujan Graphs of All Degrees, Annals of Mathematics, Volume 182, Issue 1, pp. 307-325, 2015.

[19] A. W. MARCUS, D. A. SPIELMAN, N. SRIVASTAVA, Interlacing Families II: Mixed Characteristic Polynomials and The Kadison-Singer Problem, Annals of Mathematics, Volume 182, Issue 1, pp. 327-350, 2015.

[20] S. NITZAN, A. OLEVSKII, A. ULANOVSKII, Exponential frames on unbounded sets, Proceedings of the American Mathematical Society, Volume 144, pp. 109-118, 2014.

[21] J. RADON, Mengen konvexer Körper, die einen gemeinsamen Punkt enthalten, Mathematische Annalen, Volume 83, pp. 281-302, 1921.

[22] H. RAUHUT, Compressive Sensing and Structured Random Matrices, Radon Series on Computational and Applied Mathematics, Issue 9, pp. 1–94, 2010.

[23] T. TAO, Real stable polynomials and the Kadison-Singer problem, www.terrytao.wordpress.com/2013/11/04/real-stable-polynomials-and-the-kadison-singer-problem/, 2013.

[24] V. N. TEMLYAKOV, The Marcinkiewicz-type discretization theorems for the hyperbolic cross polynomials, Jaen Journal on Approximation, Volume 9, Number 1, 2017.

[25] D. TIMOTIN, The solution to the Kadison–Singer Problem: yet another presentation, arXiv:1501.00464 [math.FA], 2015.

[26] D. G. Wagner, Multivariate stable polynomials: Theory and applications, Bulletin of the American Mathematical Society, Volume 48, pp. 53-84, 2009.

[27] N. Weaver, The Kadison-Singer problem in discrepancy theory, Discrete Mathematics Volume 278, Issues 1–3, pp. 227-239, 2004.

[28] R. M. Young, An introduction to nonharmonic Fourier series, Academic Press, 2. Edition, 2001.

[29] J. F. Zawallich, Das Kadison-Singer Problem, Technische Universität Berlin, 2013.

Studentenservice – Zentrales Prüfungsamt
Selbstständigkeitserklärung

TECHNISCHE UNIVERSITÄT CHEMNITZ

Name: Vorname: geb. am: Matr.-Nr.:	Bitte beachten: 1. Bitte binden Sie dieses Blatt am Ende Ihrer Arbeit ein.

Selbstständigkeitserklärung*

Ich erkläre gegenüber der Technischen Universität Chemnitz, dass ich die vorliegende
selbstständig und ohne Benutzung anderer als der angegebenen Quellen und Hilfsmittel angefertigt habe.

Die vorliegende Arbeit ist frei von Plagiaten. Alle Ausführungen, die wörtlich oder inhaltlich aus anderen Schriften entnommen sind, habe ich als solche kenntlich gemacht.

Diese Arbeit wurde in gleicher oder ähnlicher Form noch nicht als Prüfungsleistung eingereicht und ist auch noch nicht veröffentlicht.

Datum: …………………………………….   Unterschrift: ………………………………………………………………………

* Statement of Authorship

I hereby certify to the Technische Universität Chemnitz that this thesis is all my own work and uses no external material other than that acknowledged in the text.

This work contains no plagiarism and all sentences or passages directly quoted from other people's work or including content derived from such work have been specifically credited to the authors and sources.

This paper has neither been submitted in the same or a similar form to any other examiner nor for the award of any other degree, nor has it previously been published.