Characterisations (cf. [5], [6])

- $\|T_{\cos} f\|_{S^r_{p,q}F(\mathbb{T}^n)} \asymp \|T_{\cos} f\|_{L^p(\mathbb{T}^n)}$
  $$+ \sum_{e \subset \{1,\dots,n\}} \left\| 2^{|j| r} \mathcal{R}^e_m(T_{\cos} f, 2^{-j}, \cdot) \right\|_{L^p(\ell^q, \mathbb{T}^n)}$$
- $\|T_{\cos} f\|_{S^r_{p,q}B(\mathbb{T}^n)} \asymp \|T_{\cos} f\|_{L^p(\mathbb{T}^n)}$
  $$+ \sum_{e \subset \{1,\dots,n\}} \left(\sum_{j \in \mathbb{N}^n_0(e)} 2^{|j| rq} \, \omega^e_m(T_{\cos} f, 2^{-j})^q_p \right)^{\frac{1}{q}}$$
- $\Delta^{m,e}_h = \prod_{i \in e} \Delta^m_{h_i,i}$ and $\Delta^{m,\emptyset}_h = \mathrm{Id}$

Estimate of p-Norm

- $S^r_{p,q}A \subset S^{1/p}_{p,1}B \subset S^0_{\infty,1}B \subset L^\infty$ for $r > 1/p$, $1 < p < \infty$ and $0 < q \leq \infty$
- $\|T_{\cos} f\|_{L^p(\mathbb{T}^n)} \lesssim \|f\|_{L^\infty(\mathbb{R}^n)} \lesssim \|f\|_{S^r_{p,q}A(\mathbb{R}^n)}$

Useful Trick (cf. [3], [4])

Write $f = \sum_{k \in \mathbb{N}^n_0} f_k = \sum_{k \in \mathbb{Z}^n} f_{j+k}$ with

$$f_{j+k} = \begin{cases} 0\,, & \text{if } j + k < 0\,, \\ \left(\varphi_{j_1+k_1} \cdot \ldots \cdot \varphi_{j_n+k_n} \widehat{f} \right)^{\vee}\,, & \text{else.} \end{cases}$$

Other Techniques

- Estimate differences $\Delta^{m,e}_h T_{\cos} f_{j+k}$ by derivatives
- Chain rule generates factors of $\sin$ for change of variable
- Riesz-Thorin interpolation theorem
- $|g(\cos(x))|$
  $$\leq \sup_{-1 \leq y \leq 1} \frac{|g(y)|}{\langle 2^{j+k}(y - \cos(x)) \rangle^a}$$
  $$\lesssim 2^{(j+k)a} \sup_{y \in \mathbb{R}} \frac{|g(y)|}{\langle 2^{j+k}(y - x) \rangle^a}$$

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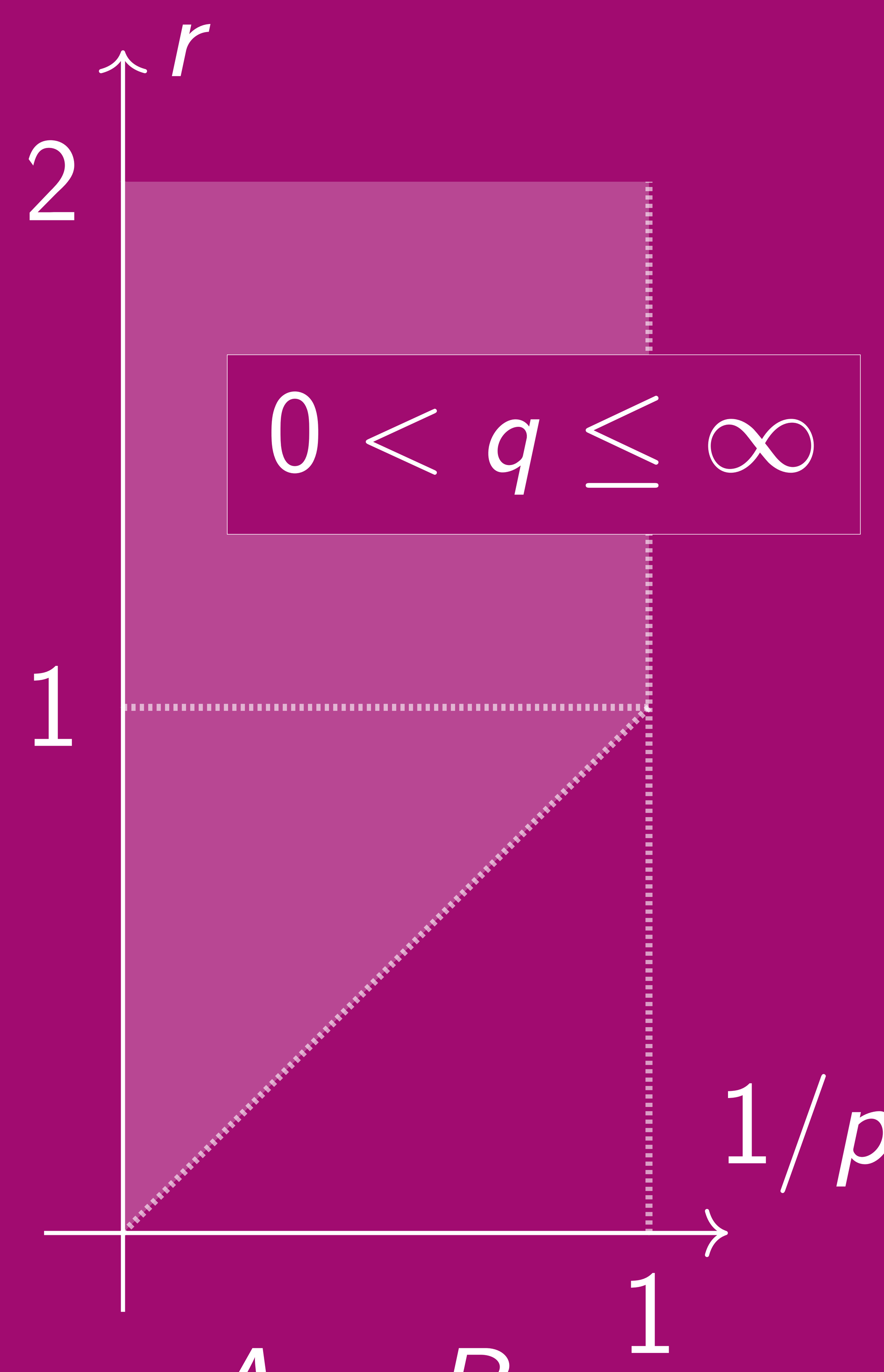
The transformation

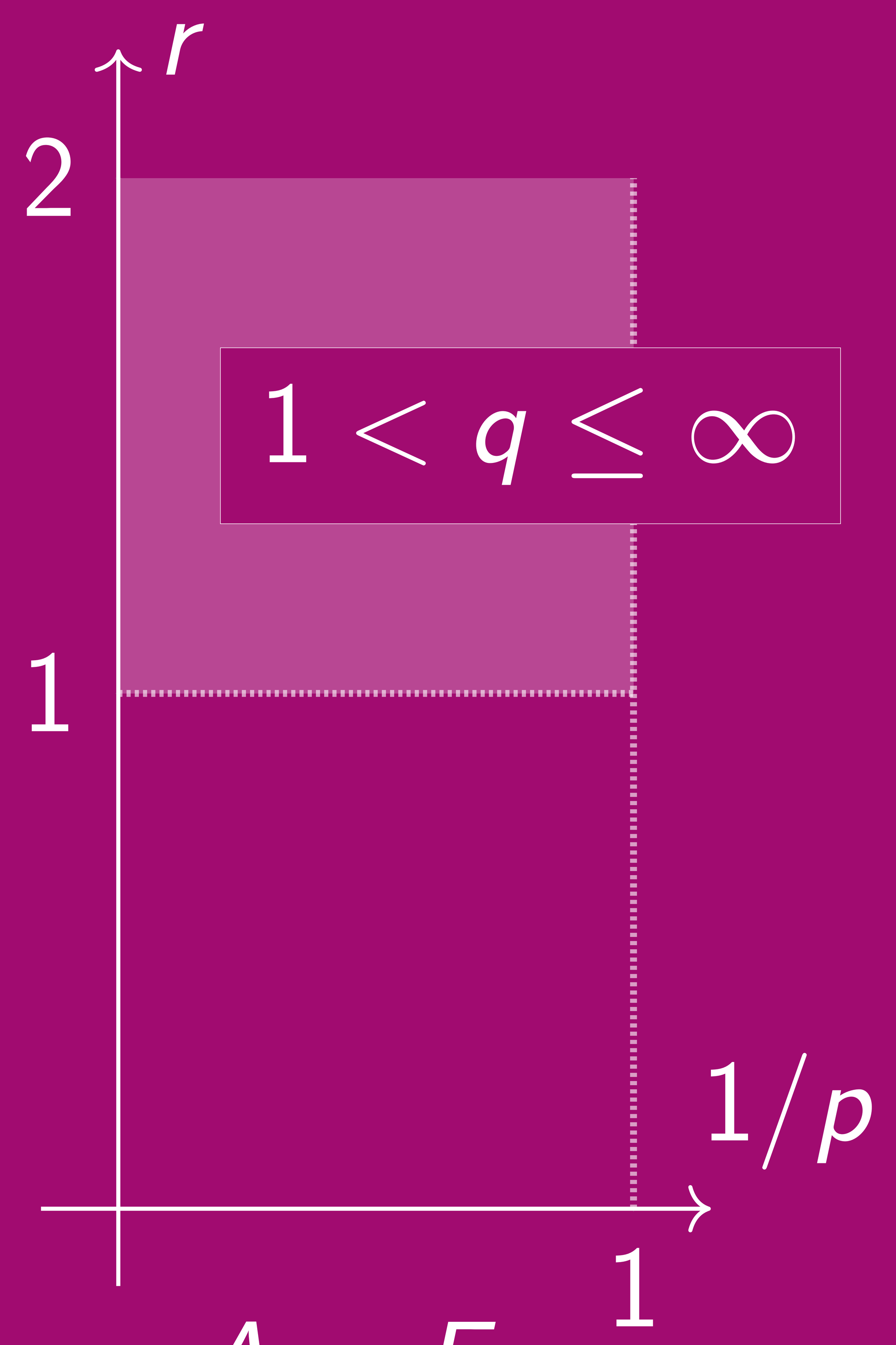
$$f(x_1, \dots, x_n) \to f(\cos(x_1), \dots, \cos(x_n))$$

is bounded from

$S^r_{p,q}A(\mathbb{R}^n)$ to $S^r_{p,q}A(\mathbb{T}^n)$:



$A = B$



$A = F$

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Limiting Cases if $A = B$

- $S^r_{p,q}B(\mathbb{R}^n) \to S^r_{p,q}B(\mathbb{T}^n)$
  for $p = 1, \infty$, $0 < q \leq \infty$, $r > 1/p$
- $S^{1/p}_{p,1}B(\mathbb{R}^n) \to S^{1/p}_{p,\infty}B(\mathbb{T}^n)$
  for $1 < p < \infty$

Regarding Sharpness

- $r > 1/2p$ is a necessary condition for boundedness if $A \in \{B, F\}$
- Show this using characteristic functions similar to [1]

An Application

- $f \in H^s([-1,1]) = B^s_{2,2}([-1,1])$
  $\Rightarrow T_{\cos} f \in H^s(\mathbb{T})$, if $s > 1/2$.
- $(T_{\cos} f)^\wedge(k)$
  $$= 2 \int_{-1}^{1} f(x)\, p_{|k|}(x)\, (1 - x^2)^{-1/2}\, dx$$
  $$= 2 \langle f, p_{|k|} \rangle_{L^2(w\,dx)}$$
  with $p_{|k|}(x) = \cos(|k| \arccos(x))$
- $\sum_{n \geq 0} (1 + n)^{2s} |\langle f, p_n \rangle|^2$
  $$\approx \sum_{k \in \mathbb{Z}} (1 + |k|)^{2s} \left| (T_{\cos} f)^\wedge(k) \right|^2$$
  $$= \|T_{\cos} f\|_{H^s(\mathbb{T})}$$
- This means $H^s([-1,1])$, $s > 1/2$ can be embedded into the Chebyshev-Sobolev-type space $L^{2,s}_{-\frac{1}{2},-\frac{1}{2}}([-1,1])$, cf. [2].

Contact

- Kai Lüttgen
  kai.luettgen@math.tu-chemnitz.de
- Tino Ullrich
  tino.ullrich@math.tu-chemnitz.de