Kai Lüttgen
(joint work with Martin Schäfer & Tino Ullrich)

Smoothness, Reflections and the Half-Period Cosine Basis

MSIA23 Raitenhaslach

20.03.2023

UNIVERSITY OF TECHNOLOGY
IN THE EUROPEAN CAPITAL OF CULTURE
CHEMNITZ

- The half-period cosine basis $\{1\} \cup \left\{\sqrt{2}\cos(\pi k\cdot) \;\middle|\; k \in \mathbb{N}\right\}$ is an ONB of $L^2([0,1])$.
- Following Dick, Nuyens & Pillichshammer[1], for $s > 0$ we define the Sobolev-type spaces

$$H^s_{\cos}([0,1]) = \left\{ f \in L^2([0,1]) \;\middle|\; \sum_{k \in \mathbb{N}_0} \langle k \rangle^{2s} \left| \langle f, \cos(\pi k\cdot) \rangle \right|^2 < \infty \right\}$$

- Here $\langle k \rangle^{2s} = (1 + |k|^2)^s$.

[1] *Lattice rules for nonperiodic smooth integrands*, Numer. Math. (2014) 126:259–291

- Consider $f(x) = x - \frac{1}{2}$.
- Then $f \in C^\infty([0,1])$ and

$$f(x) \sim -\frac{4}{\pi^2} \sum_{k \text{ odd}} \frac{1}{k^2} \cos(\pi k x) \; .$$

- Therefore, $f \in H^s_{\cos}([0,1])$ only for $0 < s < 3/2$.
- In general we have

$$\langle f, \cos(\pi k \cdot) \rangle = \frac{1}{\pi^2 k^2} \Big((-1)^k f'(1) - f'(0) \Big) - \frac{1}{\pi^2 k^2} \langle f'', \cos(\pi k \cdot) \rangle \; .$$

- Dick et al. prove, that

$$H^1_{\cos}([0,1]) \simeq W^{1,2}([0,1])$$
$$H^m_{\cos}([0,1]) \hookrightarrow W^{m,2}([0,1]) \text{ for } m \in \mathbb{N}\backslash\{1\}\ .$$

- For $s > 0$ we consider Besov spaces:

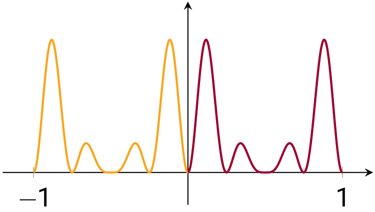
$$B^s_{2,2}(\mathbb{R}) = \left\{ f \in L^2(\mathbb{R}) \;\middle|\; \sum_{j\geq 0} 2^{2js} \left\|\varphi_j(D)f\right\|^2_{L^2(\mathbb{R})} < \infty \right\}$$

$$B^s_{2,2}([0,1]) = \left\{ f \in L^2([0,1]) \;\middle|\; \exists F \in B^s_{2,2}(\mathbb{R}) : F|_{[0,1]} = f \right\}$$

What is the relationship between the spaces $H^s_{\cos}([0,1])$ and $B^s_{2,2}([0,1])$?

On the one hand...

- Let $\mathbb{T} \simeq [-1,1]_{/\{-1,1\}}$
- For $s > 0$ consider $B^s_{2,2}(\mathbb{T}) = \left\{ f \in L^2(\mathbb{T}) \;\middle|\; \sum_{k\in\mathbb{Z}} \langle k\rangle^{2s} \left|\widehat{f}(k)\right|^2 < \infty \right\}$
- Suppose $f \in B^s_{2,2}(\mathbb{T})$ is even. Then:



$$\widehat{f}(k) \approx \int_{-1}^{1} f(x)\, e^{-\pi i k x}\, dx$$
$$\approx \int_{0}^{1} f(x)\, \cos(\pi k x)\, dx$$
$$\approx \langle f, \cos(\pi k \cdot)\rangle$$

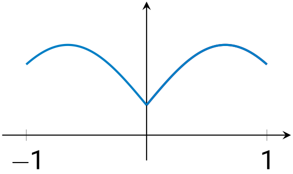
On the other hand...

- Reflection operator $\rho : f(x) \mapsto f(|x|)$
- Let $f \in H^s_{\cos}([0,1])$.
- Then, clearly, $\widehat{(\rho f)}(k) \approx \langle f, cos(\pi k \cdot) \rangle$ and so $\rho f \in B^s_{2,2}(\mathbb{T})_{even}$.
- We can identify $H^s_{cos}([0,1])$ and $B^s_{2,2}(\mathbb{T})_{even}$.

What is the relationship between the spaces $B^s_{2,2}(\mathbb{T})_{even}$ and $B^s_{2,2}([0,1])$?

Once more: periodisation by reflection

- Let $f \in B^s_{2,2}([0,1])$.
- Then consider again $\rho : f(x) \mapsto f(|x|)$.



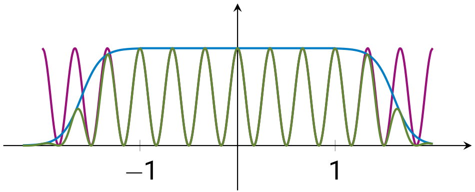
Theorem (L., Schäfer, Ullrich)

The reflection operator $\rho : B^s_{2,2}([0,1]) \to B^s_{2,2}(\mathbb{T})_{even}$ is bounded if and only if $0 < s < 3/2$.

- As a byproduct of our method of proof, we obtain a novel characterisation of Besov spaces in terms of Chui-Wang wavelets.

The other direction

- Let $f \in B^s_{2,2}(\mathbb{T})_{even}$.
- Multiply with appropriate bump function $\alpha \in C^\infty_c(\mathbb{R})$.



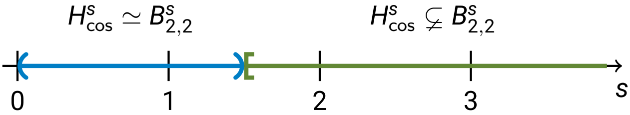
$$\|f\|_{B^s_{2,2}([0,1])} = \inf_{F|_{[0,1]}=f} \|F\|_{B^s_{2,2}(\mathbb{R})} \leq \|\alpha f\|_{B^s_{2,2}(\mathbb{R})} \lesssim \|f\|_{B^s_{2,2}(\mathbb{T})} \approx \|f\|_{H^s_{\cos}([0,1])}$$

Summary

Theorem (L., Schäfer, Ullrich)

For $s > 0$ we have $H^s_{\cos}([0,1]) \hookrightarrow B^s_{2,2}([0,1])$.

If $0 < s < 3/2$, then also $B^s_{2,2}([0,1]) \hookrightarrow H^s_{\cos}([0,1])$.



Outlook on higher dimensions

- In the setting of dominating mixed smoothness we have the following:

Theorem (L., Schäfer, Ullrich)

For $r > 0$ we have $S^r_{\cos}H([0,1]^n) \hookrightarrow S^r_{2,2}B([0,1]^n)$.

If $0 < r < 3/2$, then also $S^r_{2,2}B([0,1]^n) \hookrightarrow S^r_{\cos}H([0,1]^n)$.

- Also have results regarding the non-Hilbert space case.

Outlook on higher dimensions

- In the setting of dominating mixed smoothness we have the following:

Theorem (L., Schäfer, Ullrich)

For $r > 0$ we have $S^r_{\cos}H([0,1]^n) \hookrightarrow S^r_{2,2}B([0,1]^n)$.

If $0 < r < 3/2$, then also $S^r_{2,2}B([0,1]^n) \hookrightarrow S^r_{\cos}H([0,1]^n)$.

- Also have results regarding the non-Hilbert space case.

Thank you for your attention!