On SOBOLEV spaces of dominating mixed smoothness with $p = 1$

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Let $\Omega \subset \mathbb{R}^2$ be open and nonempty. Furthermore, let $1 \leq p \leq \infty$ and $m \in \mathbb{N}$.

- For a multi-index $\alpha = (\alpha_1, \alpha_2) \in \mathbb{N}_0^2$ we write $|\alpha|_1 = \alpha_1 + \alpha_2$ and $|\alpha|_\infty = \max\{\alpha_1, \alpha_2\}$.
- $W^{m,p}(\Omega) = \{u \in L^p(\Omega) \mid \partial^\alpha u \in L^p(\Omega)\ \forall \alpha \in \mathbb{N}_0^2 : |\alpha|_1 \leq m\}$
- $\|u\|_{W^{m,p}(\Omega)} = \sum_{0 \leq |\alpha|_1 \leq m} \|\partial^\alpha u\|_{L^p(\Omega)}$
- $S_p^m W(\Omega) = \{u \in L^p(\Omega) \mid \partial^\alpha u \in L^p(\Omega)\ \forall \alpha \in \mathbb{N}_0^2 : |\alpha|_\infty \leq m\}$
- $\|u\|_{S_p^m W(\Omega)} = \sum_{0 \leq |\alpha|_\infty \leq m} \|\partial^\alpha u\|_{L^p(\Omega)}$

[Draw picture]

- $S_p^m W(\Omega)$ are called SOBOLEV spaces of **domi**nating **mix**ed smoothness.
- For this talk: "domix spaces"
- Introduced by NIKOL'SKII in 1962
- Appear as solution spaces for certain hyperbolic PDEs , see e.g. [MAMEDOV]...
- ...but are also widely used in approximation theory.

Some easy facts:

- $S_p^m W(\Omega)$ is a separable BANACH space if $1 \leq p < \infty$ and a HILBERT space if $p = 2$.
- Trivial embedding: $W^{2,p}(\Omega) \hookrightarrow S_p^1 W(\Omega) \hookrightarrow W^{1,p}(\Omega)$

What else is known?

- For $1 < p < \infty$: $S^m_p W(\mathbb{R}^2) = S^m_{p,2} F(\mathbb{R}^2)$, see [SCHMEISSER], [VYBIRAL], [NGUYEN] and many more...
- Some "indirect" results, for example reduced SOBOLEV inequalities in [ADAMS]
- Some embeddings:
  - [ABDULLA]: $S^m_p W(\mathbb{R}^2) \hookrightarrow C^{m-1,1-\frac{1}{p}}(\mathbb{R}^2)$, $m \in \mathbb{N}$, $1 \leq p < \infty$
  - [NAJAFOV, RUSTAMOVA]: $S^1_p W(\Omega) \hookrightarrow C(\Omega)$ for certain domains Ω and $1 \leq p < \infty$.

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Questions:

- Is the completion of $\mathcal{H}_1 \otimes \mathcal{H}_2$ the entire space $S_1^1 W\left((0,1)^2\right)$?
- Is this space embedded into the continuous functions?

Lemma (cf. [Adams, Fournier])

Suppose $(j_\varepsilon)_{\varepsilon>0}$ is an approximate identity. Let $1 \le p < \infty$ and $u \in S_p^m W\left((0,1)^2\right)$. If $\Omega \subset (0,1)^2$ is nonempty, open, and with compact closure in $(0,1)^2$, then

$$\|j_\varepsilon * u - u\|_{S_p^m W(\Omega)} \longrightarrow 0\,, \quad \text{as } \varepsilon \downarrow 0\,.$$

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Proof

Let $0 < \varepsilon < \operatorname{dist}\big(\Omega, \partial(0,1)^2\big)$ be sufficiently small. Further, let $\widetilde{u}$ denote the trivial extension of u to $\mathbb{R}^2$. If $\varphi \in C_c^\infty(\Omega)$, then for any multi-index $\alpha \in \mathbb{N}_0^2$ with $|\alpha|_\infty \leq m$ we have:

$$\begin{aligned}
\int_\Omega j_\varepsilon * u(x)\partial^\alpha \varphi(x)\,\mathrm{d}x &= \int_\Omega \int_{\mathbb{R}^2} j_\varepsilon(y)\widetilde{u}(x-y)\partial^\alpha\varphi(x)\,\mathrm{d}y\,\mathrm{d}x \\
&= \int_{\mathbb{R}^2} j_\varepsilon(y) \int_\Omega \widetilde{u}(x-y)\partial^\alpha\varphi(x)\,\mathrm{d}x\,\mathrm{d}y \\
&= (-1)^{|\alpha|_1} \int_{\mathbb{R}^2} j_\varepsilon(y) \int_\Omega \partial^\alpha u(x-y)\varphi(x)\,\mathrm{d}x\,\mathrm{d}y \\
&= (-1)^{|\alpha|_1} \int_\Omega j_\varepsilon * \partial^\alpha u(x)\varphi(x)\,\mathrm{d}x
\end{aligned}$$

This shows that $\partial^\alpha(j_\varepsilon * u) = j_\varepsilon * \partial^\alpha u$ in $\mathscr{D}'(\Omega)$.

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Proof (continued)

Since $u \in S_p^m W\left((0,1)^2\right)$, we have $\partial^\alpha u \in L^p(\Omega)$, $0 \le |\alpha|_\infty \le m$. The properties of the approximate identity j_ε lead to

$$\begin{aligned}
\|j_\varepsilon * u - u\|_{S_p^m W(\Omega)} &= \sum_{0 \le |\alpha|_\infty \le m} \|\partial^\alpha (j_\varepsilon * u) - \partial^\alpha u\|_{L^p(\Omega)} \\
&= \sum_{0 \le |\alpha|_\infty \le m} \|j_\varepsilon * \partial^\alpha u - \partial^\alpha u\|_{L^p(\Omega)} \\
&\longrightarrow 0\,, \quad \text{as } \varepsilon \downarrow 0\,.
\end{aligned}$$

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Lemma (cf. [Adams, Fournier])

For any $1 \leq p < \infty$ the set of restrictions of functions $C_c^\infty(\mathbb{R}^2)\big|_{(0,1)^2}$ is dense in $S_p^m W\left((0,1)^2\right)$.

Lemma

For any $1 \leq p < \infty$ the set of restrictions of functions $(C_c^\infty(\mathbb{R}) \otimes C_c^\infty(\mathbb{R}))|_{(0,1)^2}$ is dense in $S_p^m W\left((0,1)^2\right)$.

Proof

Let $u \in S_p^m W\left((0,1)^2\right)$. For a given $\varepsilon > 0$ we can find a $\varphi \in C_c^\infty(\mathbb{R}^2)$ such that

$$\|u - \varphi\|_{S_p^m W((0,1)^2)} < \varepsilon \,.$$

Then, by a result of [Treves], there exists a function $\psi_1 \otimes \psi_2 \in C_c^\infty(\mathbb{R}) \otimes C_c^\infty(\mathbb{R})$ such that

$$\|\partial^\alpha \varphi - \partial^\alpha \left(\psi_1 \otimes \psi_2\right)\|_{L^\infty(\mathbb{R}^2)} < \varepsilon$$

for every multi-index $0 \leq |\alpha|_\infty \leq m$.

Proof (continued)

But this implies:

$$\begin{aligned}\|\varphi - \psi_1 \otimes \psi_2\|_{S_p^m W((0,1)^2)} &= \sum_{0\leq|\alpha|\leq m} \|\partial^\alpha \varphi - \partial^\alpha (\psi_1 \otimes \psi_2)\|_{L^p((0,1)^2)} \\ &\leq \sum_{0\leq|\alpha|\leq m} \|\partial^\alpha \varphi - \partial^\alpha (\psi_1 \otimes \psi_2)\|_{L^\infty((0,1)^2)} \\ &\leq \sum_{0\leq|\alpha|\leq m} \|\partial^\alpha \varphi - \partial^\alpha (\psi_1 \otimes \psi_2)\|_{L^\infty(\mathbb{R}^2)} \\ &\leq C\varepsilon\end{aligned}$$

Finally, by the triangle inequality:

$$\begin{aligned}\|u - \psi_1 \otimes \psi_2\|_{S_p^m W((0,1)^2)} &\leq \|u - \varphi\|_{S_p^m W((0,1)^2)} + \|\varphi - \psi_1 \otimes \psi_2\|_{S_p^m W((0,1)^2)} \\ &\leq (C+1)\varepsilon\end{aligned}$$

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Conclusion

For any $m \in \mathbb{N}$ and $1 \leq p < \infty$, the domix space $S_p^m W\left((0,1)^2\right)$ coincides with the completion of $\left(C_c^\infty(\mathbb{R}) \otimes C_c^\infty(\mathbb{R})\right)|_{(0,1)^2}$ with respect to the $\|\cdot\|_{S_p^m W((0,1)^2)}$-norm. In particular, this is true for $m = p = 1$, i.e. in the "HICKERNELL-case".

- Answer to our first question
- Let us also consider the embedding $S_p^1 W\left((0,1)^2\right) \hookrightarrow C\left((0,1)^2\right)$.

Lemma (cf. [Lieb, Loss])

Let $T \in \mathscr{D}'(\mathbb{R}^2)$ be a distribution and $\varphi \in C_c^\infty(\mathbb{R}^2)$ be a test function. If we set $\tau_y\varphi(x) = \varphi(x-y)$, then we have

$$\langle T, \tau_y\varphi\rangle - \langle T, \varphi\rangle = \int_0^1 \big(y_1\langle \partial_1 T, \tau_{ty}\varphi\rangle + y_2\langle \partial_2 T, \tau_{ty}\varphi\rangle\big)\,\mathrm{d}t$$

Lemma

Let $e_1 = (1,0)^{\mathrm{T}}$ and $e_2 = (0,1)^{\mathrm{T}}$. Then we also have

$$\begin{aligned}\langle T, \tau_y\varphi\rangle - \langle T, \varphi\rangle &= \int_0^1 y_1\langle \partial_1 T, \tau_{t_1 y_1 e_1}\varphi\rangle\,\mathrm{d}t_1 + \int_0^1 y_2\langle \partial_2 T, \tau_{t_2 y_2 e_2}\varphi\rangle\,\mathrm{d}t_2\\ &\quad + \int_0^1\int_0^1 y_1 y_2\langle \partial_1\partial_2 T, \tau_{t_1 y_1 e_1}\tau_{t_2 y_2 e_2}\varphi\rangle\,\mathrm{d}t_1\,\mathrm{d}t_2\ .\end{aligned}$$

Corollary (cf. [Abdulla], [Hickernell et al.])

Suppose $u \in S_1^1 W_{loc}(\mathbb{R}^2)$. Then, for each $y \in \mathbb{R}^2$ and a.e. $x \in \mathbb{R}^2$,

$$\begin{aligned} u(x+y) - u(x) &= \int_0^1 y_1\, \partial_1 u(x + t_1 y_1 e_1)\, \mathrm{d}t_1 + \int_0^1 y_2\, \partial_2 u(x + t_2 y_2 e_2)\, \mathrm{d}t_2 \\ &\quad + \int_0^1 \int_0^1 y_1 y_2\, \partial_1 \partial_2 u(x + t_1 y_1 e_1 + t_2 y_2 e_2)\, \mathrm{d}t_1\, \mathrm{d}t_2 \ . \end{aligned}$$

Theorem (cf. [Najafov, Rustamova])

$S_p^1 W\left((0,1)^2\right) \hookrightarrow C\left((0,1)^2\right)$ for all $1 \le p < \infty$.

Thank you for your attention!

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