Characterising mixed Besov spaces with the half-period cosine basis

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- The following results stem from joint work with **Martin Schäfer**, **Serhii Stasyuk** and **Tino Ullrich**.

The Half-Period Cosine Basis

- For $k = (k_1, \ldots, k_n) \in \mathbb{N}_0^n$ let

$$c_k(x) := \cos(\pi k_1 x_1) \cdot \ldots \cdot \cos(\pi k_n x_n) \ , \quad x = (x_1, \ldots, x_n) \in \mathbb{R}^n \ .$$

- Augmented with appropriate normalisation factors, the family $(c_k)_{k \in \mathbb{N}_0^n}$ is an orthonormal basis of $L^2\big([0,1]^n\big)$.
- We call $(c_k)_{k \in \mathbb{N}_0^n}$ the half-period cosine basis.
- In practice, this basis is used, for example, in the context of numerical integration. See e.g. [Dick et al.], [Kuo et al.], [Suryanarayana et al.].
- The connection between decay rate of basis coefficients and smoothness of f is of interest.

- So far the focus has been mostly on $H^r_{\text{mix}} = S^r_{2,2}B$.
- Goal: generalisation to mixed BESOV spaces
- Fix a dyadic partition of unity $(\varphi_j)_{j\in\mathbb{N}_0}$ on the real line such that φ_j is even $\forall j \in \mathbb{N}_0$.
- If $f \in C\big([0,1]^n\big)$ and $j = (j_1, \ldots, j_n) \in \mathbb{N}_0^n$ we define

$$f_j^{\cos} := \sum_{k\in\mathbb{N}_0^n} \varphi_{j_1}(k_1) \cdot \ldots \cdot \varphi_{j_n}(k_n) \, \langle f, c_k \rangle_{L^2([0,1]^n)} \cdot c_k \ .$$

Main Results

Theorem

Suppose $f \in C\big([0,1]^n\big)$.

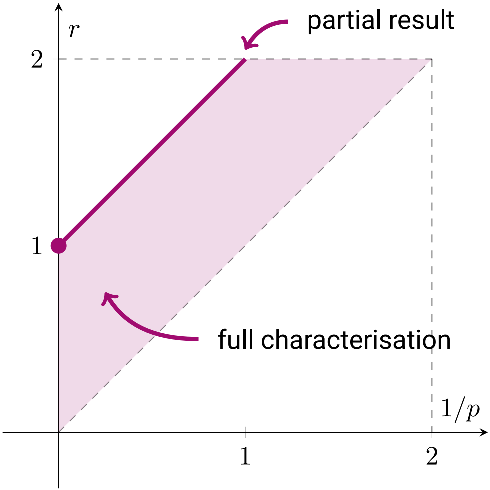
- Let $1/2 < p \leq \infty$, $0 < q \leq \infty$ and $1/p < r < \min\{2, 1 + 1/p\}$. Then

$$\|f\|_{S^r_{p,q}B([0,1]^n)} \asymp \left(\sum_{j \in \mathbb{N}_0^n} 2^{rq|j|_1} \, \|f_j^{\cos}\|^q_{L^p([0,1]^n)} \right)^{1/q},$$

  with the usual modification if $q = \infty$.

- Let $1 < p \leq \infty$. Then:

$$\sup_{j \in \mathbb{N}_0^n} 2^{(1+1/p)|j|_1} \, \|f_j^{\cos}\|_p \lesssim \|f\|_{S^{1+1/p}_{p,1}B([0,1]^n)} \lesssim \sum_{j \in \mathbb{N}_0^n} 2^{(1+1/p)|j|_1} \, \|f_j^{\cos}\|_p$$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Our Tool: the FABER-SCHAUDER Basis

- Define the hat function Λ on the real line by setting

$$\Lambda(x) = \max\{0, 1 - |x|\} \ .$$

- For $j \in \mathbb{N}_0^n$ and $k \in \mathbb{Z}^n$ we define

$$\Lambda_{j,k}(x) := \Lambda\big(2^{j_1}x_1 - k_1\big) \cdot \ldots \cdot \Lambda\big(2^{j_n}x_n - k_n\big) \ , \quad x = (x_1, \ldots, x_n) \in \mathbb{R}^n \ .$$

- By choosing appropriate (finite) index sets $D(j) \subset \mathbb{Z}^n$ the family $(\Lambda_{j,k})_{j\in\mathbb{N}_0^n, k\in D(j)}$ becomes a (conditional) basis for the space $C\big([0,1]^n\big)$ or $C\big(\mathbb{T}^n\big)$.
- In our context the torus is $\mathbb{T} = [-1,1]_{/\{-1,1\}}$.

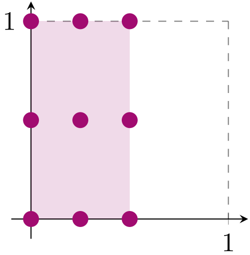
The FABER-SCHAUDER Basis

- Both on $[0,1]^n$ and on $\mathbb{T}^n$ the coefficients $d_{j,k}(f)$ of f in the basis expansion

$$f = \sum_{j \in \mathbb{N}_0^n} \sum_{k \in D(j)} d_{j,k}(f) \Lambda_{j,k}$$

are given by mixed second differences of f (point evaluation)

- Example: $[0,1]^2$, $j = (2,1)$, $k = (1,1)$



Sequence Spaces

Definition

Let $0 < p, q \leq \infty$ and $r \in \mathbb{R}$. The space $s^r_{p,q}b$ consists of all sequences $(a_{j,k})_{j\in\mathbb{N}_0^n, k\in\mathbb{Z}^n}$ of complex numbers such that

$$\left\| (a_{j,k})_{j\in\mathbb{N}_0^n, k\in\mathbb{Z}^n} \right\|^q_{s^r_{p,q}b} := \sum_{j\in\mathbb{N}_0^n} 2^{q|j|_1(r-1/p)} \left(\sum_{k\in\mathbb{Z}^n} |a_{j,k}|^p \right)^{q/p} < \infty \, .$$

Known Theorem (cf. [Triebel], [Hinrichs et al.], [Byrenheid])

Let $1/2 < p \leq \infty$, $0 < q \leq \infty$ and $1/p < r < 2$. Then there is a constant $C > 0$ such that

$$\left\| \left(d_{j,k}(f) \right)_{j,k} \right\|_{s^r_{p,q} b} \leq C \cdot \|f\|_{S^r_{p,q} B(\Omega^n)}$$

for all $f \in S^r_{p,q} B(\Omega^n)$, where Ω is either $\mathbb{T}$ or $[0,1]$.

New Result for the Limiting Case $r = 2$

Let $1/2 < p \leq \infty$. Then there is a constant $C > 0$ such that

$$\left\| \left(d_{j,k}(f) \right)_{j,k} \right\|_{s^2_{p,\infty} b} \leq C \cdot \|f\|_{S^2_{p,\min\{p,1\}} B(\Omega^n)}$$

for all $f \in S^2_{p,\min\{p,1\}} B(\Omega^n)$, where Ω is either $\mathbb{T}$ or $[0,1]$.

Known Theorem (cf. [Triebel], [Hinrichs et al.], [Byrenheid])

Let $0 < p, q \leq \infty$ and $\max\{0, 1/p - 1\} < r < 1 + 1/p$. Let further $(a_{j,k})_{j,k} \in s^r_{p,q}b$. Then the series

$$f := \sum_{j \in \mathbb{N}_0^n} \sum_{k \in D(j)} a_{j,k} \Lambda_{j,k}$$

converges unconditionally in $S^{r-\varepsilon}_{p,q}B(\Omega^n)$ (and in $S^r_{p,q}B(\Omega^n)$ if $\max\{p, q\} < \infty$). The function f belongs to $S^r_{p,q}B(\Omega^n)$ and there is a constant $C > 0$ such that

$$\|f\|_{S^r_{p,q}B(\Omega^n)} \leq C \cdot \left\| \left(a_{j,k}\right)_{j,k} \right\|_{s^r_{p,q}b} ,$$

where, again, Ω is either $\mathbb{T}$ or $[0, 1]$.

New Result for the Limiting Case $r = 1 + 1/p$

Let $0 < p \leq \infty$. Let further $(a_{j,k})_{j,k} \in s^{1+1/p}_{p,\min\{p,1\}}b$. Then the series

$$f := \sum_{j \in \mathbb{N}_0^n} \sum_{k \in D(j)} a_{j,k} \Lambda_{j,k}$$

converges unconditionally in $S^{1+1/p-\varepsilon}_{p,\infty}B(\Omega^n)$. The function f belongs to $S^{1+1/p}_{p,\infty}B(\Omega^n)$ and there is a constant $C > 0$ such that

$$\|f\|_{S^{1+1/p}_{p,\infty}B(\Omega^n)} \leq C \cdot \left\| \left(a_{j,k}\right)_{j,k} \right\|_{s^{1+1/p}_{p,\min\{p,1\}}b} ,$$

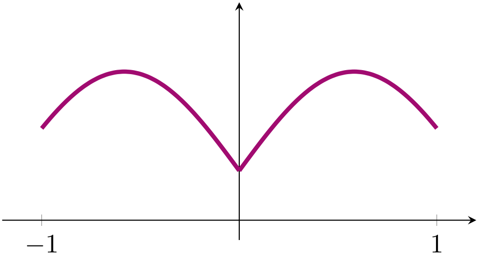
where, again, Ω is either $\mathbb{T}$ or $[0, 1]$.

A Reflection Operator

- Define a reflection operator $R : C\big([0,1]^n\big) \to C(\mathbb{T}^n)$ by setting

$$Rf(x) := f(|x_1|, \ldots, |x_n|)\ , \quad x = (x_1, \ldots, x_n) \in \mathbb{T}^n\ .$$

- Similar trick has been used by [Bočkarev] to solve a different problem.



Theorem

- Suppose $1/2 < p \le \infty$, $0 < q \le \infty$ and $1/p < r < \min\{2, 1 + 1/p\}$. Then

$$R : S^r_{p,q}B\big([0,1]^n\big) \to S^r_{p,q}B\big(\mathbb{T}^n\big) \ .$$

- In case $1 < p \le \infty$ we have

$$R : S^{1+1/p}_{p,1}B\big([0,1]^n\big) \to S^{1+1/p}_{p,\infty}B\big(\mathbb{T}^n\big) \ .$$

Corollary

Let $1/2 < r < 3/2$. Then

$$R : H^r_{\text{mix}}\big([0,1]^n\big) \to H^r_{\text{mix}}\big(\mathbb{T}^n\big) \ .$$

Final Step

- Due to symmetry:

$$
\begin{aligned}
&\|Rf\|^q_{S^r_{p,q}B(\mathbb{T}^n)} \\
&= \sum_{j\in\mathbb{N}_0^n} 2^{rq|j|_1} \left\| \sum_{k\in\mathbb{Z}^n} \varphi_{j_1}(k_1)\cdot\ldots\cdot\varphi_{j_n}(k_n)\widehat{Rf}_k\, \mathrm{e}^{\pi \mathrm{i} k\cdot\bullet} \right\|^q_{L^p(\mathbb{T}^n)} \\
&\approx \sum_{j\in\mathbb{N}_0^n} 2^{rq|j|_1} \left\| \sum_{k\in\mathbb{N}_0^n} \varphi_{j_1}(k_1)\cdot\ldots\cdot\varphi_{j_n}(k_n)\langle f, c_k\rangle_{L^2([0,1]^n)} c_k \right\|^q_{L^p([0,1]^n)} \\
&= \sum_{j\in\mathbb{N}_0^n} 2^{rq|j|_1} \left\| f_j^{\cos} \right\|^q_{L^p([0,1]^n)}
\end{aligned}
$$

Main Results

Theorem

Suppose $f \in C\big([0,1]^n\big)$.

- Let $1/2 < p \leq \infty$, $0 < q \leq \infty$ and $1/p < r < \min\{2, 1 + 1/p\}$. Then

$$\|f\|_{S^r_{p,q}B([0,1]^n)} \asymp \left(\sum_{j \in \mathbb{N}_0^n} 2^{rq|j|_1} \, \|f_j^{\cos}\|^q_{L^p([0,1]^n)} \right)^{1/q} ,$$

  with the usual modification if $q = \infty$.

- Let $1 < p \leq \infty$. Then:

$$\sup_{j \in \mathbb{N}_0^n} 2^{(1+1/p)|j|_1} \, \|f_j^{\cos}\|_p \lesssim \|f\|_{S^{1+1/p}_{p,1}B([0,1]^n)} \lesssim \sum_{j \in \mathbb{N}_0^n} 2^{(1+1/p)|j|_1} \, \|f_j^{\cos}\|_p$$

Thank you for your attention!

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