RECONSTRUCTING HYPERBOLIC CROSS TRIGONOMETRIC POLYNOMIALS BY SAMPLING ALONG RANK-1 LATTICES

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Abstract. The evaluation of multivariate trigonometric polynomials at the nodes of a rank-1 lattice leads to a onedimensional discrete Fourier transform. Often, one is also interested in the reconstruction of the Fourier coefficients from their samples. We present necessary and sufficient conditions on rank-1 lattices allowing a stable reconstruction of trigonometric polynomials supported on hyperbolic crosses. In addition, we suggest approaches for determining suitable rank-1 lattices using a component-by-component algorithm. We present numerical results for reconstructing trigonometric polynomials up to spatial dimension 100.

Key words. trigonometric approximation, hyperbolic cross, lattice rule, fast Fourier transform

AMS subject classifications. 65T40, 42B05, 65D30.

1. Introduction. Full grid discretisations of problems in d spatial dimensions lead to an exponential growth in the number of degrees of freedom. Hence, even an efficient algorithm like the fast Fourier transform (FFT) suffers from the curse of dimensionality. For moderately high dimensional problems the approximation with trigonometric polynomials with frequencies supported on hyperbolic crosses decreases the problem sizes strongly. In addition, many applications allow an arrangement of the different dimensions in descending order according to their importance. In other words, we assume that the components of the variable $\boldsymbol{x} = (x_1, \ldots, x_d)$ are ordered being x_1 the most important.

As discretisation in the frequency domain we consider so called *weighted symmetric hyperbolic crosses*

$$H_N^{d,\boldsymbol{\gamma}} := \left\{ \boldsymbol{k} \in \mathbb{Z}^d : \prod_{s=1}^{d} \max\left(1, \frac{|k_s|}{\gamma_s}\right) \leq N \right\}$$

with $N \in \mathbb{R}$, $N \geq 1$, $d \in \mathbb{N}$, and weights $\boldsymbol{\gamma} = (\gamma_s)_{s\in\mathbb{N}} \subset \mathbb{R}$, $1 \geq \gamma_1 \geq \gamma_2 \geq \ldots \geq 0$. The sequence of weights $\boldsymbol{\gamma}$ characterises the importance of the corresponding components of $\boldsymbol{x}$. In the case $\gamma_s = 0$, all components x_j with $j \in \mathbb{N}, j \geq s$, are of no relevance and we set

$$\frac{|k_j|}{\gamma_j} := \begin{cases} 0, & \text{for } k_j = 0, \\ \infty, & \text{for } k_j \neq 0. \end{cases}$$

Often the frequency grids $H_N^{d,\boldsymbol{\gamma}}$ are called weighted Zaremba crosses in the context of numerical integration.

The natural spatial discretisation corresponding to $H_N^{d,\boldsymbol{\gamma}}$ are sparse grids. In general, the evaluation of trigonometric polynomials with frequencies supported on weighted hyperbolic crosses $H_N^{d,\boldsymbol{\gamma}}$ at all sparse grid nodes and the reconstruction of the trigonometric polynomial from the samples at the sparse grid nodes do not provide stability. More precisely, the corresponding Fourier matrices suffers from growing condition numbers, which implicates a loss of accuracy, cf. [7]. Consequently, we look for a stable spatial discretisation here.

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Throughout this paper we make no distinction between row and column vectors. In particular, the product $\boldsymbol{a} \cdot \boldsymbol{b} = \sum_{s=1}^{d} a_s b_s$ of two d-dimensional vectors $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{R}^d$ denotes the corresponding scalar product.

In order to reconstruct multivariate trigonometric polynomials

$$f(\boldsymbol{x}) = \sum_{\boldsymbol{h} \in H_N^{d,\boldsymbol{\gamma}}} \hat{f}_{\boldsymbol{h}} \mathrm{e}^{2\pi\mathrm{i}\boldsymbol{h}\cdot\boldsymbol{x}}$$

we have to reconstruct all involved Fourier coefficients

$$\hat{f}_{\boldsymbol{h}} := \int_{\boldsymbol{x}\in[0,1)^d} f(\boldsymbol{x}) \mathrm{e}^{-2\pi\mathrm{i}\boldsymbol{h}\cdot\boldsymbol{x}} d\boldsymbol{x} = \sum_{\boldsymbol{k} \in H_N^{d,\boldsymbol{\gamma}}} \hat{f}_{\boldsymbol{h}} \int_{\boldsymbol{x}\in[0,1)^d} \mathrm{e}^{2\pi\mathrm{i}(\boldsymbol{k}-\boldsymbol{h})\cdot\boldsymbol{x}} d\boldsymbol{x}$$

exactly. We want to do this by sampling the trigonometric polynomial f. In the words of numerical integration, we construct cubature formulas that integrates all trigonometric polynomials with frequencies supported on the *difference set*

$$\mathcal{H}_N^{d,\boldsymbol{\gamma}} := \left\{ \boldsymbol{h} - \boldsymbol{k} \in \mathbb{Z}^d : \ \boldsymbol{h}, \boldsymbol{k} \in H_N^{d,\boldsymbol{\gamma}} \right\}$$

exactly. Furthermore, the fast evaluation and the fast reconstruction of the considered multivariate trigonometric polynomials are of our interest. For that reason, we restrict ourself to *rank-1 lattices* in this paper. We take advantage of their useful structure. The evaluation of multivariate trigonometric polynomials at all nodes of a rank-1 lattice simplifies to a onedimensional FFT if the Fourier coefficients $\hat{f}_{\boldsymbol{h}}$ are given. We address the problem of the reconstruction of $\hat{f}_{\boldsymbol{h}}$ from samples on a rank-1 lattice. In Corollary 2.4 we prove that $\lfloor \gamma_1 N \rfloor \lfloor \gamma_2 N \rfloor$ samples are necessary for the reconstruction and give a constructive proof for a reconstruction with approximately $c_{d,\boldsymbol{\gamma}} N^2 \log^{d-2} N$ points, see Theorem 3.2 and Corollary 4.8 for details.

The paper is organised as follows: In Section 2 we introduce the necessary notation and collect some basic facts about rank-1 lattices as spatial discretisation for hyperbolic cross trigonometric polynomials. In Section 3 we show that there exists a rank-1 lattice of relatively small size allowing the exact integration of trigonometric polynomials with frequencies supported on the difference set $\mathcal{H}_N^{d,\boldsymbol{\gamma}}$. This sampling scheme allows a perfectly stable reconstruction of trigonometric polynomials with frequencies supported on the weighted hyperbolic cross $H_N^{d,\boldsymbol{\gamma}}$. The constructive proof describes a component-by-component algorithm. We specify this algorithm in detail. Moreover, we present a simple algorithm to reduce the cardinality of our sampling set while retaining the desired properties. The result of Section 3 mainly depends on the cardinalities of the difference sets $\mathcal{H}_N^{d,\boldsymbol{\gamma}}$. For that reason, we consider these sets and especially their cardinalities in Section 4 in detail. Section 5 compares the results of this paper with known results of random sampling concerning oversampling, stability, and fast computation. Each section contains at least one example.

2. Prerequisite. Let a spatial dimension $d \in \mathbb{N}$ be given. We consider periodic functions f mapping from the d-dimensional torus $[0,1)^d$ in the complex numbers $\mathbb{C}$, $f(\boldsymbol{x}) = \sum_{\boldsymbol{k}\in\mathbb{Z}^d} \hat{f}_{\boldsymbol{k}} \mathrm{e}^{2\pi\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}}$, with Fourier coefficients $\hat{f}_{\boldsymbol{k}} \in \mathbb{C}$. All such functions with Fourier coefficients supported on finite sets are trigonometric polynomials. For a fixed index set $I \subset \mathbb{Z}^d$ with a finite cardinality $|I|$ we call $\Pi_I = \operatorname{span}\{\mathrm{e}^{2\pi\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}} : \ \boldsymbol{k} \in I\}$ the space of trigonometric polynomials supported on I.

Assuming I is a suitable discretisation in frequency domain for approximating functions, e.g. functions with dominating mixed smoothness, cf. [9, 11], we are interested in evaluating the corresponding trigonometric polynomials at sampling nodes and reconstructing the Fourier coefficients from samples.

In this paper we focus on trigonometric polynomials with frequencies supported on hyperbolic crosses $H_N^{d,\boldsymbol{\gamma}}$. For $d \in \mathbb{N}$, $N_1, N_2 \in \mathbb{R}$, $N_1 \leq N_2$, and $\boldsymbol{\gamma}$ like above, the inclusion $H_{N_1}^{d,\boldsymbol{\gamma}} \subset H_{N_2}^{d,\boldsymbol{\gamma}}$ obviously holds.

2.1. Rank-1 lattices. For given $M \in \mathbb{N}$ and $\boldsymbol{z} \in \mathbb{Z}^d$ we define the rank-1 lattice

$$\Lambda(\boldsymbol{z}, M) := \left\{\boldsymbol{x}_j = M^{-1} j \boldsymbol{z} \bmod 1, j = 0, \dots, M-1\right\}, \text{ cf. [1, 2, 12, 13].}$$

The evaluation of the trigonometric polynomial $f \in \Pi_{H_N^{d,\boldsymbol{\gamma}}}$ at the nodes $\boldsymbol{x}_j \in \Lambda(\boldsymbol{z}, M)$ simplifies to a onedimensional discrete Fourier transform

$$f(\boldsymbol{x}_j) = \sum_{\boldsymbol{k} \in H_N^{d,\boldsymbol{\gamma}}} \hat{f}_{\boldsymbol{k}} \mathrm{e}^{2\pi \mathrm{i} j \frac{\boldsymbol{k}\cdot\boldsymbol{z}}{M}} = \sum_{l=0}^{M-1} \Big(\sum_{\boldsymbol{k}\cdot\boldsymbol{z} \equiv l \pmod{M}} \hat{f}_{\boldsymbol{k}} \Big) e^{2\pi \mathrm{i} \frac{jl}{M}}, \qquad j = 0, \dots, M-1.$$

One evaluates f at all nodes $\boldsymbol{x}_j \in \Lambda(\boldsymbol{z}, M)$, $j = 0, \dots, M-1$, by the precomputation of all $\hat{g}_l := \sum_{\boldsymbol{k}\cdot\boldsymbol{z} \equiv l \pmod{M}} \hat{f}_{\boldsymbol{k}}$ and a onedimensional fast Fourier transform in $C(M \log M + d|H_N^{d,\boldsymbol{\gamma}}|)$ floating point operations with a constant C that does not depend on the spatial dimension d. Hence, a fast evaluation of trigonometric polynomials at all sampling nodes $\boldsymbol{x}_j$ of the rank-1 lattice $\Lambda(\boldsymbol{z}, M)$ is guaranteed.

So we shift our attention to the reconstruction of a trigonometric polynomial f with frequencies supported on $H_N^{d,\boldsymbol{\gamma}}$ from function values at the nodes $\boldsymbol{x}_j$ of a rank-1 lattice $\Lambda(\boldsymbol{z}, M)$. We consider the corresponding Fourier matrix $\boldsymbol{A}$ and its adjoint $\boldsymbol{A}^*$,

$$\boldsymbol{A} := \left(\mathrm{e}^{2\pi \mathrm{i} \boldsymbol{k}\cdot\boldsymbol{x}}\right)_{\boldsymbol{x} \in \Lambda(\boldsymbol{z}, M),\, \boldsymbol{k} \in H_N^{d,\boldsymbol{\gamma}}} \quad \text{and} \quad \boldsymbol{A}^* := \left(\mathrm{e}^{-2\pi \mathrm{i} \boldsymbol{k}\cdot\boldsymbol{x}}\right)_{\boldsymbol{k} \in H_N^{d,\boldsymbol{\gamma}},\, \boldsymbol{x} \in \Lambda(\boldsymbol{z}, M)},$$

to conclude necessary and sufficient conditions on rank-1 lattices $\Lambda(\boldsymbol{z}, M)$ allowing a unique reconstruction of the Fourier coefficients $\hat{f}_{\boldsymbol{k}}$, $\boldsymbol{k} \in H_N^{d,\boldsymbol{\gamma}}$. In particular, we purpose to find rank-1 lattices $\Lambda(\boldsymbol{z}, M)$ that allow even a stable reconstruction of the Fourier coefficients of specific trigonometric polynomials. We generalise some known results from [8] in the following two lemmas.

LEMMA 2.1. *Let $N \in \mathbb{R}$, $N \geq 1$, $d \in \mathbb{N}$, $\boldsymbol{\gamma}$ like above, and $\mathcal{X} = \{\boldsymbol{x}_j,\ j = 0, \dots, M-1\} \subset [0,1)^d$ an arbitrary set of sampling nodes. In order to obtain orthogonal columns in $\boldsymbol{A} = \left(\mathrm{e}^{2\pi \mathrm{i} \boldsymbol{k}\cdot\boldsymbol{x}}\right)_{\boldsymbol{x} \in \mathcal{X};\, \boldsymbol{k} \in H_N^{d,\boldsymbol{\gamma}}}$, i.e. $\boldsymbol{A}^*\boldsymbol{A} = M\boldsymbol{I}$, one needs at least $M = \lfloor \gamma_1 N \rfloor \lfloor \gamma_2 N \rfloor$ different sampling nodes in $\mathcal{X}$.*

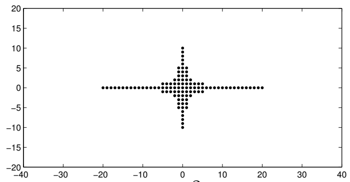
Proof. Let $\mathcal{X}$ be an arbitrary sampling scheme. $\boldsymbol{A}^*\boldsymbol{A} = M\boldsymbol{I}$ reads as

$$\sum_{j=0}^{M-1} \mathrm{e}^{2\pi \mathrm{i} (\boldsymbol{k}-\boldsymbol{l})\cdot\boldsymbol{x}_j} = M \delta_{\boldsymbol{k}-\boldsymbol{l}}, \qquad \text{for all } \boldsymbol{k}, \boldsymbol{l} \in H_N^{d,\boldsymbol{\gamma}}. \tag{2.1}$$

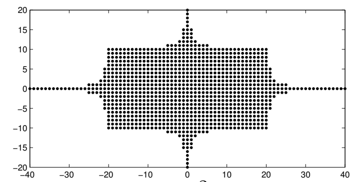
We follow the proof of [8, Theorem 3.5] and consider the two-dimensional case $d = 2$, see Figure 2.1 for an illustrating example. The set of differences of two elements of the hyperbolic cross fulfils

$$\begin{aligned} \mathcal{H}_N^{2,\boldsymbol{\gamma}} &:= \{\boldsymbol{k}-\boldsymbol{l} :\ \boldsymbol{k}, \boldsymbol{l} \in H_N^{2,\boldsymbol{\gamma}}\} \supset [-\lfloor \gamma_1 N \rfloor, \lfloor \gamma_1 N \rfloor] \times [-\lfloor \gamma_2 N \rfloor, \lfloor \gamma_2 N \rfloor] \cap \mathbb{Z}^2 \\ &= \{\boldsymbol{k}-\boldsymbol{l} :\ \boldsymbol{k}, \boldsymbol{l} \in \hat{G}_N^{2,\boldsymbol{\gamma}}\}, \text{ with } \hat{G}_N^{2,\boldsymbol{\gamma}} = \mathop{\times}_{j=1}^{2} \left(\left[-\left\lfloor 2^{-1} \lfloor \gamma_j N \rfloor\right\rfloor, \left\lceil 2^{-1} \lfloor \gamma_j N \rfloor\right\rceil\right] \cap \mathbb{Z}\right). \end{aligned}$$

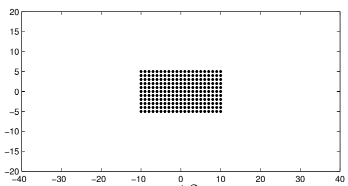
Obviously, equation (2.1) have to hold for all $\boldsymbol{k}, \boldsymbol{l} \in \hat{G}_N^{2,\boldsymbol{\gamma}}$. In matrix notation we get $\tilde{\boldsymbol{A}}^* \tilde{\boldsymbol{A}} = M\boldsymbol{I}$ with $\tilde{\boldsymbol{A}} = \left(\mathrm{e}^{2\pi \mathrm{i} \boldsymbol{k}\cdot\boldsymbol{x}_j}\right)_{j=0,\dots,M-1;\, \boldsymbol{k} \in \hat{G}_N^{2,\boldsymbol{\gamma}}}$. In order to obtain a full column



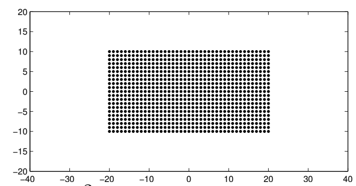
(a) $H_N^{2,\boldsymbol{\gamma}}$



(b) $\mathcal{H}_N^{2,\boldsymbol{\gamma}}$

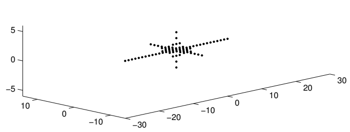


(d) $\hat{G}_N^{2,\boldsymbol{\gamma}}$

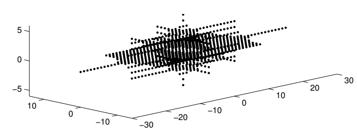


(c) $\bigtimes_{j=1}^{2} \left([-\lfloor \gamma_j N \rfloor, \lfloor \gamma_j N \rfloor] \cap \mathbb{Z} \right)$

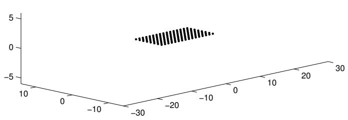
Fig. 2.1. *The set* $H_N^{2,\boldsymbol{\gamma}}$ *generates* $\mathcal{H}_N^{2,\boldsymbol{\gamma}}$. *Its subset* $\bigtimes_{j=1}^{2} \left([-\lfloor \gamma_j N \rfloor, \lfloor \gamma_j N \rfloor] \cap \mathbb{Z} \right)$ *can also be generated by all possible differences of two elements of the set* $\hat{G}_N^{2,\boldsymbol{\gamma}}$. *The figures show the corresponding sets for* $N = 40$ *and* $\boldsymbol{\gamma} = (\frac{1}{2}, \frac{1}{4}, 0, \ldots)$.



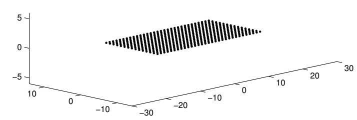
(a) $H_N^{3,\boldsymbol{\gamma}}$



(b) $\mathcal{H}_N^{3,\boldsymbol{\gamma}}$



(d) $\hat{G}_N^{2,\boldsymbol{\gamma}} \times \{0\}$



(c) $\left(\bigtimes_{j=1}^{2} \left([-\lfloor \gamma_j N \rfloor, \lfloor \gamma_j N \rfloor] \cap \mathbb{Z} \right) \right) \times \{0\}$

Fig. 2.2. *The set* $\mathcal{H}_N^{3,\boldsymbol{\gamma}}$ *is based on* $H_N^{3,\boldsymbol{\gamma}}$ *like above. Its subset* $\left(\bigtimes_{j=1}^{2} \left([-\lfloor \gamma_j N \rfloor, \lfloor \gamma_j N \rfloor] \cap \mathbb{Z} \right) \right) \times \{0\}$ *can also be generated by all differences of two elements of the set* $\hat{G}_N^{2,\boldsymbol{\gamma}} \times \{0\}$. *The figures illustrate the sets for* $N = 30$ *and* $\boldsymbol{\gamma} = (\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, 0, \ldots)$.

rank matrix $\tilde{\boldsymbol{A}}$ the cardinality M of the sampling set $\mathcal{X}$ has to fulfil $M \geq |\hat{G}_N^{2,\boldsymbol{\gamma}}| = \lfloor \gamma_1 N \rfloor \lfloor \gamma_2 N \rfloor$. The inclusions

$$\mathcal{H}_N^{d,\boldsymbol{\gamma}} \supset \{\boldsymbol{k} - \boldsymbol{l} : \; \boldsymbol{k}, \boldsymbol{l} \in H_N^{2,\boldsymbol{\gamma}} \times \{0\}^{d-2}\} \supset \{\boldsymbol{k} - \boldsymbol{l} : \; \boldsymbol{k}, \boldsymbol{l} \in \hat{G}_N^{2,\boldsymbol{\gamma}} \times \{0\}^{d-2}\}$$

yield the assertion for spatial dimensions $d > 2$. Figure 2.2 shows an example for the spatial dimension $d = 3$. ◻

Remark 2.2. Note that in fact tools of the proof of Lemma 2.1 can be generalised to arbitrary index sets $I \subset \mathbb{Z}^d$ in place of $H_N^{d,\boldsymbol{\gamma}}$. The strategy is to find an index set $\tilde{\mathcal{I}} \subset \mathcal{I} = \{\boldsymbol{k} - \boldsymbol{l} : \; \boldsymbol{k}, \boldsymbol{l} \in I\}$ with $\tilde{\mathcal{I}} = \{\boldsymbol{k} - \boldsymbol{l} : \; \boldsymbol{k}, \boldsymbol{l} \in \tilde{I}\}$ and a cardinality of $\tilde{I}$ as large as possible. Figures 2.1 and 2.2 illustrate this strategy applied to weighted hyperbolic crosses of dimension $d = 2$ and $d = 3$. In contrast to our result, in general, the dimensionality of the index sets $\tilde{\mathcal{I}}$ and $\tilde{I}$ can be larger than two. ◻

Lemma 2.3. *Let* $d \in \mathbb{N}$, $N \in \mathbb{R}$, $N \geq 1$, $\boldsymbol{\gamma}$ *like above, and* $\Lambda(\boldsymbol{z}, M)$ *a rank-1* *lattice. The Fourier matrix* $\boldsymbol{A} = \left(\mathrm{e}^{2\pi\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}}\right)_{\boldsymbol{x}\in\Lambda(\boldsymbol{z},M);\ \boldsymbol{k}\in H_N^{d,\boldsymbol{\gamma}}}$ *fulfils either* $\boldsymbol{A}^*\boldsymbol{A} = M\boldsymbol{I}$*, or* $\boldsymbol{A}^*\boldsymbol{A}$ *is rank deficient.*

Proof. Let $H_N^{d,\boldsymbol{\gamma}} \subset \mathbb{Z}^d$ be a weighted hyperbolic cross and $\Lambda(\boldsymbol{z}, M)$ a rank-1 lattice. We consider the corresponding Fourier matrix $\boldsymbol{A} = \left(\mathrm{e}^{2\pi\mathrm{i}j\frac{\boldsymbol{k}\cdot\boldsymbol{z}}{M}}\right)_{j=0,\dots,M-1;\ \boldsymbol{k}\in H_N^{d,\boldsymbol{\gamma}}}$. If there exist two elements $\boldsymbol{k}, \boldsymbol{k}' \in H_N^{d,\boldsymbol{\gamma}}$ with $\boldsymbol{k} \neq \boldsymbol{k}'$ and $\boldsymbol{k}\cdot\boldsymbol{z} \equiv \boldsymbol{k}'\cdot\boldsymbol{z} \pmod M$, the matrix $\boldsymbol{A}$ contains at least two identical columns and has not full column rank and so $\operatorname{rank}(\boldsymbol{A}^*\boldsymbol{A}) < |H_N^{d,\boldsymbol{\gamma}}|$. On the other hand, we assume that $\boldsymbol{k}\cdot\boldsymbol{z} \not\equiv \boldsymbol{k}'\cdot\boldsymbol{z} \pmod M$ for all $\boldsymbol{k}, \boldsymbol{k}' \in H_N^{d,\boldsymbol{\gamma}}$ with $\boldsymbol{k} \neq \boldsymbol{k}'$. With $\boldsymbol{z} \in \mathbb{Z}^d$ we obtain

$$(\boldsymbol{A}^*\boldsymbol{A})_{\boldsymbol{k},\boldsymbol{k}'\in H_N^{d,\boldsymbol{\gamma}}} = \sum_{j=0}^{M-1} \mathrm{e}^{2\pi\mathrm{i}\frac{j(\boldsymbol{k}'-\boldsymbol{k})\cdot\boldsymbol{z}}{M}} = \begin{cases} M, & \text{for } \boldsymbol{k} = \boldsymbol{k}', \\ 0, & \text{else.} \end{cases} \qquad \square$$

We summarise the last two lemmas.

COROLLARY 2.4. *Using a rank-1 lattice as sampling scheme for reconstructing trigonometric polynomials with Fourier coefficients supported on hyperbolic crosses* $H_N^{d,\boldsymbol{\gamma}}$ *we need at least* $\lfloor\gamma_1 N\rfloor\, \lfloor\gamma_2 N\rfloor$ *sampling points. Once we have found a rank-1 lattice allowing this reconstruction, the computation is perfectly stable.*

EXAMPLE 2.5. We consider the hyperbolic cross $H_2^{d,\boldsymbol{\gamma}}$ with $\boldsymbol{\gamma} = \left(\frac{1}{2}\right)_{s\in\mathbb{N}}$. Beside its definition this frequency set fulfils $H_2^{d,\boldsymbol{\gamma}} = \left\{\boldsymbol{h} \in \mathbb{Z}^d :\ \sum_{s=1}^d |h_s| \leq 1\right\}$ and $|H_2^{d,\boldsymbol{\gamma}}| = 2d+1$. Consequently, we consider trigonometric polynomials of trigonometric degree 1, and the difference set $\mathcal{H}_2^{d,\boldsymbol{\gamma}} = \left\{\boldsymbol{h} \in \mathbb{Z}^d :\ \sum_{s=1}^d |h_s| \leq 2\right\}$ with $|\mathcal{H}_2^{d,\boldsymbol{\gamma}}| = 2d(d+1)+1$ can be interpreted as the frequency set of trigonometric polynomials of trigonometric degree 2. Following [2, Theorem 3.1], the rank-1 lattice $\Lambda(\boldsymbol{z}, M)$ with $M = 2d+1$ and $\boldsymbol{z} = (1, 2, \dots, d)$ exactly integrates all trigonometric polynomials with frequencies supported on $\mathcal{H}_2^{d,\boldsymbol{\gamma}}$. So all Fourier coefficients of trigonometric polynomials with frequencies supported on $H_2^{d,\boldsymbol{\gamma}}$ can be reconstructed by sampling along $\Lambda(\boldsymbol{z}, M)$. The corresponding Fourier matrix is a square matrix and contains orthogonal columns. Accordingly, we obtain a unitary discrete Fourier transform up to normalisation. $\square$

3. A component-by-component proof. In this section we apply some results of numerical integration. In particular, we formulate a constructive theorem. Its proof describes a component-by-component construction of a rank-1 lattice $\Lambda(\boldsymbol{z}, M)$ that exactly integrates all trigonometric polynomials with frequencies supported on

$$\mathcal{H}_N^{d,\boldsymbol{\gamma}} = \{\boldsymbol{l} \in \mathbb{Z}^d : \boldsymbol{l} = \boldsymbol{k}_1 - \boldsymbol{k}_2;\ \boldsymbol{k}_1, \boldsymbol{k}_2 \in H_N^{d,\boldsymbol{\gamma}}\},$$

cf. [1, Theorem 3]. This difference set contains the frequency supports of all

$$f_{\boldsymbol{h}}(\cdot) := f(\cdot)\mathrm{e}^{-2\pi\mathrm{i}\boldsymbol{h}\cdot(\cdot)}, \quad f \in \Pi_{H_N^{d,\boldsymbol{\gamma}}},\ \boldsymbol{h} \in H_N^{d,\boldsymbol{\gamma}}.$$

By integrating $f_{\boldsymbol{h}}$ we gain the Fourier coefficient $\hat{f}_{\boldsymbol{h}}$ from f. Consequently we get an exact integration of all functions $f_{\boldsymbol{h}}$ by the lattice rule based on $\Lambda(\boldsymbol{z}, M)$ and an exact reconstruction of the Fourier coefficients of f, respectively.

To exactly integrate all trigonometric polynomials $f \in \Pi_{\mathcal{H}_N^{d,\boldsymbol{\gamma}}}$ the rank-1 lattice has to fulfil the condition $0 \notin \{\boldsymbol{k}\cdot\boldsymbol{z} \bmod M :\ \boldsymbol{k} \in \mathcal{H}_N^{d,\boldsymbol{\gamma}} \setminus \{\boldsymbol{0}\}\}$, cf. [13]. This is equivalent to $\boldsymbol{k}\cdot\boldsymbol{z} \not\equiv \boldsymbol{k}'\cdot\boldsymbol{z} \pmod M$, for all $\boldsymbol{k}, \boldsymbol{k}' \in H_N^{d,\boldsymbol{\gamma}},\ \boldsymbol{k} \neq \boldsymbol{k}'$.

LEMMA 3.1. *Let* $d \in \mathbb{N}$*,* $d \geq 2$*, and* $N \in \mathbb{R}$*. We obtain the following identity*

$$\left\{\boldsymbol{l} \in \mathcal{H}_N^{d,\boldsymbol{\gamma}} :\ l_d = 0\right\} = \left\{(l_1, \dots, l_{d-1}, 0)^\top \in \mathbb{Z}^d :\ (l_j)_{j=1}^{d-1} \in \mathcal{H}_N^{d-1,\boldsymbol{\gamma}}\right\}.$$

Proof. We note

$$\begin{aligned}
&\left\{\boldsymbol{l} \in \mathcal{H}_N^{d,\boldsymbol{\gamma}} :\ l_d = 0\right\} = \left\{\boldsymbol{k}_1 - \boldsymbol{k}_2 \in \mathbb{Z}^d :\ \boldsymbol{k}_1, \boldsymbol{k}_2 \in H_N^{d,\boldsymbol{\gamma}};\ k_{1,d} = k_{2,d}\right\} \\
&= \bigcup_{k_{1,d}=-\lfloor \gamma_d N \rfloor}^{\lfloor \gamma_d N \rfloor} \left\{\boldsymbol{k}_1 - \boldsymbol{k}_2 \in \mathbb{Z}^d :\ (k_{i,j})_{j=1}^{d-1} \in H^{d-1,\boldsymbol{\gamma}}_{\frac{N}{\max\left(1, \gamma_d^{-1} k_{1,d}\right)}};\ i = 1, 2;\ k_{1,d} = k_{2,d}\right\} \\
&= \left\{\boldsymbol{k}_1 - \boldsymbol{k}_2 \in \mathbb{Z}^d :\ (k_{i,j})_{j=1}^{d-1} \in H_N^{d-1,\boldsymbol{\gamma}};\ i = 1, 2;\ k_{1,d} - k_{2,d} = 0\right\} \\
&= \left\{\boldsymbol{l} \in \mathbb{Z}^d :\ (l_j)_{j=1}^{d-1} \in \mathcal{H}_N^{d-1,\boldsymbol{\gamma}};\ l_d = 0\right\}. \qquad \square
\end{aligned}$$

We denote $(a_1, \ldots, a_{d-1}, b) = (\boldsymbol{a}, b) \in \mathbb{R}^d$ for $d \in \mathbb{N}$, $d \geq 2$, $\boldsymbol{a} \in \mathbb{R}^{d-1}$, $b \in \mathbb{R}$, and formulate the theorem of this section.

THEOREM 3.2. *Let the dimension $d \in \mathbb{N}$, $d \geq 2$, $N \in \mathbb{R}$, $\boldsymbol{\gamma}$ like above, and $M \in \mathbb{N}$ be a prime satisfying*

$$M \geq \frac{|\mathcal{H}_N^{d,\boldsymbol{\gamma}}| - |\mathcal{H}_N^{d-1,\boldsymbol{\gamma}}| - 4\lfloor \gamma_d N \rfloor + 4}{2}$$

and assume there exists a rank-1 lattice $\Lambda(\boldsymbol{z}^*, M)$ with $\boldsymbol{z}^* \in \mathbb{Z}^{d-1}$ and

$$\boldsymbol{h} \cdot \boldsymbol{z}^* \not\equiv 0 \pmod{M} \text{ for all } \boldsymbol{h} \in \mathcal{H}_N^{d-1,\boldsymbol{\gamma}} \setminus \{\boldsymbol{0}\}.$$

Then there exists a $z_d \in \{1, \ldots, M-1\}$ such that

$$(\boldsymbol{h}, h_d) \cdot (\boldsymbol{z}^*, z_d) \not\equiv 0 \pmod{M} \text{ for all } (\boldsymbol{h}, h_d) \in \mathcal{H}_N^{d,\boldsymbol{\gamma}} \setminus \{\boldsymbol{0}\}.$$

Proof. We adapt the proof of [1, Theorem 1] to our needs.
Let us assume that

$$\boldsymbol{h} \cdot \boldsymbol{z}^* \not\equiv 0 \pmod{M} \text{ for all } \boldsymbol{h} \in \mathcal{H}_N^{d-1,\boldsymbol{\gamma}} \setminus \{\boldsymbol{0}\}.$$

Now we determine an upper bound for the number of elements $z_d \in \mathbb{Z}_M^*$ with

$$\begin{aligned}
(\boldsymbol{h}, h_d) \cdot (\boldsymbol{z}^*, z_d) \equiv 0 \pmod{M} \text{ for at least one } (\boldsymbol{h}, h_d) \in \mathcal{H}_N^{d,\boldsymbol{\gamma}} \setminus \{\boldsymbol{0}\} \qquad &\text{or} \\
\boldsymbol{h} \cdot \boldsymbol{z}^* \equiv -h_d z_d \pmod{M} \text{ for at least one } (\boldsymbol{h}, h_d) \in \mathcal{H}_N^{d,\boldsymbol{\gamma}} \setminus \{\boldsymbol{0}\},&
\end{aligned}$$

equivalently. Like in [1] we consider three cases.

$h_d = 0$: $\boldsymbol{h} \cdot \boldsymbol{z}^* \equiv -0 z_d$ never holds because of $\boldsymbol{h} \cdot \boldsymbol{z}^* \not\equiv 0 \pmod{M}$ for all $\boldsymbol{h} \in \mathcal{H}_N^{d-1,\boldsymbol{\gamma}} \setminus \{\boldsymbol{0}\}$.

$\boldsymbol{h} = \boldsymbol{0}$: We obtain $|h_d|, z_d \in \mathbb{Z}_M^*$ and M prime. Thus, M cannot be a prime factor of $h_d z_d \in \mathbb{Z}$. So the conditions $\boldsymbol{0} \cdot \boldsymbol{z}^* \equiv -h_d z_d \pmod{M}$ never holds. The number of elements of $\mathcal{H}_N^{d,\boldsymbol{\gamma}} \setminus \{\boldsymbol{0}\}$ of that type is $|\{k_1 - k_2 : k_1, k_2 \in \mathbb{Z} \cap [-\lfloor \gamma_d N \rfloor, \lfloor \gamma_d N \rfloor], k_1 \neq k_2\}| = |\{-2\lfloor \gamma_d N \rfloor, \ldots, 2\lfloor \gamma_d N \rfloor\} \setminus \{0\}| = 4\lfloor \gamma_d N \rfloor$.

else: Since M is prime, $h_d \not\equiv 0 \pmod{M}$, and $\boldsymbol{h} \cdot \boldsymbol{z}^* \not\equiv 0 \pmod{M}$ there is exactly one $z_d \in \mathbb{Z}_M^*$ that fulfils $\boldsymbol{h} \cdot \boldsymbol{z}^* \equiv -h_d z_d \pmod{M}$. Due to the symmetry of the index set $\{(\boldsymbol{h}, h_d) \in \mathcal{H}_N^{d,\boldsymbol{\gamma}} \setminus \{\boldsymbol{0}\} : \boldsymbol{h} \neq \boldsymbol{0} \text{ and } h_d \neq 0\}$ we have to count only one z_d for the two elements $(\boldsymbol{h}, h_d), -(\boldsymbol{h}, h_d)$.

Hence, we have at most

$$\frac{|\{(\boldsymbol{h}, h_d) \in \mathcal{H}_N^{d,\boldsymbol{\gamma}} \setminus \{\boldsymbol{0}\} : \boldsymbol{h} \neq \boldsymbol{0} \text{ and } h_d \neq 0\}|}{2} = \frac{|\mathcal{H}_N^{d,\boldsymbol{\gamma}}| - |\mathcal{H}_N^{d-1,\boldsymbol{\gamma}}| - 4\lfloor \gamma_d N \rfloor}{2}$$

elements of $\mathbb{Z}_M^*$ with

$$\boldsymbol{h} \cdot \boldsymbol{z}^* \equiv -h_d z_d \pmod{M} \text{ for at least one } (\boldsymbol{h}, h_d) \in \mathcal{H}_N^{d,\boldsymbol{\gamma}} \setminus \{\boldsymbol{0}\}.$$

We want to provide a rank-1 lattice that allows an exact integration of all monomials supported on $\mathcal{H}_N^{d,\boldsymbol{\gamma}}$. Consequently, we need more elements in $\mathbb{Z}_M^*$ than we have counted above

$$|\mathbb{Z}_M^*| = M - 1 \geq \frac{|\mathcal{H}_N^{d,\boldsymbol{\gamma}}| - |\mathcal{H}_N^{d-1,\boldsymbol{\gamma}}| - 4\lfloor \gamma_d N \rfloor}{2} + 1.$$

Choosing M in this way yields at least one element $z_d^* \in \mathbb{Z}_M^*$ with $(\boldsymbol{h}, h_d) \cdot (\boldsymbol{z}^*, z_d^*) \not\equiv 0 \pmod{M}$ for all $(\boldsymbol{h}, h_d) \in \mathcal{H}_N^{d,\boldsymbol{\gamma}} \setminus \{\boldsymbol{0}\}$. Accordingly, there exists a rank-1 lattice that allows the exact integration of all trigonometric polynomials with frequencies supported on $\mathcal{H}_N^{d,\boldsymbol{\gamma}}$. ▯

REMARK 3.3. The constructive proof of Theorem 3.2 specifies a component-by-component search algorithm. It is indicated by Algorithm 1. □

In order to get a general statement about the existence of rank-1 lattices allowing the exact reconstruction of trigonometric polynomials with Fourier coefficients supported on $H_N^{d,\boldsymbol{\gamma}}$ we have to take the maximum over the lower bounds

$$M_{s,\boldsymbol{\gamma},N}^{\text{low}} = \begin{cases} |H_N^{1,\boldsymbol{\gamma}}|, & \text{for } s = 1, \\ \frac{|\mathcal{H}_N^{s,\boldsymbol{\gamma}}| - |\mathcal{H}_N^{s-1,\boldsymbol{\gamma}}| - 4\lfloor \gamma_s N \rfloor + 4}{2}, & \text{else,} \end{cases}$$

for all s from 1 to d. We formulate the following corollary.

COROLLARY 3.4. *For an arbitrary prime number M satisfying*

$$M \geq \max_{s=1,\ldots,d} M_{s,\boldsymbol{\gamma},N}^{\text{low}}$$

there exists a rank-1 lattice that allows a perfectly stable reconstruction of all Fourier coefficients of trigonometric polynomials with frequencies supported on $H_N^{d,\boldsymbol{\gamma}}$. In particular, there exists a prime number

$$M^* \leq 2 \max_{s=1,\ldots,d} M_{s,\boldsymbol{\gamma},N}^{\text{low}}$$

and a generating vector $\boldsymbol{z} \in (\mathbb{Z}_{M^*}^*)^d$, with $\Lambda(\boldsymbol{z}, M^*)$ is such a reconstruction lattice.

Proof. The first assertion is a simple consequence of Theorem 3.2. Bertrand's postulate ensures that there exists a prime M^* with $\max_{s=1,\ldots,d} M_{s,\boldsymbol{\gamma},N}^{\text{low}} \leq M^* \leq 2\max_{s=1,\ldots,d} M_{s,\boldsymbol{\gamma},N}^{\text{low}}$. This M^* fulfils Theorem 3.2 for all required index sets $\mathcal{H}_N^{s,\boldsymbol{\gamma}}$, $s = 1, \ldots, d$. ▯

We obtain the following approach to construct a reconstruction lattice.

1. Compute or estimate the values $M_{s,\boldsymbol{\gamma},N}^{\text{low}}$ for $s = 1, \ldots, d$ and their maximum M according to Corollary 3.4. Find the nearest prime number M^* larger than M. This lattice size M^* ensures the existence of a reconstruction lattice. In addition, Theorem 3.2 guarantees that we can find a rank-1 reconstruction lattice of size M^* by the component-by-component construction.
2. Apply Algorithm 1 in order to find the generating vector $\boldsymbol{z}$.
3. Decrease the lattice size using Algorithm 2.

EXAMPLE 3.5. The example in Table 3 follows the example from [1, Table 3]. In our notation we have to fix the parameters $N = 16$ and $\boldsymbol{\gamma} = \left([\sqrt{3}/2]^{s-1}\right)_{s\in\mathbb{N}}$. Besides some other results [1, Theorem 1] ensures the existence of a rank-1 lattice of size $M = 2017$ that allows the exact integration of all trigonometric polynomials with frequencies supported on $H_{16}^{d,\boldsymbol{\gamma}}$ for all d. In other words, this sampling set only

Algorithm 1 Component-by-component lattice search

Input:	$M \in \mathbb{N}$ prime	cardinality of rank-1 lattice
	$d \in \mathbb{N}$	spatial dimension
	$N \in \mathbb{N}$, $\boldsymbol{\gamma} \in \mathbb{R}^d$	refinement and weights of $H_N^{d,\boldsymbol{\gamma}}$

$z_1 = 1$
for $s = 2, \ldots, d$ ***do***
  form the set $H_N^{s,\boldsymbol{\gamma}}$
  search for $z_s \in [1, M-1] \cap \mathbb{Z}$ with $|\{(\boldsymbol{z}, z_s) \cdot \boldsymbol{h} \bmod M : \boldsymbol{h} \in H_N^{s,\boldsymbol{\gamma}}\}| = |H_N^{s,\boldsymbol{\gamma}}|$
  $\boldsymbol{z} = (\boldsymbol{z}, z_s)$
end for

Output:	$\boldsymbol{z} \in \mathbb{Z}^d$	generating vector

Algorithm 2 Lattice size decreasing

Input:	$H_N^{d,\boldsymbol{\gamma}} \subset \mathbb{Z}^d$	index set
	$M_{\max} \in \mathbb{N}$ prime	cardinality of rank-1 lattice
	$\boldsymbol{z} \in \mathbb{N}^d$	$\Lambda(\boldsymbol{z}, M_{\max})$ is reconstruction lattice for $H_N^{d,\boldsymbol{\gamma}}$

for $j = 0, \ldots, M_{\max} - |H_N^{d,\boldsymbol{\gamma}}|$ ***do***
  if $|\{\boldsymbol{z} \cdot \boldsymbol{h} \bmod (M_{\max} - j) : \boldsymbol{h} \in H_N^{d,\boldsymbol{\gamma}}\}| = |H_N^{d,\boldsymbol{\gamma}}|$ ***then***
    $M_{\min} = M - j$
  end if
end for

Output:	$M_{\min}$	lattice size

guarantees the reconstruction of the Fourier coefficient $\hat{f}_{\boldsymbol{0}}$ assuming $f \in \Pi_{H_{16}^{21,\boldsymbol{\gamma}}}$. Our Table presents rank-1 lattices allowing the unique reconstruction of all Fourier coefficients $\hat{f}_{\boldsymbol{h}}$, $\boldsymbol{h} \in H_{16}^{21,\boldsymbol{\gamma}}$. The first seven columns of Table 3 show the strategy to find rank-1 lattices that allow a reconstruction in our sense. We generated all frequency sets $H_{16}^{s,\boldsymbol{\gamma}}$ and the corresponding difference sets $\mathcal{H}_{16}^{s,\boldsymbol{\gamma}}$, counted the cardinalities, and calculated all $M_{s,\boldsymbol{\gamma},16}^{\text{low}}$, $s = 1, \ldots, 21$. Then we searched the smallest prime number M^* not smaller than the maximum of all $M_{s,\boldsymbol{\gamma},N}^{\text{low}}$, $s = 1, \ldots, 21$. Now we used Algorithm 1 to find the components of a suitable generating vector $\boldsymbol{z}$. To compute the last column we fixed the vector $\boldsymbol{z}$ and searched for the rank-1 lattice $\Lambda(\boldsymbol{z}, M_{\boldsymbol{z}})$ of the smallest size $M_{\boldsymbol{z}}$ allowing a unique reconstruction of $f \in \Pi_{H_{16}^{21,\boldsymbol{\gamma}}}$, see Algorithm 2. That way, we reduced the lattice size from $M^* = 1\,061\,353$ to $M_{\boldsymbol{z}} = 172\,445$. Consequently, we obtain an oversampling factor of $M_{\boldsymbol{z}}/|H_{16}^{21,\boldsymbol{\gamma}}| = 172\,445/24\,341 \approx 7.0845$. So we constructed a mildly oversampled and perfectly stable spatial discretisation for trigonometric polynomials $f \in \Pi_{H_{16}^{21,\boldsymbol{\gamma}}}$. □

4. Cardinality of the difference set $\mathcal{H}_N^{d,\boldsymbol{\gamma}}$. The assertion of Corollary 3.4 concerning the existence of rank-1 lattices allowing a reconstruction mainly depends on the cardinalities of $\mathcal{H}_N^{d,\boldsymbol{\gamma}}$. To estimate these and especially their asymptotical behaviour we consider sets

$$\mathcal{H}_{N_1,N_2}^{d,\boldsymbol{\gamma}} = \{\boldsymbol{h} = \boldsymbol{a} + \boldsymbol{b} : \boldsymbol{a} \in H_{N_1}^{d,\boldsymbol{\gamma}}, \boldsymbol{b} \in H_{N_2}^{d,\boldsymbol{\gamma}}\}.$$

The set $\mathcal{H}_{N_1,N_2}^{d,\boldsymbol{\gamma}}$ can be interpreted as the symmetric hyperbolic cross of size N_1 shifted along the symmetric hyperbolic cross of size N_2. Due to the symmetry of the set $H_N^{d,\boldsymbol{\gamma}}$

s	$\lfloor 16\gamma_s \rfloor$	$\lvert H_{16}^{s,\boldsymbol{\gamma}}\rvert$	$\lvert \mathcal{H}_{16}^{s,\boldsymbol{\gamma}}\rvert$	$M_{s,\boldsymbol{\gamma},16}^{\mathrm{low}}$		z_s	
1	16	33	65	33	$M^* := \min_{p \text{ prime}} \{p : p \ge \max_{s=1,\dots,21} M_{s,\boldsymbol{\gamma},N}^{\mathrm{low}}\} = 1061353$	1	$M_{\boldsymbol{z}} = 172445$
2	13	207	1313	600		30	
3	12	903	14197	6420		345	
4	10	2587	88621	37140		1489	
5	9	5305	357433	134390		5349	
6	7	9135	1041817	342179		12403	
7	6	13179	2310889	634526		27533	
8	5	16701	4128701	908898		33342	
9	5	19391	6251369	1061326		36848	
10	4	21183	8273585	1011102		45271	
11	3	22373	10018073	872240		37422	
12	3	23159	11440521	711220		20364	
13	2	23635	12482641	521058		14565	
14	2	23947	13284889	401122		4505	
15	2	24119	13851537	283322		3342	
16	1	24221	14212477	180470		102	
17	1	24287	14489129	138326		787	
18	1	24317	14666645	88758		189	
19	1	24335	14796281	64818		82	
20	1	24341	14872141	37930		48	
21	0	24341	14872141	2		1	

TABLE 3.1

Example for component-by-component lattice search with $N = 16$ *and* $\boldsymbol{\gamma} = \left(\left[\frac{\sqrt{3}}{2}\right]^{s-1}\right)_{s\in\mathbb{N}}$.

the difference sets fulfil $\mathcal{H}_N^{d,\boldsymbol{\gamma}} = \mathcal{H}_{N,N}^{d,\boldsymbol{\gamma}}$. To prepare the theorem we collect some easy inclusion arguments.

LEMMA 4.1. *With* $N \in \mathbb{R}$, $N \ge 1$, $d \in \mathbb{N}$, $d \ge 2$, *and* $\boldsymbol{\gamma}$ *like above we have the inclusions* $\mathcal{H}_{N,N}^{d,\boldsymbol{\gamma}} \subset H_{N'}^{d,\boldsymbol{\gamma}} \subset \mathcal{H}_{N',1}^{d,\boldsymbol{\gamma}}$ *with* $N' = 2^{j_0} N^2 \prod_{s=1}^{j_0} \gamma_s$ *and* $j_0 = \max\{j \in [1,d] \cap \mathbb{N} :\ \gamma_j > \frac{1}{2},\ \prod_{s=1}^{j} \gamma_s \ge \frac{1}{N}\}$.

Proof. The second inclusion from above is a simple consequence of $\mathbf{0} \in H_1^{d,\boldsymbol{\gamma}}$ and $\boldsymbol{h} + \mathbf{0} \in \mathcal{H}_{N',1}^{d,\boldsymbol{\gamma}}$ for all $\boldsymbol{h} \in H_{N'}^{d,\boldsymbol{\gamma}}$. To prove the first inclusion we take two arbitrary vectors $\boldsymbol{a}, \boldsymbol{b} \in H_N^{d,\boldsymbol{\gamma}}$ and partition the indices of this pair of vectors in four distinct sets

$$
\begin{aligned}
I_0 &= \{s \in [1,d] \cap \mathbb{N} :\ a_s = b_s = 0\}, & I_1 &= \{s \in [1,d] \cap \mathbb{N} :\ a_s \ne 0, b_s = 0\},\\
I_2 &= \{s \in [1,d] \cap \mathbb{N} :\ a_s = 0, b_s \ne 0\}, & I_3 &= \{s \in [1,d] \cap \mathbb{N} :\ a_s \ne 0, b_s \ne 0\}
\end{aligned}
$$

with $\bigcup_{j=0}^{3} I_j = [1,d] \cap \mathbb{N}$. We calculate

$$
\begin{aligned}
\prod_{s=1}^{d} \max\left(1, \frac{|a_s + b_s|}{\gamma_s}\right) &= \prod_{s\in I_0} 1 \prod_{s\in I_1} \frac{|a_s|}{\gamma_s} \prod_{s\in I_2} \frac{|b_s|}{\gamma_s} \prod_{s\in I_3} \max\left(1, \frac{|a_s + b_s|}{\gamma_s}\right)\\
&\le \prod_{s\in I_0} 1 \prod_{s\in I_1} \frac{|a_s|}{\gamma_s} \prod_{s\in I_2} \frac{|b_s|}{\gamma_s} \prod_{s\in I_3} 2\gamma_s \frac{|a_s|}{\gamma_s} \frac{|b_s|}{\gamma_s} \le N^2 \prod_{s\in I_3} 2\gamma_s \le N^2 2^{j_0} \prod_{s=1}^{j_0} \gamma_s,
\end{aligned}
$$

where we take $|I_3| \leq \max\{j \in [1,d] \cap \mathbb{N} : \prod_{s=1}^{j} \gamma_s \geq N^{-1}\}$ into account. □

REMARK 4.2. Note that these inclusions also yield results concerning the cardinalities of the considered sets, namely $|\mathcal{H}_N^{d,\boldsymbol{\gamma}}| \leq |H_{N'}^{d,\boldsymbol{\gamma}}| \leq |\mathcal{H}_{N',1}^{d,\boldsymbol{\gamma}}|$. □

LEMMA 4.3. *Let $d \in \mathbb{N}$, $N_1, N_2, N_1', N_2' \in \mathbb{R}$ and fix $\boldsymbol{\gamma}$. Moreover, we assume $1 \leq N_i \leq N_i'$ for $i = 1, 2$. Then the inclusion $\mathcal{H}_{N_1,N_2}^{d,\boldsymbol{\gamma}} \subset \mathcal{H}_{N_1',N_2'}^{d,\boldsymbol{\gamma}}$ holds true. The cardinalities of these sets follow the corresponding inequality. Furthermore, we obtain the equality $\mathcal{H}_{N_1,N_2}^{d,\boldsymbol{\gamma}} = \mathcal{H}_{N_2,N_1}^{d,\boldsymbol{\gamma}}$.*

Proof. The inclusion results from the inclusions of the sets $H_{N_i}^{d,\boldsymbol{\gamma}} \subset H_{N_i'}^{d,\boldsymbol{\gamma}}$, $i = 1, 2$. The symmetry of the sets $H_{N_i}^{d,\boldsymbol{\gamma}}$, $i = 1, 2$, justifies the equality from above. □

In order to prove Theorem 4.7 we show some basic facts. At first we prepare an induction needed by the theorem with the following three lemmas.

LEMMA 4.4. *For $a \geq 4$ and $d \in \mathbb{N}_0$ the following inequality holds*

$$\int_2^{\frac{a}{2}} \frac{a}{x} \left(\log_2 \frac{a}{x}\right)^d dx \leq \frac{a(\log_2 \frac{a}{2})^{d+1}}{d+1} \log_{\mathrm{e}} 2.$$

Proof. We estimate

$$\begin{aligned}
\int_2^{\frac{a}{2}} \frac{a}{x} \left(\log_2 \frac{a}{x}\right)^d dx &= (-1)^d a \log_{\mathrm{e}} 2 \int_{\frac{2}{a}}^{\frac{1}{2}} \frac{1}{y \log_{\mathrm{e}} 2} \left(\log_2 y\right)^d dy \\
&= (-1)^d a \log_{\mathrm{e}} 2 \int_{-\log_2 \frac{a}{2}}^{-1} t^d dt = (-1)^d a \log_{\mathrm{e}} 2 \left[\frac{1}{d+1} t^{d+1}\right]_{-\log_2 \frac{a}{2}}^{-1} \\
&= \frac{a \log_{\mathrm{e}} 2}{d+1} \left(\left(\log_2 \frac{a}{2}\right)^{d+1} - 1\right) \leq \frac{a \log_{\mathrm{e}} 2}{d+1} \left(\log_2 \frac{a}{2}\right)^{d+1}. \qquad \square
\end{aligned}$$

LEMMA 4.5. *Let $d \geq 2$. We define the following functions*

$$\begin{aligned}
g_{d,i} &: [1,\infty)^2 \to [1,\infty), \quad i = 1, 2, \\
g_{d,1}(a,b) &:= ab \max(\log_2 a, \log_2 b, 1)^{d-2}, \\
g_{d,2}(a,b) &:= \max(a(\log_2 a)^{d-1}, b(\log_2 b)^{d-1}, a, b).
\end{aligned}$$

The following inequalities are fulfilled

$$g_{d,i}(a,b) \leq g_{d+1,i}(a,b), \quad \text{for } i = 1, 2, \tag{4.1}$$

and with $\max(a,b) \geq 2$

$$g_{d,1}(a,b) \leq ab \max(\log_2 a, \log_2 b)^{d-2}, \tag{4.2}$$
$$g_{d,2}(a,b) \leq \max(a(\log_2 a)^{d-1}, b(\log_2 b)^{d-1}). \tag{4.3}$$

Proof. Easy case-by-case analysis proves these assertions. □

LEMMA 4.6. *Let $d \in \mathbb{N}$, $a, b \geq 1$, and $\boldsymbol{\gamma}$ like above. We define the functions*

$$f_d(a,b) = \begin{cases} 2\gamma_1(a+b)+1, & \text{for } d=1,\\ c_{d,1}g_{d,1}(a,b) + c_{d,2}g_{d,2}(a,b), & \text{else,} \end{cases}$$

with $g_{d,i}$ from Lemma 4.5 and constants $c_{2,1} = 2\gamma_1\gamma_2$, $c_{2,2} = (1+4\gamma_1)\gamma_2$ and for $d > 2$

$$\begin{pmatrix} c_{d,1} \\ c_{d,2} \end{pmatrix} = \gamma_d \begin{pmatrix} \log_e 4 & 1 \\ 0 & 1 + \frac{\log_e 4}{d-1} \end{pmatrix} \begin{pmatrix} c_{d-1,1} \\ c_{d-1,2} \end{pmatrix}.$$

For $d > 1$ the following inequalities hold true

$$f_{d-1}(a,b) \leq \frac{1}{\gamma_d} f_d(a,b), \tag{4.4}$$

$$b f_{d-1}(a,2) \leq \frac{2}{\gamma_d} f_d(a,b), \tag{4.5}$$

$$\text{for } \gamma_d a, \gamma_d b \geq 1 \qquad f_{d-1}(\gamma_d a, \gamma_d b) \leq f_d(a,b), \tag{4.6}$$

$$\text{and} \qquad \sum_{l=2}^{\lfloor \gamma_d a \rfloor} f_{d-1}(2\gamma_d a l^{-1}, b) \leq f_d(a,b). \tag{4.7}$$

Proof.

(4.4) At first we show (4.4) for $d = 2$

$$\begin{aligned} f_1(a,b) &= 1 + 2\gamma_1(a+b) \leq 1 + 4\gamma_1 \max(a,b) \\ &\leq (1+4\gamma_1)\max(a,b) \leq \frac{c_{2,2}}{\gamma_2} g_{2,2}(a,b) \leq \frac{1}{\gamma_2} f_2(a,b) \end{aligned}$$

and for $d > 2$

$$\begin{aligned} f_{d-1}(a,b) &= c_{d-1,1}g_{d-1,1}(a,b) + c_{d-1,2}g_{d-1,2}(a,b) \\ &\leq \frac{c_{d,1}}{\gamma_d} g_{d,1}(a,b) + \frac{c_{d,2}}{\gamma_d} g_{d,2}(a,b) = \frac{1}{\gamma_d} f_d(a,b). \end{aligned}$$

(4.5) For $d = 2$ we get

$$\begin{aligned} b f_1(a,2) &= b(2\gamma_1(a+2)+1) = \frac{2\gamma_1\gamma_2}{\gamma_2} ab + \frac{(1+4\gamma_1)\gamma_2}{\gamma_2} b \\ &\leq \frac{c_{2,1}}{\gamma_2} g_{2,1}(a,b) + \frac{c_{2,2}}{\gamma_2} g_{2,2}(a,b) \leq \frac{1}{\gamma_2} f_2(a,b) \end{aligned}$$

and $d > 2$ yields

$$\begin{aligned} b f_{d-1}(a,2) &= c_{d-1,1} 2ab \max(\log_2 a, 1)^{d-3} + c_{d-1,2} \max(ab(\log_2 a)^{d-2}, 2b, ab) \\ &\leq 2c_{d-1,1} g_{d,1}(a,b) + 2c_{d-1,2} ab \max(\log_2 a, \log_2 b, 1)^{d-2} \\ &\leq 2\frac{c_{d,1}}{\gamma_d} g_{d,1}(a,b) \leq \frac{2}{\gamma_d} f_d(a,b). \end{aligned}$$

(4.6) Assuming $\gamma_d a, \gamma_d b \geq 1$, we obtain

$$\begin{aligned} f_1(\gamma_2 a, \gamma_2 b) &= 2\gamma_1\gamma_2(a+b) + 1 \leq 4\gamma_1\gamma_2 \max(a,b) + \gamma_2^2 ab \\ &\leq c_{2,2} g_{2,2}(a,b) + c_{2,1} g_{2,1}(a,b) = f_2(a,b). \end{aligned}$$

With $d>2$, $g_{d-1,i}(\gamma_d a,\gamma_d b)\le \gamma_d g_{d-1,i}(a,b)$, and (4.1) we estimate

$$\begin{aligned}
f_{d-1}(\gamma_d a,\gamma_d b) &= c_{d-1,1}g_{d-1,1}(\gamma_d a,\gamma_d b)+c_{d-1,2}g_{d-1,2}(\gamma_d a,\gamma_d b)\\
&\le c_{d-1,1}\gamma_d g_{d,1}(a,b)+c_{d-1,2}\gamma_d g_{d,2}(a,b)\\
&\le c_{d,1}g_{d,1}(a,b)+c_{d,2}g_{d,2}(a,b)=f_d(a,b).
\end{aligned}$$

(4.7) Obviously, for $2>\lfloor\gamma_d a\rfloor$ we have an empty sum and the inequality holds true. So let us assume $\lfloor\gamma_d a\rfloor\ge 2$. We start with $d=2$:

$$\begin{aligned}
&\sum_{l=2}^{\lfloor\gamma_2 a\rfloor} f_1(2\gamma_2 a l^{-1},b)=\sum_{l=2}^{\lfloor\gamma_2 a\rfloor}\left[2\gamma_1(2\gamma_2 a l^{-1}+b)+1\right]\\
&\quad\le 2\gamma_1\gamma_2 ab+4\gamma_1\gamma_2 a\sum_{l=2}^{\lfloor\gamma_2 a\rfloor} l^{-1}+\gamma_2 a\le 2\gamma_1\gamma_2 ab+4\gamma_1\gamma_2 a\log_{\mathrm{e}}2\log_2(\gamma_2 a)+\gamma_2 a\\
&\quad\le 2\gamma_1\gamma_2 ab+\left(\frac{1}{\log_2 a}+4\gamma_1\log_{\mathrm{e}}2\right)\gamma_2 a\log_2 a\\
&\quad\le c_{2,1}g_{2,1}(a,b)+c_{2,2}g_{2,2}(a,b)=f_2(a,b).
\end{aligned}$$

Now we consider $d\ge 3$

$$\sum_{l=2}^{\lfloor\gamma_d a\rfloor} f_{d-1}(2\gamma_d a l^{-1},b)=\sum_{l=2}^{\lfloor\gamma_d a\rfloor}\left[c_{d-1,1}g_{d-1,1}\left(\frac{2\gamma_d a}{l},b\right)+c_{d-1,2}g_{d-1,2}\left(\frac{2\gamma_d a}{l},b\right)\right].$$

Due to $\max(2\gamma_d a l^{-1},b)\ge\max\left(2\frac{\gamma_d a}{\lfloor\gamma_d a\rfloor},b\right)\ge 2$ we can apply the inequalities (4.2) and (4.3)

$$\begin{aligned}
&\le \sum_{l=2}^{\lfloor\gamma_d a\rfloor} c_{d-1,1}2\gamma_d ab l^{-1}\max(\log_2(2\gamma_d a l^{-1}),\log_2 b)^{d-3}\\
&\quad+\sum_{l=2}^{\lfloor\gamma_d a\rfloor} c_{d-1,2}\max\left(2\gamma_d a l^{-1}\log_2(2\gamma_d a l^{-1})^{d-2},b(\log_2 b)^{d-2}\right)\\
&\le c_{d-1,1}2\gamma_d ab\max(\log_2(\gamma_d a),\log_2 b)^{d-3}\sum_{l=2}^{\lfloor\gamma_d a\rfloor} l^{-1}\\
&\quad+c_{d-1,2}\sum_{l=3}^{\lfloor\gamma_d a\rfloor}\frac{2\gamma_d a}{l}\left(\log_2\left(\frac{2\gamma_d a}{l}\right)\right)^{d-2}\\
&\quad+c_{d-1,2}\gamma_d a\left(\log_2\,(\gamma_d a)\right)^{d-2}+c_{d-1,2}\gamma_d ab(\log_2 b)^{d-2},
\end{aligned}$$

plug in the result of Lemma 4.4

$$\begin{aligned}
&\le c_{d-1,1}2\gamma_d ab\max(\log_2(\gamma_d a),\log_2 b)^{d-3}\log_{\mathrm{e}}2\log_2(\gamma_d a)\\
&\quad+c_{d-1,2}\frac{\log_{\mathrm{e}}2}{d-1}2\gamma_d a\left(\log_2(\gamma_d a)\right)^{d-1}\\
&\quad+c_{d-1,2}\gamma_d a(\log_2(\gamma_d a))^{d-2}+c_{d-1,2}\gamma_d ab(\log_2 b)^{d-2},
\end{aligned}$$

and end up with

$$\le\left(\log_{\mathrm{e}}4c_{d-1,1}+c_{d-1,2}\right)\gamma_d ab\max(\log_2 a,\log_2 b)^{d-2}$$

$$
\begin{aligned}
&+\left(\frac{1}{\log_2 a}+\frac{\log_e 4}{d-1}\right) c_{d-1,2}\gamma_d a(\log_2 a)^{d-1}\\
&\le c_{d,1}g_{d,1}(a,b)+c_{d,2}g_{d,2}(a,b)=f_d(a,b). \qquad \square
\end{aligned}
$$

In the following we partition the difference set $\mathcal{H}^{d,\boldsymbol{\gamma}}_{N_1,N_2}$ with respect to its last dimension into $d-1$-dimensional subsets

$$\left\{\boldsymbol{h}\in\mathcal{H}^{d,\boldsymbol{\gamma}}_{N_1,N_2}:\ h_d=c,\ \boldsymbol{h}=\boldsymbol{a}+\boldsymbol{b},\ \boldsymbol{a}\in H^{d,\boldsymbol{\gamma}}_{N_1},\ \boldsymbol{b}\in H^{d,\boldsymbol{\gamma}}_{N_2}\right\}, c\in\mathbb{Z}.$$

The symmetry of $\mathcal{H}^{d,\boldsymbol{\gamma}}_{N_1,N_2}$ causes the equality

$$
\begin{aligned}
&\left|\left\{\boldsymbol{h}\in\mathcal{H}^{d,\boldsymbol{\gamma}}_{N_1,N_2}:\ h_d=c,\ \boldsymbol{h}=\boldsymbol{a}+\boldsymbol{b},\ \boldsymbol{a}\in H^{d,\boldsymbol{\gamma}}_{N_1},\ \boldsymbol{b}\in H^{d,\boldsymbol{\gamma}}_{N_2}\right\}\right|\\
&\quad=\left|\left\{\boldsymbol{h}\in\mathcal{H}^{d,\boldsymbol{\gamma}}_{N_1,N_2}:\ h_d=-c,\ \boldsymbol{h}=\boldsymbol{a}+\boldsymbol{b},\ \boldsymbol{a}\in H^{d,\boldsymbol{\gamma}}_{N_1},\ \boldsymbol{b}\in H^{d,\boldsymbol{\gamma}}_{N_2}\right\}\right|
\end{aligned}
$$

for all $c\in\mathbb{N}_0$. For that reason, we focus on the estimation of the cardinality of these sets with $h_d=c\in\mathbb{N}_0$. In particular, we obtain

$$
\begin{aligned}
&\{\boldsymbol{h}\in\mathcal{H}^{d,\boldsymbol{\gamma}}_{N_1,N_2}:\ h_d=c,\boldsymbol{h}=\boldsymbol{a}+\boldsymbol{b},\boldsymbol{a}\in H^{d,\boldsymbol{\gamma}}_{N_1},\boldsymbol{b}\in H^{d,\boldsymbol{\gamma}}_{N_2}\} \qquad (4.8)\\
&=\bigcup_{a_d\in\mathbb{Z}}\left\{\boldsymbol{h}=\boldsymbol{a}+\boldsymbol{b}:\ h_d=c,\ (a_s)_{s=1}^{d-1}\in H^{d-1,\boldsymbol{\gamma}}_{\frac{N_1}{\max(1,\gamma_d^{-1}|a_d|)}},\ (b_s)_{s=1}^{d-1}\in H^{d-1,\boldsymbol{\gamma}}_{\frac{N_2}{\max(1,\gamma_d^{-1}|b_d|)}}\right\}.
\end{aligned}
$$

We split up the big join into three parts

$$=\bigcup_{-a_d\in\mathbb{N}}\{\boldsymbol{h}=\boldsymbol{a}+\boldsymbol{b}:\ \ldots\}\cup\bigcup_{-b_d\in\mathbb{N}}\{\boldsymbol{h}=\boldsymbol{a}+\boldsymbol{b}:\ \ldots\}\cup\bigcup_{a_d=0}^{c}\{\boldsymbol{h}=\boldsymbol{a}+\boldsymbol{b}:\ \ldots\}$$

and use for $a_d<0$ and $b_d<0$ universal supersets

$$
\begin{aligned}
&\subset\bigcup_{-a_d\in\mathbb{N}}\left\{\boldsymbol{h}=\boldsymbol{a}+\boldsymbol{b}:\ h_d=c,\ (a_s)_{s=1}^{d-1}\in H^{d-1,\boldsymbol{\gamma}}_{\frac{N_1}{1}},\ (b_s)_{s=1}^{d-1}\in H^{d-1,\boldsymbol{\gamma}}_{\frac{N_2}{\max(1,\gamma_d^{-1}c)}}\right\}\\
&\quad\cup\bigcup_{-b_d\in\mathbb{N}}\left\{\boldsymbol{h}=\boldsymbol{a}+\boldsymbol{b}:\ h_d=c,\ (a_s)_{s=1}^{d-1}\in H^{d-1,\boldsymbol{\gamma}}_{\frac{N_1}{\max(1,\gamma_d^{-1}c)}},\ (b_s)_{s=1}^{d-1}\in H^{d-1,\boldsymbol{\gamma}}_{\frac{N_2}{1}}\right\}\\
&\quad\cup\bigcup_{a_d=0}^{c}\left\{\boldsymbol{h}=\boldsymbol{a}+\boldsymbol{b}:\ h_d=c,\ (a_s)_{s=1}^{d-1}\in H^{d-1,\boldsymbol{\gamma}}_{\frac{N_1}{\max(1,\gamma_d^{-1}|a_d|)}},\ (b_s)_{s=1}^{d-1}\in H^{d-1,\boldsymbol{\gamma}}_{\frac{N_2}{\max(1,\gamma_d^{-1}|b_d|)}}\right\},
\end{aligned}
$$

which are all subsets of

$$\subset\left\{\boldsymbol{h}\in\mathcal{H}^{d,\boldsymbol{\gamma}}_{N_1,N_2}:\ h_d=c,\boldsymbol{h}=\boldsymbol{a}+\boldsymbol{b},\boldsymbol{a}\in H^{d,\boldsymbol{\gamma}}_{N_1},\boldsymbol{b}\in H^{d,\boldsymbol{\gamma}}_{N_2},a_d,b_d\ge 0\right\}. \qquad (4.9)$$

Note that the set in (4.9) is a subset of (4.8). Hence, we verified the equality here. To produce the set in (4.8) we only have to consider the differences of all elements $\boldsymbol{a}\in H^{d,\boldsymbol{\gamma}}_{N_1}$ and $\boldsymbol{b}\in H^{d,\boldsymbol{\gamma}}_{N_2}$ with nonnegativ values in their dth component. This fact simplifies the proof of the next theorem enormously. It gives an upper bound of the cardinality of the frequency sets $\mathcal{H}^{d,\boldsymbol{\gamma}}_{N_1,N_2}$.

Theorem 4.7. *Let $d \in \mathbb{N}$ and $N_1, N_2 \in \mathbb{R}$, $N_1, N_2 \geq 1$. The cardinality of $\mathcal{H}^{d,\boldsymbol{\gamma}}_{N_1,N_2}$ is bounded by $\left|\mathcal{H}^{d,\boldsymbol{\gamma}}_{N_1,N_2}\right| \leq C_d f_d(N_1, N_2)$, where $f_d(N_1, N_2)$ is given in Lemma 4.6.*

Proof. For dimension $d = 1$ we can easily estimate

$$\mathcal{H}^{1,\boldsymbol{\gamma}}_{N_1,N_2} = 2\lfloor\gamma_1 N_1\rfloor + 2\lfloor\gamma_1 N_2\rfloor + 1 \leq 2\gamma_1(N_1 + N_2) + 1 = f_1(N_1, N_2).$$

Now we increase the dimension d and conclude by induction. W.l.o.g. we set $N_1 \geq N_2 \geq 1$. Because of the symmetry of the set $\mathcal{H}^{d,\boldsymbol{\gamma}}_{N_1,N_2}$ we consider only one half-axis. We inspect this axis for each fixed position $h_d \in [0, \gamma_d(N_1 + N_2)] \cap \mathbb{Z}$ and look for parameters N_1' and N_2' with the largest cardinality $|\mathcal{H}^{d-1,\boldsymbol{\gamma}}_{N_1',N_2'}|$. Here is $h_d = a_d + b_d$ with $a_d \geq 0$ and $b_d \geq 0$ because of the subset relation in (4.9). We have to differ three distinct cases that we analyse in detail.

$\boxed{1}$ $h_d = 0$

Accordingly, we obtain $a_d = b_d = 0$, and the equality $\{\boldsymbol{h} \in \mathcal{H}^{d,\boldsymbol{\gamma}}_{N_1,N_2} : h_d = 0\} = \mathcal{H}^{d-1,\boldsymbol{\gamma}}_{N_1,N_2}$ holds.

$\boxed{2}$ $h_d = 1$

We consider $h_d = a_d + b_d$ with $0 \leq a_d, b_d \leq h_d = 1$.
This leads to ($a_d = 0$ and $b_d = 1$) or ($a_d = 1$ and $b_d = 0$). Due to

$$\prod_{j=1}^{d-1} \max\left(1, \frac{|a_j|}{\gamma_j}\right) \leq \frac{N_1}{\max\left(1, \frac{|a_d|}{\gamma_d}\right)} \quad \text{and} \quad \prod_{j=1}^{d-1} \max\left(1, \frac{|b_j|}{\gamma_j}\right) \leq \frac{N_2}{\max\left(1, \frac{|b_d|}{\gamma_d}\right)}$$

we obtain

$$\prod_{j=1}^{d-1} \max\left(1, \frac{|a_j|}{\gamma_j}\right) \leq N_1 \quad \text{and} \quad \prod_{j=1}^{d-1} \max\left(1, \frac{|b_j|}{\gamma_j}\right) \leq \gamma_d N_2$$

for $a_d = 0$ and $b_d = 1$. Hence, our $d-1$-dimensional subset reads as follows

$$\{(h_j)_{j=1}^{d-1} : \boldsymbol{h} \in \mathcal{H}^{d,\boldsymbol{\gamma}}_{N_1,N_2}, \boldsymbol{h} = \boldsymbol{a} + \boldsymbol{b}, h_d = b_d = 1\} = \mathcal{H}^{d-1,\boldsymbol{\gamma}}_{N_1,\gamma_d N_2}.$$

In the same way we determine the $d-1$-dimensional subset for $a_d = 1$ and $b_d = 0$ and get

$$\{(h_j)_{j=1}^{d-1} : \boldsymbol{h} \in \mathcal{H}^{d,\boldsymbol{\gamma}}_{N_1,N_2}, \boldsymbol{h} = \boldsymbol{a} + \boldsymbol{b}, h_d = 1\} = \mathcal{H}^{d-1,\boldsymbol{\gamma}}_{N_1,\gamma_d N_2} \cup \mathcal{H}^{d-1,\boldsymbol{\gamma}}_{\gamma_d N_1,N_2}.$$

$\boxed{3}$ $h_d \in [2, \lfloor\gamma_d N_1\rfloor + \lfloor\gamma_d N_2\rfloor] \cap \mathbb{Z}$

Clearly, we obtain $\max(a_d, b_d) \geq \frac{h_d}{2}$ and $\min(a_d, b_d) \geq 0$. Accordingly, we consider two cases.

• $\max(a_d, b_d) = b_d \geq \frac{h_d}{2} \;\wedge\; \min(a_d, b_d) = h_d - b_d = a_d \geq \max(0, h_d - \lfloor\gamma_d N_2\rfloor)$
Obviously, we only consider $h_d \in [2, 2\lfloor\gamma_d N_2\rfloor]$. We obtain

$$\prod_{s=1}^{d-1} \max(1, \gamma_d^{-1} a_s) \leq \frac{N_1}{\max(1, \gamma_d^{-1} \max(0, h_d - \lfloor\gamma_d N_2\rfloor))},$$
$$\prod_{s=1}^{d-1} \max(1, \gamma_d^{-1} b_s) \leq \frac{2\gamma_d N_2}{h_d} \leq \begin{cases} \frac{2\gamma_d N_2}{h_d}, & \text{for } h_d \leq \lfloor\gamma_d N_2\rfloor, \\ 2, & \text{for } h_d > \lfloor\gamma_d N_2\rfloor, \end{cases}$$

and conclude

$$\left\{(h_s)_{s=1}^{d-1} \in \mathbb{Z}^{d-1} : \boldsymbol{h} \in \mathcal{H}^{d,\boldsymbol{\gamma}}_{N_1,N_2} :\; h_d \in [2, 2\lfloor\gamma_d N_2\rfloor], h_d \text{ fixed}, b_d \geq \frac{h_d}{2}\right\}$$
$$\subset \begin{cases} \mathcal{H}^{d-1,\boldsymbol{\gamma}}_{N_1, 2\gamma_d N_2 h_d^{-1}}, & \text{for } h_d \in [2, \lfloor\gamma_d N_2\rfloor], \\ \mathcal{H}^{d-1,\boldsymbol{\gamma}}_{\gamma_d N_1 (h_d - \lfloor\gamma_d N_2\rfloor)^{-1}, 2} \subset \mathcal{H}^{d-1,\boldsymbol{\gamma}}_{N_1,2}, & \text{for } h_d \in (\lfloor\gamma_d N_2\rfloor, 2\lfloor\gamma_d N_2\rfloor]. \end{cases}$$

• $\max(a_d, b_d) = a_d \geq \frac{h_d}{2} \ \wedge\ \min(a_d, b_d) = h_d - a_d = b_d \geq \max(0, h_d - \lfloor \gamma_d N_1 \rfloor)$ Here we consider $h_d \in [2, \lfloor \gamma_d N_1 \rfloor + \lfloor \gamma_d N_2 \rfloor]$ because of $N_1 \geq N_2$. We get similar conditions like above:

$$\begin{aligned}
\prod_{s=1}^{d-1} \max(1, \gamma_d^{-1} a_s) &\leq \frac{2\gamma_d N_1}{h_d} \leq \begin{cases} \frac{2\gamma_d N_1}{h_d}, & \text{for } h_d \leq \lfloor \gamma_d N_1 \rfloor, \\ 2, & \text{for } h_d > \lfloor \gamma_d N_1 \rfloor, \end{cases} \\
\prod_{s=1}^{d-1} \max(1, \gamma_d^{-1} b_s) &\leq \frac{N_2}{\max(1, \gamma_d^{-1} \max(0, h_d - \lfloor \gamma_d N_1 \rfloor))},
\end{aligned}$$

$$\begin{aligned}
&\left\{ (h_s)_{s=1}^{d-1} \in \mathbb{Z}^{d-1} : \boldsymbol{h} \in \mathcal{H}_{N_1,N_2}^{d,\boldsymbol{\gamma}} :\ h_d \in [2, \lfloor \gamma_d N_1 \rfloor + \lfloor \gamma_d N_2 \rfloor], h_d \text{ fixed}, a_d \geq \frac{h_d}{2} \right\} \\
&\subset \begin{cases} \mathcal{H}_{2\gamma_d N_1 h_d^{-1}, N_2}, & \text{for } h_d \in [2, \lfloor \gamma_d N_1 \rfloor], \\ \mathcal{H}^{d-1,\boldsymbol{\gamma}}_{2,\gamma_d N_2 (h_d - \lfloor \gamma_d N_1 \rfloor)^{-1}} \subset \mathcal{H}^{d-1,\boldsymbol{\gamma}}_{2,N_2}, & \text{for } h_d \in (\lfloor \gamma_d N_1 \rfloor, \lfloor \gamma_d N_1 \rfloor + \lfloor \gamma_d N_2 \rfloor]. \end{cases}
\end{aligned}$$

Obviously, all sets from above only depends on h_d, N_1 and N_2 but not on the exact summands a_d and b_d. So let us sum up all the cardinalities of the $d-1$-dimensional sets

$$\begin{aligned}
&\left| \mathcal{H}^{d,\boldsymbol{\gamma}}_{N_1,N_2} \right| \\
&\leq \left| \mathcal{H}^{d-1,\boldsymbol{\gamma}}_{N_1,N_2} \right| + 2\chi_{[1,\infty)}(\gamma_d N_2) \left| \mathcal{H}^{d-1,\boldsymbol{\gamma}}_{N_1,\gamma_d N_2} \right| + 2\chi_{[1,\infty)}(\gamma_d N_1) \left| \mathcal{H}^{d-1,\boldsymbol{\gamma}}_{\gamma_d N_1,N_2} \right| \\
&\quad + 2\chi_{[2,\infty)}(\gamma_d N_2) \sum_{l=2}^{\lfloor \gamma_d N_2 \rfloor} \left| \mathcal{H}^{d-1,\boldsymbol{\gamma}}_{N_1,2\gamma_d N_2 l^{-1}} \right| + 2\chi_{[2,\infty)}(\gamma_d N_1) \sum_{l=2}^{\lfloor \gamma_d N_1 \rfloor} \left| \mathcal{H}^{d-1,\boldsymbol{\gamma}}_{2\gamma_d N_1 l^{-1},N_2} \right| \\
&\quad + 2\chi_{[2,\infty)}(2\lfloor \gamma_d N_2 \rfloor) \sum_{l=\lfloor \gamma_d N_2 \rfloor + 1}^{2\lfloor \gamma_d N_2 \rfloor} \left| \mathcal{H}^{d-1,\boldsymbol{\gamma}}_{N_1,2} \right| \\
&\quad + 2\chi_{[2,\infty)}(\lfloor \gamma_d N_1 \rfloor + \lfloor \gamma_d N_2 \rfloor)\chi_{[1,\infty)}(\lfloor \gamma_d N_2 \rfloor) \sum_{l=\lfloor \gamma_d N_1 \rfloor + 1}^{\lfloor \gamma_d N_1 \rfloor + \lfloor \gamma_d N_2 \rfloor} \left| \mathcal{H}^{d-1,\boldsymbol{\gamma}}_{2,N_2} \right|.
\end{aligned}$$

We plug in the induction hypothesis $|\mathcal{H}^{d-1,\boldsymbol{\gamma}}_{N_1,N_2}| \leq C_{d-1} f_{d-1}(N_1, N_2)$, exploit the symmetry of f_d for all $d \in \mathbb{N}$, and apply the inequalities from Lemma 4.6

$$\begin{aligned}
&\leq C_{d-1} f_{d-1}(N1, N2) + 2C_{d-1}(f_{d-1}(\gamma_d N_1, N_2) + f_{d-1}(N_1, \gamma_d N_2)) \\
&\quad + 2C_{d-1} \left(\sum_{l=2}^{\lfloor \gamma_d N_2 \rfloor} f_{d-1}(N_1, 2\gamma_d N_2 l^{-1}) + \sum_{l=2}^{\lfloor \gamma_d N_1 \rfloor} f_{d-1}(2\gamma_d N_1 l^{-1}, N_2) \right) \\
&\quad + 2C_{d-1} \left(\lfloor \gamma_d N_2 \rfloor f_{d-1}(N_1, 2) + \lfloor \gamma_d N_1 \rfloor f_{d-1}(2, N_2) \right) \\
&\leq C_{d-1} f_d(N_1, N_2) \left(5\gamma_d^{-1} + 4 + 8 \right) = C_{d-1} \left(12 + 5\gamma_d^{-1} \right) f_d(N_1, N_2). \qquad \square
\end{aligned}$$

In order to consider the asymptotics of the cardinality $|\mathcal{H}^{d,\boldsymbol{\gamma}}_N|$ we fix $N = N_1 = N_2$ in Theorem 4.7. We apply Theorem 3.2 and formulate the following corollary.

COROLLARY 4.8. *There exists a rank-1 lattice $\Lambda(\boldsymbol{z}, M)$ that allows the perfectly stable reconstruction of all trigonometric polynomials with Fourier coefficients supported on $H^{d,\boldsymbol{\gamma}}_N$.*

The number M of sampling points is bounded above by $\tilde{C}_d N^2 \max(\log_2 N, 1)^{d-2}$.

d \ N	1	2	4	8	16	32	64	128	256
2	1	13	41	121	385	1313	4753	17849	68801
3	1	25	129	545	2369	10617	48785	223241	1020465
4	1	41	321	1825	9921	53281	288321	1530561	7986369
5	1	61	681	4993	32673	202705	1249985	7480225	
6	1	85	1289	11833	91201	642113	4432913	29372377	
7	1	113	2241	25201	225473	1782665	13631761		
8	1	145	3649	49409	507777	4475841	37634561		
9	1	181	5641	90673	1061665	10376673			
10	1	221	8361	157625	2088705	22539233			

TABLE 4.1
Cardinalities of $\mathcal{H}_N^{d,\boldsymbol{\gamma}}$ of different refinements N and dimensions d and fixed $\boldsymbol{\gamma} = \left(\frac{1}{2}\right)_{s=1}^{10}$.

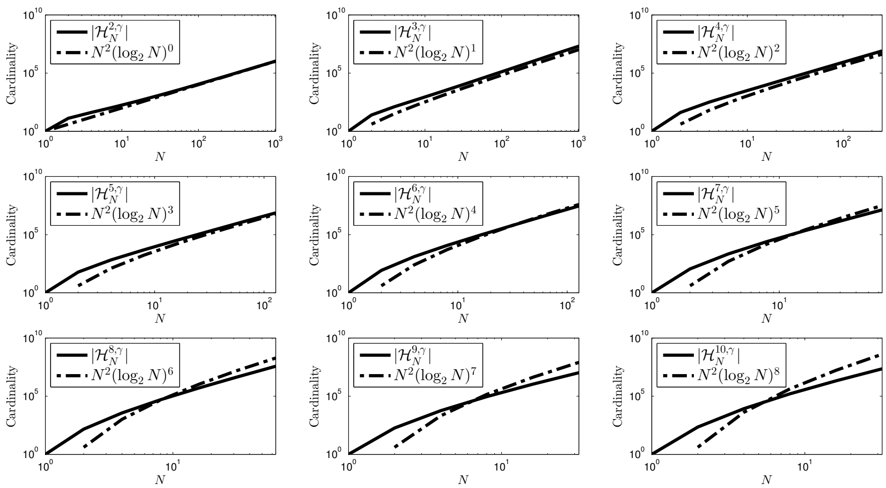


FIG. 4.1. *Cardinalities of $\mathcal{H}_N^{d,\boldsymbol{\gamma}}$, $\boldsymbol{\gamma} = (\frac{1}{2})_{s=1}^{d}$, compared against the main part of the upper bound from Theorem 4.7.*

Proof. Theorem 4.7 ensures the existence of a constant $\tilde{C}_d$ with

$$\max_{s=1,\dots,d} |\mathcal{H}_N^{d,\boldsymbol{\gamma}}| \leq 2^{-1}\tilde{C}N^2 \max(\log_2 N, 1)^{d-2}.$$

Following Corollary 3.4, we can find a prime $M^* \leq \tilde{C}_d N^2 \max(\log_2 N, 1)^{d-2}$ that fulfils Theorem 3.2 for all $s = 1, \dots, d$. ▯

REMARK 4.9. In Table 4.1 we present some exact cardinalities of difference sets $\mathcal{H}_N^{d,\boldsymbol{\gamma}}$ for special parameters $d = 2, \dots, 10$, $N = 2^k$, $k = 0, \dots, 8$ and $\boldsymbol{\gamma} = \left(\frac{1}{2}\right)_{s=1}^{10}$. Figure 4.1 shows the corresponding plots for fixed dimension d compared to the main part of our estimation. The plots lead us to believe in decreasing constants C_d, even though our theoretical results do not ensure this. In order to get better constants in our inequalities above one has to examine the union of $d-1$-dimensional sets exactly and estimate more precisely.

Taking Remark 4.2 into account, the theorem yields

$$|H_N^{d,\boldsymbol{\gamma}}| \leq |\mathcal{H}_{N,1}^{d,\boldsymbol{\gamma}}| \leq \bar{C}_d f_d(N,1) \in \mathcal{O}\big(N\,(\log N)^{d-1}\big)$$

for dimension $d \geq 2$ and refinement $N \geq 2$. The set of functions $\mathcal{O}\big(N\,(\log N)^{d-1}\big)$ does not depend on the weights $\boldsymbol{\gamma}$ but the constant $\bar{C}_d$ however. □

EXAMPLE 4.10. Corresponding to Table 4.1 we fix $\boldsymbol{\gamma} = \left(\frac{1}{2}\right)_{s=1}^{d}$. In the following, we investigate some important cases for fixed N and growing dimension d.

$N = 2$. We consider the symmetric hyperbolic crosses with parameters $N = 2$, $\boldsymbol{\gamma} = \left(\frac{1}{2}\right)_{s\in\mathbb{N}}$ and growing dimension d. Table 4.1 shows the corresponding cardinalities of the difference sets $\mathcal{H}_2^{d,\boldsymbol{\gamma}}$ for $d = 2, \dots, 10$. One easily verifies that the sequence of these cardinalities follows $|\mathcal{H}_2^{d,\boldsymbol{\gamma}}| = 1+4\sum_{s=1}^d d$. We apply Corollary 3.4 and calculate $\max_{s=2,\dots,d} M_{s,\boldsymbol{\gamma},2}^{\text{low}} = 2d$. Clearly, M^* has to be a prime not smaller than $2d$ and so $M^* \geq 2d+1$ holds. In fact, we obtain that the rank-1 lattice $\Lambda(\boldsymbol{z}, 2d+1)$, with $\boldsymbol{z} = (1, \dots, d)$, is a reconstruction lattice for $H_2^{d,\boldsymbol{\gamma}}$. We gain an oversampling factor

$$|\Lambda(\boldsymbol{z}, 2d+1)|/|H_2^{d,\gamma}| = (2d+1)/(2d+1) = 1.$$

Obviously, the corresponding discrete Fourier transform is a unitary transform and computable with a complexity of $\mathcal{O}\left(d \log d\right)$.

Note that the sets $\mathcal{H}_2^{d,\boldsymbol{\gamma}}$ fulfil $\mathcal{H}_2^{d,\boldsymbol{\gamma}} = \left\{\boldsymbol{h} \in \mathbb{Z}^d : \sum_{s=1}^d |h_s| \leq 2\right\}$, and as a consequence the given rank-1 lattice exactly integrates all d-dimensional trigonometric polynomials of trigonometric degree not larger than two, cf. [2, Theorem 3.1].

$N = 4$. We go straightforward and consider the case $N = 4$. The difference sets and their cardinalities are specified by $\mathcal{H}_4^{d,\boldsymbol{\gamma}} = \left\{\boldsymbol{h} \in \mathbb{Z}^d : \sum_{s=1}^d |h_s| \leq 4\right\}$ and $|\mathcal{H}_4^{d,\boldsymbol{\gamma}}| = 1 + \frac{1}{3}\sum_{s=1}^d 8s(s^2+2)$. This yields

$$\max_{s=2,\dots,d} M_{s,\boldsymbol{\gamma},4}^{\text{low}} = M_{d,\boldsymbol{\gamma},4}^{\text{low}} = \tfrac{4}{3}d(d^2+2) - 2.$$

For $d = 100$ we get $M^* = 1\,333\,601 \geq 1\,333\,598 = M_{100,\boldsymbol{\gamma},4}^{\text{low}}$, and we can construct a rank-1 lattice of this size using Algorithm 1. Applying the lattice size reduction (Algorithm 2) we decrease M^* to $M_{\boldsymbol{z}} = 124\,347$. Thus, we obtain a small oversampling factor of

$$M_{\boldsymbol{z}}/|H_4^{100,\gamma}| = 124\,347/20\,201 \approx 6.16.$$

In general, we can give an upper bound for the oversampling factor for arbitrary dimensions d here. With $|H_4^{d,\boldsymbol{\gamma}}| = 1 + \sum_{s=1}^d 4d = 2d^2 + 2d + 1$ we obtain

$$\frac{M_d^*}{|H_4^{d,\gamma}|} \leq \frac{2M_{d,\boldsymbol{\gamma},4}^{\text{low}}}{|H_4^{d,\gamma}|} = \frac{8d(d^2+2) - 12}{6d^2+6d+3} \leq \frac{4}{3}d$$

for $d \in \mathbb{N},\ d \geq 2$, because of Bertrand's postulate. Up to date results about primes even allow better estimations with factors mildly larger than $\frac{2}{3}$ instead of $\frac{4}{3}$, cf. [4, 5]. Note that reconstruction rank-1 lattices for $H_4^{d,\boldsymbol{\gamma}}$ integrate all trigonometric polynomials up to trigonometric degree 4 exactly.

$N = 8$. Last but not least, we want to consider the case $N = 8$. We fix the dimension $d = 50$, construct the frequency set $H_8^{50,\boldsymbol{\gamma}}$ and obtain a cardinality of $|H_8^{50,\boldsymbol{\gamma}}| = 171\,901$. In order to apply Corollary 3.4 we have to compute the cardinalities of the difference sets $\mathcal{H}_8^{s,\boldsymbol{\gamma}}$ for $s = 2, \cdots, 50$. This brings out big problems in large dimensions, e.g. $|\mathcal{H}_8^{50,\boldsymbol{\gamma}}| > 3 \cdot 10^8$. Even the estimation from above brings some difficulties. Accordingly, we cannot easily give a useful a priori lattice size M^*. So, we change our approach in the following way.

1. Search for a generating integer vector $\boldsymbol{z}$ with the result that $\boldsymbol{h}_1 \cdot \boldsymbol{z} \neq \boldsymbol{h}_2 \cdot \boldsymbol{z}$ for all $\boldsymbol{h}_1, \boldsymbol{h}_2 \in H_8^{50,\boldsymbol{\gamma}}$. For example, one can use Algorithm 1 with a large M such as the maximum of all integer values available on the used machine.

$\lvert H_N^{d,\gamma}\rvert$	10	10^2	10^3	10^4	10^5	10^6	10^7
$M_l(\delta_{10})$	266	5 021	73 677	971 190	12 054 971	143 974 975	1 673 970 291
$M_l(\delta_3)$	500	9 706	144 031	1 909 852	23 793 460	284 881 895	3 318 279 975

TABLE 5.1
Cardinalities $|H_N^{d,\gamma}|$ and corresponding bounds $M_l(\delta_{10})$ and $M_l(\delta_3)$ from inequality (5.2).

2. Compute $m_{\max} = \max\{\boldsymbol{h}\cdot\boldsymbol{z} :\ \boldsymbol{h}\in H_8^{50,\gamma}\}$, $m_{\min} = \min\{\boldsymbol{h}\cdot\boldsymbol{z} :\ \boldsymbol{h}\in H_8^{50,\gamma}\}$, and $M^* = m_{\max} - m_{\min} + 1$. Consequently, we obtain $|\{\boldsymbol{h}\cdot\boldsymbol{z} \bmod M^* :\ \boldsymbol{h}\in H_8^{50,\gamma}\}| = |H_8^{50,\gamma}|$ and $\Lambda(\boldsymbol{z}, M^*)$ is a reconstruction lattice.
3. Apply Algorithm 2 with input parameters $H_8^{50,\gamma}$, M^*, and $\boldsymbol{z}$.

This strategy leads to $M^* = 12\,214\,721$ and an output $M_{\boldsymbol{z}} = 3\,739\,059$ of Algorithm 2, which yields to a reasonable oversampling factor of $M_{\boldsymbol{z}}/|H_8^{50,\gamma}| \approx 21.75$. □

5. Comparison with random sampling. In this section we compare theoretical results of this paper and theoretical results from random sampling. To go into numerical experiments in detail would take us too far from the topic of this paper.

Of course, one can evaluate hyperbolic cross trigonometric polynomials at arbitrary sampling nodes $\mathcal{X} := \{\boldsymbol{x}_j \in [0,1)^d :\ j = 0,\dots,M-1\}$ and reconstruct it from the corresponding samples. Certainly, the condition number of $M^{-1}\boldsymbol{A}^*\boldsymbol{A}$ and its upper bounds based on the Frobenius norm $\|M^{-1}\boldsymbol{A}^*\boldsymbol{A} - \boldsymbol{I}\|_{\mathrm{F}}$ with

$$\|M^{-1}\boldsymbol{A}^*\boldsymbol{A} - \boldsymbol{I}\|_{\mathrm{F}}^2 = M^{-2} \sum_{\boldsymbol{h},\boldsymbol{l}\in H_N^{d,\gamma},\, \boldsymbol{h}\neq\boldsymbol{l}} |\,(\boldsymbol{A}^*\boldsymbol{A})_{\boldsymbol{h},\boldsymbol{k}}\,|^2$$

are of large interest. In [6], the authors estimate the Frobenius norm of matrices of that kind. To apply their results let us assume that the elements of $\mathcal{X}$ are independent identically and uniformly distributed on $[0,1)^d$. Let $0 < \delta < 1$, $0 < \alpha < \delta^2$, $\varepsilon > 0$, and

$$\left\lfloor \frac{\alpha M}{3|H_N^{d,\gamma}|} \right\rfloor \geq \left[\log_{\mathrm{e}}\left(\frac{\delta^2}{\alpha}\right)\right]^{-1} \log_{\mathrm{e}}\left(\frac{|H_N^{d,\gamma}|}{\varepsilon(1-\alpha)}\right), \tag{5.1}$$

then the Frobenius norm $\|M^{-1}\boldsymbol{A}^*\boldsymbol{A} - \boldsymbol{I}\|_{\mathrm{F}}$ is bounded above by δ with a probability of at least $1-\varepsilon$. We rearrange (5.1) and obtain that M necessarily fulfils

$$M \geq M_{\mathrm{l}}(\delta) = \left\lceil \min_{\alpha\in(0,\delta^2)} \frac{3|H_N^{d,\gamma}|(\log_{\mathrm{e}}|H_N^{d,\gamma}| - \log_{\mathrm{e}}(1-\alpha))}{\alpha(2\log_{\mathrm{e}}\delta - \log_{\mathrm{e}}\alpha)} \right\rceil \tag{5.2}$$

in order to apply [6, Theorem 4.1]. Note that $M_{\mathrm{l}}(\delta)$ does not depend on the probability $1-\varepsilon$. It is a uniform lower bound with respect to ε and the cited theoretical result.

Obviously, the two inequalities mainly depends on the cardinality of the frequency set $H_N^{d,\gamma}$. In Table 5.1 we show the corresponding values $M_l(\delta_{10})$ and $M_l(\delta_3)$ of the right hand side of (5.2) for cardinalities that are powers of ten. Here we chose $\delta_{10} = 0.895533$. The Frobenius norm $\|M^{-1}\boldsymbol{A}^*\boldsymbol{A} - \boldsymbol{I}\|_{\mathrm{F}}$ bounded by δ_{10} ensures a condition number of $M^{-1}\boldsymbol{A}^*\boldsymbol{A}$ smaller or equal to ten. In the same way we determined $\delta_3 = 0.632455$. The Frobenius norm $\|M^{-1}\boldsymbol{A}^*\boldsymbol{A} - \boldsymbol{I}\|_{\mathrm{F}}$ bounded by δ_3 guarantees a condition number of $\boldsymbol{A}^*\boldsymbol{A}$ of at most three. Table 5.1 presents mildly increasing oversampling factors. Nevertheless, we obtain a large oversampling for reasonable problem sizes. Reducing δ_{10} to δ_3 in (5.2) approximately doubles the lower bound

d	N	$\lvert H_N^{d,\gamma}\rvert$	$M_l(\delta_{10})$	$M_l(\delta_3)$	M^*
2	64	329	20 514	39 927	2 251
3	64	1 097	81 858	160 074	21 961
4	64	2 977	252 406	495 009	119 723
5	64	7 073	661 993	1 300 894	480 773
6	64	15 241	1 545 555	3 041 765	1 591 417
7	64	30 409	3 297 460	6 497 254	4 599 407
8	64	56 961	6 540 453	12 899352	12 001 349
6	2	13	381	720	13
6	4	85	4 127	7 969	307
6	8	389	24 919	48 538	3 433
6	16	1 457	112 932	221 037	29 251
6	32	4 865	436 805	857 666	219 677
6	64	15 241	1 545 555	3 041 765	1 591 417
6	128	46 069	5 190 314	10 233 406	10 945 973

TABLE 5.2
Comparison of theoretical results from random sampling against rank-1 lattices.

M_l. In contrast to our existence bound from Corollary 3.4, the inequalities (5.1) and (5.2) do not take account of the dimensionality d of the corresponding frequency set. Moreover, note that Algorithm 1 presents a deterministic way to find perfectly stable spatial discretisations, i.e. $\|M^{-1}\boldsymbol{A}^*\boldsymbol{A} - \boldsymbol{I}\|_{\mathrm{F}} = 0$.

In Table 5.2 we compared the theoretical results from random sampling and sampling along rank-1 lattices by means of some chosen examples. Like described above, we denote by $M_l(\delta_c)$ a lower bound of the theoretically determined number of random samples needed to obtain a condition number of at most c with a suitable probability. Moreover, the value M^* is the theoretical lattice size guaranteeing the existence of a rank-1 lattice allowing the reconstruction of hyperbolic cross trigonometric polynomials. Note that the corresponding rank-1 lattices guarantee $\boldsymbol{A}^*\boldsymbol{A} = M\boldsymbol{I}$. Thus, the condition number of $\boldsymbol{A}^*\boldsymbol{A}$ is exactly one.

First we fix $N = 64$ and consider growing dimensions $d = 2, \ldots, 8$. We see that the oversampling of the theoretical results for random sampling $M_l(\delta_{10})/|H_N^{d,\gamma}|$ beats the theoretical results for rank-1 lattices $M^*/|H_N^{d,\gamma}|$ starting from dimension $d = 6$. The second part of the table fixes the dimension $d = 6$ and considers different refinements $N = 2^n$, $n = 1, \ldots, 7$. Taking the theoretical results into account, random sampling providing a condition number of at most ten yield lower oversampling for $N \geq 64$. Even for relatively large problem sizes the theoretical results of rank-1 lattices are close to the theoretical results from random sampling providing a condition number not larger than three.

To evaluate hyperbolic cross trigonometric polynomials at arbitrary sampling nodes one can apply a matrix vector product with a complexity of $\mathcal{O}(M|H_N^{d,\gamma}|)$. One uses approximative algorithms as described in [3] in order to reduce the complexity to one almost linear in M and $|H_N^{d,\gamma}|$ up to some constant depending on the spatial dimension d and some logarithmic factors to the power of d. Briefly, one considers the trigonometric polynomial $f \in \Pi_{H_N^{d,\gamma}}$ as a trigonometric polynomial $g \in \Pi_{H_{\beta N}^{d,\gamma}}$, $\beta \in \mathbb{N}$, $\beta \geq 2$, evaluates g at its natural spatial discretisation, constructs the corresponding interpolant $\tilde{g}$ using locally supported basis functions, and evaluates $\tilde{g}$ at all sampling nodes $\boldsymbol{x}_j$. In order to obtain the desired stability from above one has to ensure stability in each step of the fast algorithm. Consequently, one has to provide a fixed stable spatial discretisation for all trigonometric polynomials with frequencies supported on $H_N^{d,\gamma}$ and a corresponding fast transform. In general, sparse grids, the natural spatial discretisations of hyperbolic cross trigonometric polynomials, do not guarantee this stability, cf. [7]. Possibly, the fast algorithm destroys the nice stability properties of $\mathcal{X}$ and , as a consequence, limits the usability of $\mathcal{X}$ here.

Summary. The concept of rank-1 lattices provides mildly oversampled and stable spatial discretisations for reasonable cardinalities of hyperbolic crosses. In addition, the FFT and some simple precomputations allows the fast and stable evaluation of multivariate trigonometric polynomials f at all sampling nodes of rank-1 lattices $\Lambda(\boldsymbol{z}, M)$. Assuming $\boldsymbol{A}^*\boldsymbol{A} = M\boldsymbol{I}$, the inverse FFT provide the fast, stable, and unique reconstruction of f from the samples at $\Lambda(\boldsymbol{z}, M)$.

Most of the results of this paper can be generalised. More precisely, one considers trigonometric polynomials supported on arbitrary d-dimensional frequency sets I instead of $H_N^{d,\gamma}$. In order to determine a rank-1 lattice allowing the reconstruction of trigonometric polynomials supported on an arbitrary frequency set I one can apply the approach from Example 4.10 ($N = 8$). Here, one attains a perfectly stable spatial discretisation.

Of course, besides the reconstruction of trigonometric polynomials one can also approximate functions of appropriate smoothness by sampling along rank-1 lattices, cf. [10]. Our results can be used to classify some convergence properties of these approximations.

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