NFFT basierte schnelle Ewald-Summation für gemischt periodische Randbedingungen

Franziska Nestler
TECHNISCHE UNIVERSITÄT CHEMNITZ
Fakultät für Mathematik

Mecklenburger Workshop
Approximationsmethoden und schnelle Algorithmen
Hasenwinkel, 18. März 2014

Contents

The 3d-NFFT (FFT for nonequispaced data)

Notation

- define the torus $\mathbb{T} := \mathbb{R}/\mathbb{Z} \simeq [-1/2, 1/2)$
- for $M \in 2\mathbb{N}$ set $\mathcal{I}_M := \{-M/2, \dots, M/2 - 1\}^3 \subset \mathbb{Z}^3$

$$\text{FFT:} \qquad f(\boldsymbol{j}) \quad := \sum_{\boldsymbol{k} \in \mathcal{I}_M} \hat{f}_{\boldsymbol{k}} \mathrm{e}^{2\pi \mathrm{i} \boldsymbol{k} \cdot \boldsymbol{j}/M} \qquad \boldsymbol{j} \in \mathcal{I}_M$$

[Dutt, Rokhlin 1993] [Beylkin 1995]
[Potts, Steidl, Tasche 2001] [Greengard, Lee 2004]

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\end{aligned}$$

Complexity: $\mathcal{O}(|\mathcal{I}_M| \log |\mathcal{I}_M| + N)$

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\text{adjoint NFFT:} \quad & h(\boldsymbol{k}) && := \sum_{j=1}^{N} f_j \mathrm{e}^{-2\pi \mathrm{i} \boldsymbol{k} \cdot \boldsymbol{x}_j} && \boldsymbol{k} \in \mathcal{I}_M
\end{aligned}
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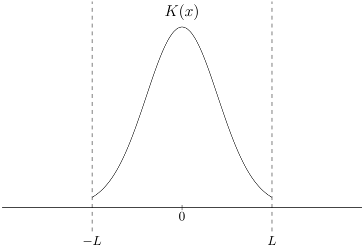
Fast summation based on NFFT in 1d [Potts, Steidl 2003]

Compute $f(x_j) := \sum_{i=1}^{N} c_i K(x_i - x_j)$ for $x_j \in [-L/2, L/2], j = 1, \ldots, N$.



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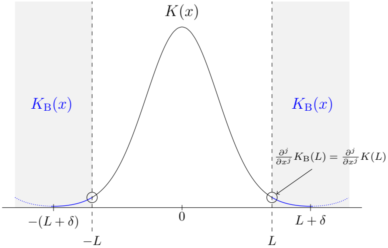

- $x_i - x_j \in [-L, L]$
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- construct K_B, claim smoothness in $x = \pm L$

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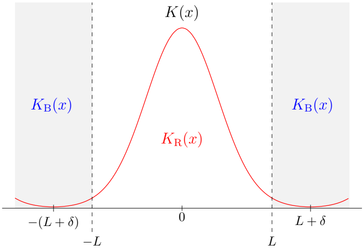

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- regularized kernel K_{R} is smooth and periodic with period $h := 2(L + \delta)$

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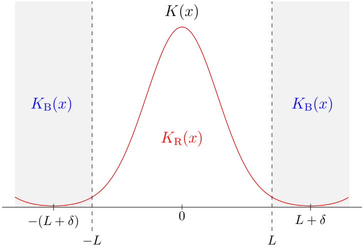

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$$f(x_j) = \sum_{i=1}^{N} c_i K_{\mathrm{R}}(x_i - x_j)$$
$$\approx \sum_{i=1}^{N} c_i \sum_{l=-M/2}^{M/2-1} \hat{b}_l \mathrm{e}^{2\pi \mathrm{i} l (x_i - x_j)/h}$$

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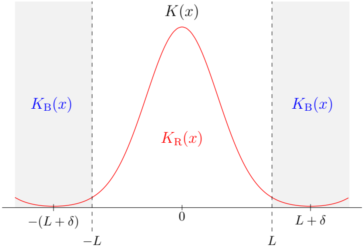

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Definition of the Coulomb interaction energy

Let N charges $q_j \in \mathbb{R}$ at positions $\boldsymbol{x}_j \in \mathbb{R}^3$ be given.
Coulomb interaction energy ($\boldsymbol{x}_{ij} := \boldsymbol{x}_i - \boldsymbol{x}_j$):

$$E := \frac{1}{2} \sum_{\substack{i,j=1 \\ i \neq j}}^{N} \frac{q_i q_j}{\|\boldsymbol{x}_{ij}\|} = \frac{1}{2} \sum_{j=1}^{N} q_j \phi(\boldsymbol{x}_j) \quad \text{with} \quad \phi(\boldsymbol{x}_j) := \sum_{\substack{i=1 \\ i \neq j}}^{N} \frac{q_i}{\|\boldsymbol{x}_{ij}\|}.$$

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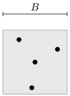
3d-periodic boundary conditions

choose $S := \mathbb{Z}^3$, $\boldsymbol{x}_j \in B\mathbb{T}^3$ and set

$$\phi(\boldsymbol{x}_j) := \phi_{\mathcal{S}}(\boldsymbol{x}_j)$$
$$:= \sum_{\boldsymbol{n}\in\mathcal{S}} \sum_{\substack{i=1\\ i\neq j \text{ if } \boldsymbol{n}=\boldsymbol{0}}}^{N} \frac{q_i}{\|\boldsymbol{x}_{ij} + B\boldsymbol{n}\|}$$

$\rightarrow$ crystals, ...



Fast NFFT based algorithm: P^{2}NFFT, $\mathcal{O}(N \log N)$ [Pippig, Potts 2011]

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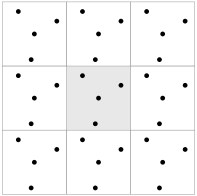

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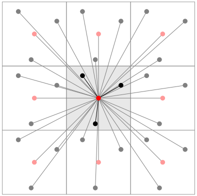

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Coulomb energy s.t. periodic boundary conditions

$$E(\mathcal{S}) := \frac{1}{2}\sum_{j=1}^{N} q_j \phi_{\mathcal{S}}(\boldsymbol{x}_j) \quad \text{with} \quad \phi_{\mathcal{S}}(\boldsymbol{x}_j) := \sum_{\boldsymbol{n}\in\mathcal{S}} \sum_{\substack{i=1 \\ i\neq j \text{ if } \boldsymbol{n}=\boldsymbol{0}}}^{N} \frac{q_i}{\|\boldsymbol{x}_{ij} + B\boldsymbol{n}\|}$$

Mixed periodic boundary conditions:

2d-periodic

$\boldsymbol{x}_j \in B\mathbb{T}^2 \times \mathbb{R}$, $\mathcal{S} := \mathbb{Z}^2 \times \{0\}$
→ thin liquid films, ...

1d-periodic

$\boldsymbol{x}_j \in B\mathbb{T} \times \mathbb{R}^2$, $\mathcal{S} := \mathbb{Z} \times \{0\}^2$
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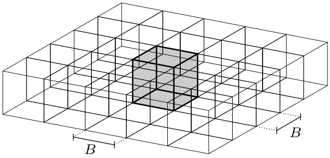

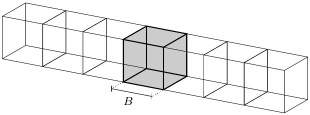

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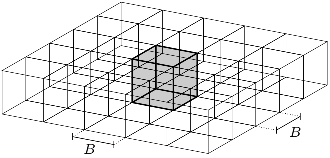

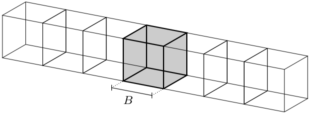

Assume $\sum_{j=1}^{N} q_j = 0 \Rightarrow$ conditional convergence

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FMM: $\mathcal{O}(N)$
[Greengard, Rokhlin 1987]
MMM2D: $\mathcal{O}(N^{5/3})$
[Arnold, Holm 2002]
SE2P: $\mathcal{O}(N \log N)$
[Lindbo, Tornberg 2011]

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FMM: $\mathcal{O}(N)$
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MMM1D: $\mathcal{O}(N^2)$
[Arnold, Holm 2005]

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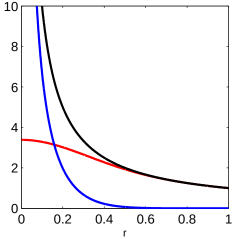
Ewald summation

Idea of the Ewald summation [Ewald 1921]:

Ewald splitting

$$\frac{1}{r} = \underbrace{\frac{\mathrm{erf}(\alpha r)}{r}}_{\text{long ranged, continuous}} + \underbrace{\frac{\mathrm{erfc}(\alpha r)}{r}}_{\text{singular in 0, short ranged}}$$



- $\mathrm{erf}(x) := \frac{2}{\sqrt{\pi}} \int_0^x \mathrm{e}^{-t^2} \mathrm{d}t$ (error function)
- $\mathrm{erfc}(x) := 1 - \mathrm{erf}(x)$ (complementary error function)
- $\alpha > 0$ (scaling parameter)

Ewald Summation

We apply $\boxed{\frac{1}{r} = \frac{\operatorname{erfc}(\alpha r)}{r} + \frac{\operatorname{erf}(\alpha r)}{r}}$ with $r := \|\boldsymbol{x}_{ij} + B\boldsymbol{n}\|$ and obtain

$$\sum_{\boldsymbol{n}\in S} \sum_{\substack{i=1 \\ i\neq j \text{ if } \boldsymbol{n}=\boldsymbol{0}}}^{N} \frac{q_i}{\|\boldsymbol{x}_{ij} + B\boldsymbol{n}\|}$$

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- short range part can be obtained via direct evaluation after truncation

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- short range part can be obtained via direct evaluation after truncation
- $\lim_{r\to 0} \frac{\mathrm{erf}(\alpha r)}{r} = \frac{2\alpha}{\sqrt{\pi}} \Rightarrow$ substract self potential
- write the long range part as a sum in Fourier space

Ewald Summation

$$\phi_S^{\mathrm{L}}(\boldsymbol{x}_j) := \boxed{\sum_{\boldsymbol{n}\in S} \sum_{i=1}^{N} q_i \frac{\operatorname{erf}(\alpha\|\boldsymbol{x}_{ij} + B\boldsymbol{n}\|)}{\|\boldsymbol{x}_{ij} + B\boldsymbol{n}\|}}$$

- write the long range part as a sum in Fourier space

Ewald summation s.t. 2d-periodic boundary conditions

Long range part under 2d-periodic b.c. [Grzybowski, Gwóźdź, Bródka 2000]

We write $\boldsymbol{x}_{ij} = (\tilde{\boldsymbol{x}}_{ij}, x_{ij,3})$ and obtain for $\phi^{\mathrm{L}}_{\mathbb{Z}^2\times\{0\}}(\boldsymbol{x}_j)$

$$\frac{1}{2B}\sum_{\boldsymbol{k}\in\mathbb{Z}^2\setminus\{\boldsymbol{0}\}}\sum_{i=1}^{N} q_i \mathrm{e}^{2\pi\mathrm{i}\boldsymbol{k}\cdot\tilde{\boldsymbol{x}}_{ij}/B}\Theta^{\mathrm{p2}}(\|\boldsymbol{k}\|, x_{ij,3}) - \frac{2\sqrt{\pi}}{B^2}\sum_{i=1}^{N} q_i \Theta^{\mathrm{p2}}_0(x_{ij,3})$$

2d Fourier series

$\boldsymbol{k} = \boldsymbol{0}$ part

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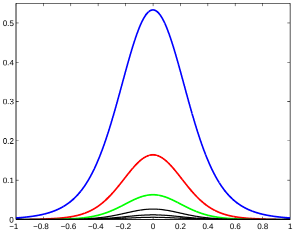

$\Theta^{\mathrm{p2}}(k,r)$

$k=1$
$k=\sqrt{2}$
$k=\sqrt{3}$
$\ldots$

$$\frac{\mathrm{e}^{2\pi kr/B}\operatorname{erfc}\left(\frac{\pi k}{\alpha B}+\alpha r\right)+\mathrm{e}^{-2\pi kr/B}\operatorname{erfc}\left(\frac{\pi k}{\alpha B}-\alpha r\right)}{k}$$

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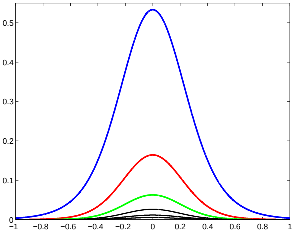

$\Theta^{\mathrm{p2}}(k, r)$

$k = 1$

$k = \sqrt{2}$

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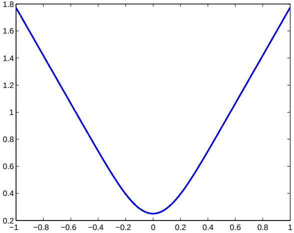
$\dots$

$$\frac{\mathrm{e}^{2\pi k r/B}\operatorname{erfc}\left(\frac{\pi k}{\alpha B}+\alpha r\right)+\mathrm{e}^{-2\pi k r/B}\operatorname{erfc}\left(\frac{\pi k}{\alpha B}-\alpha r\right)}{k}$$

$\boldsymbol{k} = \boldsymbol{0}$ part



$$\Theta^{\mathrm{p2}}_0(r) = \frac{1}{\alpha}\mathrm{e}^{-\alpha^2 r^2} + r\sqrt{\pi}\operatorname{erf}(\alpha r)$$

A fast algorithm I

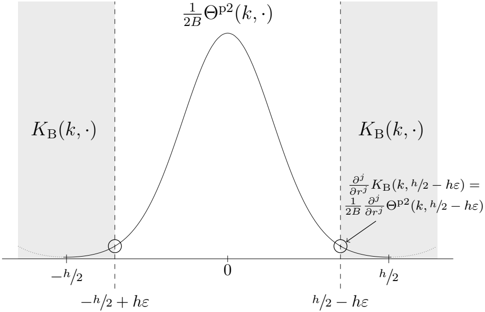
The 2d Fourier sum

$$\frac{1}{2B}\sum_{\boldsymbol{k}\in\mathbb{Z}^2\setminus\{\boldsymbol{0}\}} \mathrm{e}^{2\pi\mathrm{i}\boldsymbol{k}\cdot\tilde{\boldsymbol{x}}_{ij}/B} \sum_{i=1}^{N} q_i \Theta^{\mathrm{p2}}(\|\boldsymbol{k}\|, x_{ij,3})$$

$$\boxed{\forall\, k : \Theta^{\mathrm{p2}}(k,\cdot) \in \mathrm{C}^{\infty}(\mathbb{R})}$$

Regularization:

- $x_{ij,3}$ are absolutely bounded
- we are just interested in $\Theta^{\mathrm{p2}}(k,\cdot)$ over a finite interval (white area)



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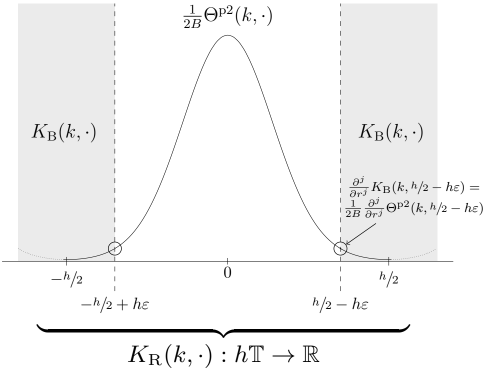
Regularization:

- $x_{ij,3}$ are absolutely bounded
- we are just interested in $\Theta^{\mathrm{p2}}(k,\cdot)$ over a finite interval (white area)
- construct a smooth and periodic function $K_{\mathrm{R}}(k,\cdot)$
- $\varepsilon \in (0, 1/2)$ determines the size of the areas at the boundaries



A fast algorithm II

The 2d Fourier sum (∗)

$$\frac{1}{2B}\sum_{\boldsymbol{k}\in\mathbb{Z}^2\setminus\{\boldsymbol{0}\}} \mathrm{e}^{2\pi\mathrm{i}\boldsymbol{k}\cdot\tilde{\boldsymbol{x}}_{ij}/B}\sum_{i=1}^{N} q_i\Theta^{\mathrm{p2}}(\|\boldsymbol{k}\|, x_{ij,3})$$

- truncate the infinite sum, $\mathbb{Z}^2 \to \mathcal{I}_{\tilde{M}} := \{-\tilde{M}/2\,\ldots,\tilde{M}/2-1\}^2$

A fast algorithm II

The 2d Fourier sum $(*)$

$$\frac{1}{2B} \sum_{\boldsymbol{k} \in \mathbb{Z}^2 \setminus \{\boldsymbol{0}\}} \mathrm{e}^{2\pi \mathrm{i} \boldsymbol{k} \cdot \tilde{\boldsymbol{x}}_{ij}/B} \sum_{i=1}^{N} q_i \Theta^{\mathrm{p2}}(\|\boldsymbol{k}\|, x_{ij,3})$$

- truncate the infinite sum, $\mathbb{Z}^2 \to \mathcal{I}_{\tilde{M}} := \{-\tilde{M}/2 \ldots, \tilde{M}/2 - 1\}^2$
- for $M \in 2\mathbb{N}$ large enough we find for each $k = \|\boldsymbol{k}\| \neq 0$ ($\boldsymbol{k} \in \mathcal{I}_{\tilde{M}}$)

$$\frac{1}{2B} \Theta^{\mathrm{p2}}(k, x_{ij,3}) = K_{\mathrm{R}}(k, x_{ij,3}) \approx \sum_{l=-M/2}^{M/2-1} \hat{b}_{k,l} \mathrm{e}^{2\pi \mathrm{i} l x_{ij,3}/h}$$

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$$(*)\ \approx \sum_{\boldsymbol{k}\in\mathcal{I}_{\tilde{M}}\setminus\{\boldsymbol{0}\}}\ \sum_{l=-M/2}^{M/2-1}\hat{b}_{\|\boldsymbol{k}\|,l}\left(\sum_{i=1}^{N} q_i \mathrm{e}^{2\pi\mathrm{i}\begin{bmatrix}k_1\\k_2\\l\end{bmatrix}\cdot\hat{\boldsymbol{x}}_i}\right)\mathrm{e}^{-2\pi\mathrm{i}\begin{bmatrix}k_1\\k_2\\l\end{bmatrix}\cdot\hat{\boldsymbol{x}}_j}$$

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- analog for the $\boldsymbol{k} = \boldsymbol{0}$ term

Ewald summation s.t. 1d periodic boundary conditions

Long range part under 1d-periodic b.c. [Porto 2000]

We write $\boldsymbol{x}_{ij} = (x_{ij,1}, \tilde{\boldsymbol{x}}_{ij})$ and obtain for $\phi^{\mathrm{L}}_{\mathbb{Z}\times\{0\}^2}(\boldsymbol{x}_j)$

$$\frac{2}{B}\sum_{k\in\mathbb{Z}\setminus\{0\}}\sum_{i=1}^{N} q_i \mathrm{e}^{2\pi\mathrm{i}k x_{ij,1}/B}\Theta^{\mathrm{p1}}(|k|, \|\tilde{\boldsymbol{x}}_{ij}\|) - \frac{1}{B}\sum_{i=1}^{N} q_i \Theta^{\mathrm{p1}}_0(\|\tilde{\boldsymbol{x}}_{ij}\|)$$

1d Fourier series

$k = 0$ part

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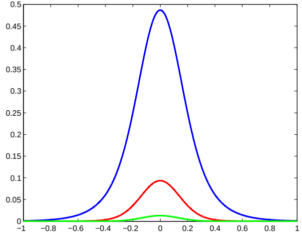

$\Theta^{\mathrm{p1}}(k, r)$

$k = 1$

$k = 2$

$k = 3$

$\dots$

$$\Theta^{\mathrm{p1}}(k, r) = \int_0^{\alpha} \frac{1}{t} \exp\left(-\frac{\pi^2 k^2}{B^2 t^2} - r^2 t^2\right) \mathrm{d}t$$

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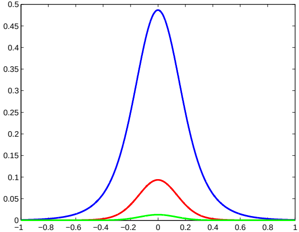

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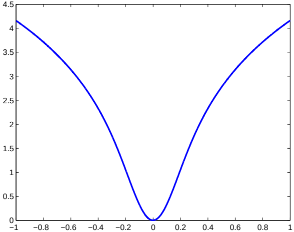


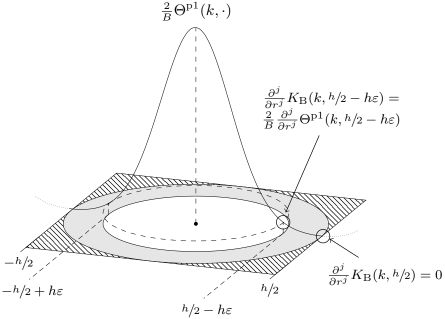
$$\Theta^{\mathrm{p1}}_0(r) = \gamma + \Gamma(0, \alpha^2 r^2) + \ln(\alpha^2 r^2)$$

A fast algorithm

The 1d Fourier sum

$$\frac{2}{B}\sum_{k\in\mathbb{Z}\setminus\{0\}} \mathrm{e}^{2\pi\mathrm{i}kx_{ij,1}/B}\sum_{i=1}^{N} q_i\Theta^{\mathrm{p1}}(|k|,\|\tilde{\boldsymbol{x}}_{ij}\|)$$

- regularization of $\Theta^{\mathrm{p1}}(|k|,\cdot)$ on $[-h/2, h/2]$



A fast algorithm

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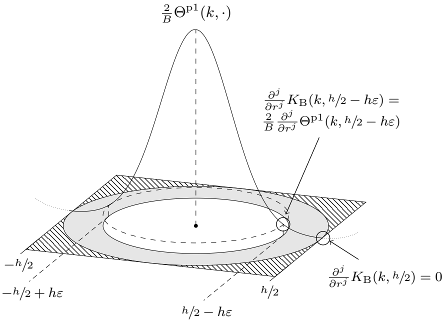
- claim vanishing derivatives at the boundary
- rotate and extend the function to the torus $h\mathbb{T}^2$ (constant value over the striped area)



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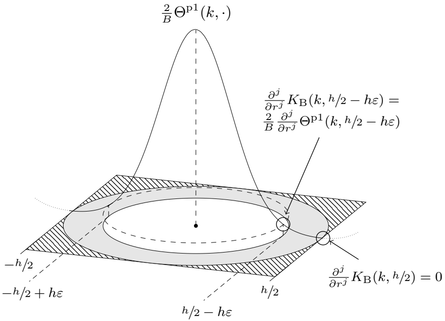
- claim vanishing derivatives at the boundary
- rotate and extend the function to the torus $h\mathbb{T}^2$ (constant value over the striped area)
- result: periodically smooth function in 2 variables
- approximate by bivariate trigonometric polynomials
- proceed as for 2d-periodic b.c.



Summary

Structure is the same as for 3d-periodic systems

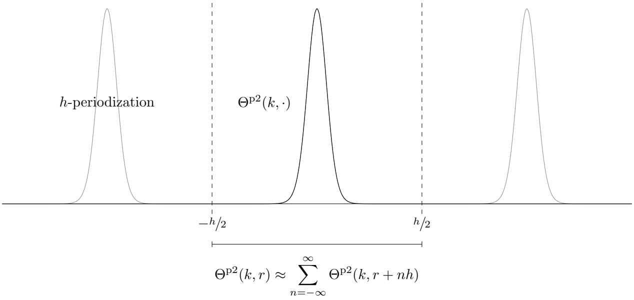
- Near field, self interactions
- Far field
  1. adjoint 3d NFFT
  2. Multiplikation with the Fourier coefficients
  3. 3d NFFT

Challenges

- Many precomputation steps: construct regularizations and compute the Fourier coefficients via the FFT.
- additional parameters throw regularization:
  degree of smoothness, regularization parameter ε, ...
  $\rightarrow$ optimal choice? exact error control?
- Especially choice of ε is important.
  To obtain first numerical results: tuned ε by hand.

How to reduce precomputation costs?

For k large enough:



Poisson summation:

$$\sum_{n=-\infty}^{\infty} \Theta^{\mathrm{p2}}(k, r+nh) = \frac{1}{h} \sum_{l=-\infty}^{\infty} \frac{2Bh^2}{\pi(h^2k^2 + B^2l^2)} \mathrm{e}^{-\pi^2 l^2/(\alpha^2 h^2) - \pi^2 k^2/(\alpha^2 B^2)} \mathrm{e}^{2\pi \mathrm{i} l r/h}$$

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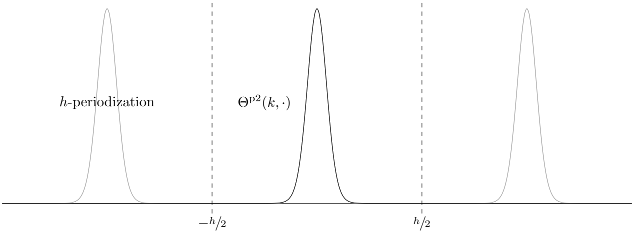

$$\Theta^{\mathrm{p2}}(k,r) \approx \sum_{n=-\infty}^{\infty} \Theta^{\mathrm{p2}}(k, r+nh)$$

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$\rightarrow$ Fourier coefficients are given analytically

Numerical results I [Nestler, Pippig, Potts 2013 (Preprint)]

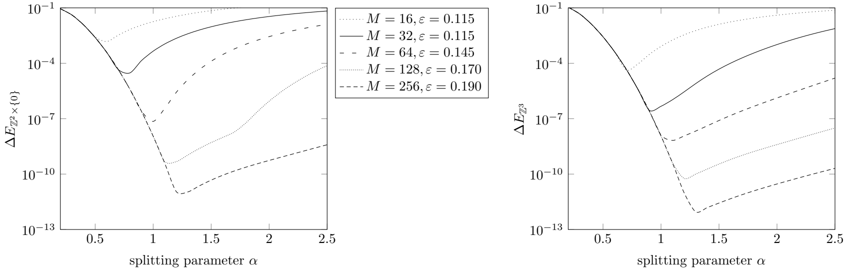

Figure : Achieved rms errors for a 2d-periodic (left) and the corresponding 3d-periodic computation (right) with $N = 300$ particles.

Numerical results II [Nestler, Pippig, Potts 2013 (Preprint)]

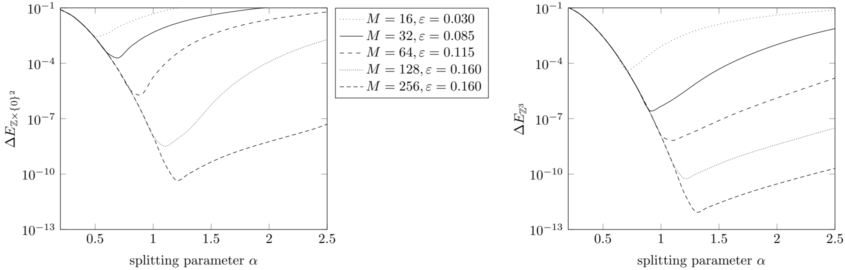

Figure : Achieved rms errors for a 1d-periodic (left) and the corresponding 3d-periodic computation (right) with $N = 300$ particles.

Numerical results III [Nestler, Pippig, Potts 2013 (Preprint)]

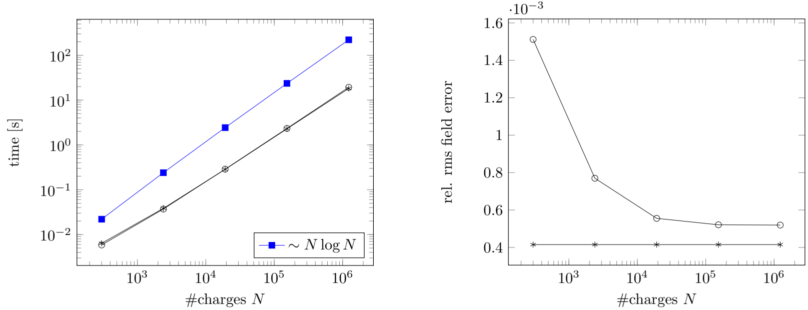

Figure : Run times without precomputations (left) and accuracy (right) for 2d- (o) and 3d-periodic (*) boundary conditions.

# charges	300	$8 \cdot 300$	$8^2 \cdot 300$	$8^3 \cdot 300$	$8^4 \cdot 300$
FFT size	16^3	32^3	64^3	128^3	256^3

Conclusion

- new approach to fast Ewald summation under mixed boundary conditions: [Nestler, Pippig, Potts 2013 (Preprint)]
  - 2d-periodic: Ewald + fast summation in 1d
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Thank you for your attention!