Properties and embeddings of SOBOLEV spaces of dominating mixed smoothness

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- For this talk: "dominating mixed" shortened to just "domix"
- Original idea of domix smoothness can be traced back to [BABENKO] and [NIKOL'SKII]
- Since then domix spaces appeared in many contexts
  - Statistics and probability theory [CLEANTHOUS, GEORGIADIS, PORCU]
  - Solution spaces for hyperbolic PDEs [MAMEDOV]
  - Quantum Mechanics [YSERENTANT]
  - Approximation theory [DŨNG, TEMLYAKOV, ULLRICH]

- A domain $\Omega \subset \mathbb{R}^n$ is a nonempty, open and connected set.
- For $m \in \mathbb{N}_0$ and $1 \leq p < \infty$ the (isotropic) SOBOLEV spaces on a domain Ω are defined as

$$W^{m,p}(\Omega) = \{u \in L^p(\Omega) \mid \partial^\alpha u \in L^p(\Omega) \text{ for all } \alpha \in \mathbb{N}_0^n : |\alpha|_1 \leq m\}$$

with norm $\|u\|_{W^{m,p}(\Omega)} = \sum_{0 \leq |\alpha|_1 \leq m} \|\partial^\alpha u\|_{L^p(\Omega)}$.

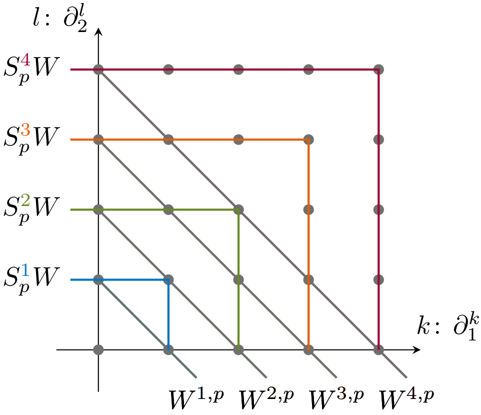
- Similarly, the domix SOBOLEV are defined as

$$S_p^m W(\Omega) = \{u \in L^p(\Omega) \mid \partial^\alpha u \in L^p(\Omega) \text{ for all } \alpha \in \mathbb{N}_0^n : |\alpha|_\infty \leq m\}$$

with norm $\|u\|_{S_p^m W(\Omega)} = \sum_{0 \leq |\alpha|_\infty \leq m} \|\partial^\alpha u\|_{L^p(\Omega)}$.

- These are *intrinsic* definitions.

Visualisation for $n = 2$



- By construction: $W^{m,p}(\Omega) \supset S_p^m W(\Omega) \supset W^{nm,p}(\Omega)$

- Literature on properties of $S_p^m W(\Omega)$ is not as expansive as for $W^{m,p}(\Omega)$.
- What usually happens in the literature:
  - $S_p^m W(\mathbb{R}^n) = S_{p,2}^m F(\mathbb{R}^n)$ for $1 < p < \infty$, then continue with $S_{p,q}^r F(\mathbb{R}^n)$ (see for example [Vybíral])
  - On Ω only *extrinsically* defined spaces

$$\left\{u \in L^p(\Omega) \;\middle|\; \exists v \in S_p^m W(\mathbb{R}^n) \text{ such that } v|_\Omega = u\right\}$$

    are considered (see for example [Abdulla]).

"Problems"

- Case $p = 1$ is rarely covered.
- Extrinsically defined spaces are usually smaller than their intrinsic counterparts.

A density problem

- [HICKERNELL, SLOAN, WASILKOWSKI] consider the spaces

$$\operatorname{clos}_{S_1^m W((0,1)^n)} W^{m,1}\big((0,1)\big) \otimes \ldots \otimes W^{m,1}\big((0,1)\big) \,.$$

- Does this construction actually yield $S_1^m W((0,1)^n)$?

Domix variant of the MEYERS-SERRIN theorem

$C^\infty(\Omega)$ is dense in $S_1^m W(\Omega)$.

Lemma [L., ULLRICH]

If $\partial\Omega$ is continuous, then the restrictions to Ω of functions in $C_c^\infty(\mathbb{R}^n)$ are also dense in $S_1^m W(\Omega)$.

Lemma [L., ULLRICH]

If $\partial\Omega$ is continuous, then restrictions to Ω of functions in the n-fold tensor product $C_c^\infty(\mathbb{R}) \otimes \ldots \otimes C_c^\infty(\mathbb{R})$ are dense in $S_1^m W(\Omega)$.

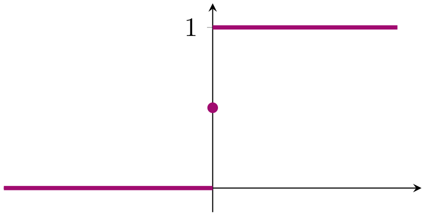
Corollary [L., ULLRICH]

In the above setting, restrictions to Ω of functions in the n-fold tensor product $W^{m,1}(\mathbb{R}) \otimes \ldots \otimes W^{m,1}(\mathbb{R})$ are dense in $S_1^m W(\Omega)$ as well.

Distribution theory lends a hand

- Consider the HEAVISIDE step function:

$$\Theta(t) = \begin{cases} 1\,, & t > 0 \\ 1/2\,, & t = 0 \\ 0\,, & t < 0 \end{cases}$$



- Define $H_1 : \mathbb{R}^n \to \mathbb{R}$ by setting $H_1(x) = \Theta(x_1) \cdot \ldots \cdot \Theta(x_n)$.
- H_1 is a fundamental solution of $\partial\!\!\!/ = \partial_1 \ldots \partial_n$, i.e. we have

$$\partial\!\!\!/ H_1 = \delta_0 \quad \text{in } \mathscr{D}'(\mathbb{R}^n)\,.$$

- Let now $u \in S_1^1 W(\Omega)$. For any $\eta \in C_c^\infty(\Omega)$ the pointwise product ηu belongs to $\mathscr{E}'(\mathbb{R}^n)$, i.e. it is a compactly supported distribution.
- By a general result of distribution theory we have

$$\eta u = H_1 * \partial\!\!\!/(\eta u) \quad \text{in } \mathscr{E}'(\mathbb{R}^n) \ .$$

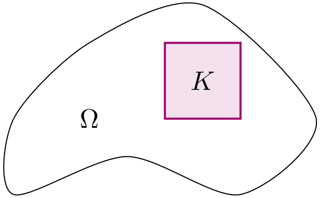
- As it turns out, this equality also holds classically, that is for a.e. $x \in \mathbb{R}^n$ we have

$$\eta(x)u(x) = H_1 * \partial\!\!\!/(\eta u)(x) = \int_{-\infty}^{x_n} \dots \int_{-\infty}^{x_1} \partial\!\!\!/(\eta u)(t)\, \mathrm{d}t_1 \dots \mathrm{d}t_n \ .$$

- It can also be shown that $H_1 * \partial\!\!\!/(\eta u) \in C(\mathbb{R}^n)$.

A simple embedding

- Consider now a compact set $K \subset \Omega$.



- By choosing $\eta \in C_c^\infty(\Omega)$ such that $\eta \equiv 1$ on K we see that

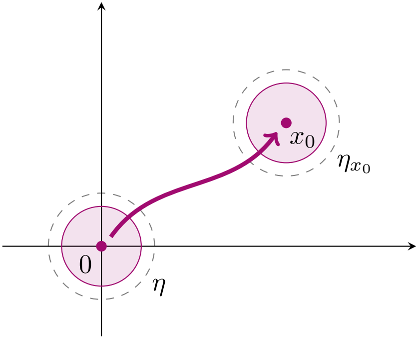
$$u = \eta u = H_1 * \bar{\partial}(\eta u) \in C(K)\ .$$

Lemma [L., ULLRICH]

$S_1^1 W(\Omega) \subset C(\Omega)$

The case $\Omega = \mathbb{R}^n$

- Fix a cutoff function $\eta \in C_c^\infty\big(B_{1/3}(0)\big)$ such that $\eta \equiv 1$ on $\overline{B_{1/4}(0)}$.
- Translate this fixed function η to get η_{x_0}.



- Let $u \in S_1^1 W(\mathbb{R}^n)$. Then:

$$\begin{aligned}
\sup_{x \in B_{1/4}(x_0)} |u(x)| &\leq \int_{B_{1/3}(x_0)} \left| \not\partial(\eta_{x_0} u) \right| \\
&\leq \sum_{0 \leq |\alpha|_\infty \leq 1} c_\alpha \int_{B_{1/3}(x_0)} \left| \partial^{1-\alpha} \eta_{x_0} \right| \cdot |\partial^\alpha u| \\
&\leq \sup_{0 \leq |\alpha|_\infty \leq 1} \left\| \partial^\alpha \eta_{x_0} \right\|_{L^\infty \left(B_{1/3}(x_0) \right)} \sum_{0 \leq |\alpha|_\infty \leq 1} c_\alpha \int_{B_{1/3}(x_0)} |\partial^\alpha u| \\
&\leq C_{n,\eta} \, \|u\|_{S_1^1 W \left(B_{1/3}(x_0) \right)} \\
&\leq C_{n,\eta} \, \|u\|_{S_1^1 W(\mathbb{R}^n)}
\end{aligned}$$

- Since

$$\sup_{x\in\mathbb{R}^n} |u(x)| = \sup_{x_0\in\mathbb{R}^n} \sup_{x\in B_{1/3}(x_0)} |u(x)| \le C_{n,\eta} \, \|u\|_{S_1^1W(\mathbb{R}^n)}$$

we can improve on the previous embedding theorem:

Theorem [L., ULLRICH]

$S_1^1W(\mathbb{R}^n) \subset C(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$

- We are able to derive a sampling inequality as well:

Lemma [L., ULLRICH]

$$\sum_{k\in\mathbb{Z}^n} |u(k)| \le C_{n,\eta} \sum_{k\in\mathbb{Z}^n} \|u\|_{S_1^1W\left(B_{1/3}(k)\right)} \le C_{n,\eta} \, \|u\|_{S_1^1W(\mathbb{R}^n)}$$

Thank you for your attention!

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References

[ABDULLA] U. G. Abdulla, "Generalized Newton-Leibniz Formula and the Embedding of the Sobolev Functions into Hölder Spaces", *arXiv preprint arXiv:2101.09132*, 2022.
[BABENKO] K. I. Babenko, "Approximation of a certain class of periodical functions of many variables by trigonometric polynomials", *Dokl. Akad. Nauk SSSR*, 1960, Volume 132, Number 5, 982–985.
[CLEANTHOUS, GEORGIADIS, PORCU] G. Cleanthous, A. G. Georgiadis & E. Porcu, "Minimax density estimation on Sobolev spaces with dominating mixed smoothness", *arXiv preprint arXiv:1906.06835*, 2019.
[DŨNG, TEMLYAKOV, ULLRICH] D. Dũng, V. Temlyakov & T. Ullrich, "Hyperbolic cross approximation", Advanced Courses in Mathematics. CRM Barcelona. Birkhäuser/Springer, Cham, 2018. Edited and with a foreword by Sergey Tikhonov.
[HICKERNELL, SLOAN, WASILKOWSKI] F. Hickernell, I. Sloan & G. Wasilkowski, "On tractability of weighted integration over bounded and unbounded regions in $\mathbb{R}^s$". *Mathematics of Computation*, 2004, 73(248), 1885-1901.
[L., ULLRICH] K. Lüttgen, T. Ullrich, "Sobolev spaces with dominating mixed smoothness for $p = 1$", *in preparation*, 2023.
[MAMEDOV] I. G. Mamedov, "On the well-posed solvability of the Dirichlet problem for a generalized Mangeron equation with nonsmooth coefficients", *Differential Equations*, 2015, 51(6), 745-754.
[NIKOL'SKII] S.M. Nikol'skii, "Functions with dominant mixed derivative, satisfying a multiple Hölder condition", *Sibirsk. Mat. Zh.*, 4:6 (1963), 1342–1364.
[VYBÍRAL] J. Vybíral, "Characterization of function spaces with dominating mixed smoothness properties", Jenaer Schriften Math. u. Inf., Math/Inf/15/03, 1–42.
[YSERENTANT] H. Yserentant, "On the regularity of the electronic Schrödinger equation in Hilbert spaces of mixed derivatives", Numer. Math. 98 (2004), no. 4, 731–759. Zbl 1062.35100, 2005j:35032.