Reconstructing Multivariate Trigonometric Polynomials by Sampling along Generated Sets

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Abstract The approximation of problems in d spatial dimensions by sparse trigonometric polynomials supported on known or unknown frequency index sets $I \subset \mathbb{Z}^d$ is an important task with a variety of applications. The use of a generalization of rank-1 lattices as spatial discretizations offers a suitable possibility for sampling such sparse trigonometric polynomials. Given an index set of frequencies, we construct corresponding sampling sets that allow a stable and unique discrete Fourier transform. Applying the one-dimensional non-equispaced fast Fourier transform (NFFT) enables the fast evaluation and reconstruction of the multivariate trigonometric polynomials.

1 Introduction

Given a spatial dimension $d \in \mathbb{N}$, we consider Fourier series of continuous functions $f(\mathbf{x}) = \sum_{\mathbf{k}\in\mathbb{Z}^d} \hat{f}_{\mathbf{k}} \mathrm{e}^{2\pi\mathrm{i}\mathbf{k}\cdot\mathbf{x}}$ mapping the d-dimensional torus $[0,1)^d$ into the complex numbers $\mathbb{C}$, where $\left(\hat{f}_{\mathbf{k}}\right)_{\mathbf{k}\in\mathbb{Z}^d} \subset \mathbb{C}$ are the Fourier coefficients. A sequence $\left(\hat{f}_{\mathbf{k}}\right)_{\mathbf{k}\in\mathbb{Z}^d}$ with a finite number of nonzero elements specifies a trigonometric polynomial. We call the index set of the nonzero elements the frequency index set of the corresponding trigonometric polynomial. For a fixed index set $I \subset \mathbb{Z}^d$ with a finite cardinality $|I|$, $\Pi_I = \operatorname{span}\{\mathrm{e}^{2\pi\mathrm{i}\mathbf{k}\cdot\mathbf{x}} : \mathbf{k} \in I\}$ is called the space of trigonometric polynomials with frequencies supported by I.

Assuming the index set I is of finite cardinality and a suitable discretization in frequency domain for approximating functions, e.g. functions of dominating mixed smoothness, cf. [13], we are interested in evaluating the corresponding trigonometric polynomials at sampling nodes and reconstructing the Fourier coefficients $\left(\hat{f}_{\mathbf{k}}\right)_{\mathbf{k}\in I}$ from samples. Accordingly, we consider (sparse) multivariate trigonomet-

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ric polynomials

$$f(\mathbf{x}) = \sum_{\mathbf{k}\in I} \hat{f}_{\mathbf{k}} \mathrm{e}^{2\pi \mathrm{i}\mathbf{k}\cdot\mathbf{x}}$$

and assume the frequency index set I is given.

For different specific index sets I there has been done some related work using rank-1 lattices as spatial discretizations [11, 6]. A multivariate trigonometric polynomial evaluated at all nodes of a rank-1 lattice essentially simplifies to a one-dimensional fast Fourier transform of the length of the cardinality of the rank-1 lattice, cf. [10]. Allowing for some oversampling one can find a rank-1 lattice, which even allows the reconstruction of the trigonometric polynomial from the samples at the rank-1 lattice nodes. A suitable strategy to search for such reconstructing rank-1 lattices can be adapted from numerical integration. In particular, a modification of the component-by-component constructions of lattice rules based on various weighted trigonometric degrees of exactness described in [2] allows one to find adequate rank-1 lattices in a relatively fast way, cf. [6]. The search strategy specified in [6] uses discrete optimization techniques.

In this paper we consider so-called generated sets, which generalize the concept of rank-1 lattices. The structure of these spatial discretizations allows for the evaluation of multivariate trigonometric polynomials by means of some simple precomputations and a one-dimensional non-equispaced discrete Fourier transform (NDFT). The fast computation can be realized by using the non-equispaced fast Fourier transform (NFFT), cf. [7]. The stability of the computation mainly depends on the Fourier matrices of this one-dimensional NFFT. Similar to the approaches known from rank-1 lattices, we have to search for suitable generating vectors guaranteeing a Fourier matrix of full column rank and, in addition, stability. In contrast to searching for suitable rank-1 lattices, we can use continuous optimization methods. Our search algorithm is based on the minimization of an upper bound of the maximum Gerschgorin circle radii, cf. [4], via a simplex search method.

The paper is organized as follows: In Section 2 we define generated sets, explain their advantages in computation, and give a basic example. To estimate the stability of the corresponding discrete Fourier transform, we specify an upper bound on the condition number of the involved Fourier matrices in Section 3. Algorithm 1 describes how to compute this upper bound in a simple and fast way. We optimize the generating vector by applying a nonlinear optimization technique as described in [12]. In practice, we (locally) minimize the theoretical number of samples needed to achieve at least a fixed stability. Some numerical examples can be found in Section 4.

The given examples include frequency index sets called weighted hyperbolic crosses

$$H_N^{d,\boldsymbol{\gamma}} := \{\mathbf{h} \in \mathbb{Z}^d : \prod_{s=1}^{d} \max(1, \gamma_s^{-1}|h_j|) \leq N\}$$

with parameters $d \in \mathbb{N}$, $N \in \mathbb{R}$, $\boldsymbol{\gamma} \subset \mathbb{R}^{\mathbb{N}}$, $1 \geq \gamma_1 \geq \gamma_2 \geq \ldots \geq 0$, and $0^{-1} := \infty$. More general index sets called generalized hyperbolic crosses were discussed in [5, 8].

Note that our approach is universally applicable. Accordingly, the theoretical statements of this paper also treat the frequency index sets specified in [5, 8].

2 Generated Sets

For given $M \in \mathbb{N}$ and $\mathbf{r} \in \mathbb{R}^d$ we define the *generated set*

$$\Lambda(\mathbf{r},M) := \{\mathbf{x}_j = j\mathbf{r} \bmod 1, j = 0,\dots,M-1\}$$

as a generalization of rank-1 lattices. We stress the fact that the restriction of the generating vector $\mathbf{r} \in M^{-1}\mathbb{Z}^d$ results in rank-1 lattices, cf. [2, 3, 14, 15]. Note, in contrast to rank-1 lattices, generated sets do not retain the group structure of the sampling sets, i.e. in general we have $\mathbf{x}_j \neq \mathbf{x}_{j+M}$.

However, we take advantage of the rank-1 structure of the generated set. In a similar way as described in [10], the evaluation of the trigonometric polynomial $f \in \Pi_I$ at all nodes $\mathbf{x}_j \in \Lambda(\mathbf{r},M)$ simplifies to a one-dimensional NDFT. For $\mathscr{Y} = \{\mathbf{k}\cdot\mathbf{r} \bmod 1 : \mathbf{k} \in I\}$ is the set of all scalar products of the elements of the frequency index set I with the generating vector $\mathbf{r}$ we obtain

$$f(\mathbf{x}_j) = \sum_{\mathbf{k}\in I} \hat{f}_{\mathbf{k}} \mathrm{e}^{2\pi \mathrm{i} j\mathbf{k}\cdot\mathbf{r}} = \sum_{y\in\mathscr{Y}} \left(\sum_{\mathbf{k}\cdot\mathbf{r}\equiv y \pmod 1} \hat{f}_{\mathbf{k}} \right) \mathrm{e}^{2\pi\mathrm{i}jy}.$$

We evaluate f at all nodes $\mathbf{x}_j \in \Lambda(\mathbf{r},M)$, $j = 0,\dots,M-1$, by the precomputation of all $\hat{g}_y := \sum_{\mathbf{k}\cdot\mathbf{r}\equiv y \pmod 1} \hat{f}_{\mathbf{k}}$ and a one-dimensional NFFT in $\mathscr{O}(M\log M + (|\log\varepsilon| + d)|I|)$ floating point operations, [7]. The parameter ε determines the accuracy of the computation and is independent of the dimension d.

As the fast evaluation of trigonometric polynomials at all sampling nodes $\mathbf{x}_j$ of the generated set $\Lambda(\mathbf{r},M)$ is guaranteed, we draw our attention to the reconstruction of a trigonometric polynomial f with frequencies supported on I using function values at the nodes $\mathbf{x}_j$ of a generated set $\Lambda(\mathbf{r},M)$. We consider the corresponding Fourier matrix $\mathbf{A}$ and its adjoint $\mathbf{A}^*$,

$$\mathbf{A} := \left(\mathrm{e}^{2\pi\mathrm{i}\mathbf{k}\cdot\mathbf{x}}\right)_{\mathbf{x}\in\Lambda(\mathbf{r},M),\,\mathbf{k}\in I} \quad \text{and} \quad \mathbf{A}^* := \left(\mathrm{e}^{-2\pi\mathrm{i}\mathbf{k}\cdot\mathbf{x}}\right)_{\mathbf{k}\in I,\,\mathbf{x}\in\Lambda(\mathbf{r},M)},$$

to determine necessary and sufficient conditions on generated sets $\Lambda(\mathbf{r},M)$ allowing for a unique reconstruction of all Fourier coefficients of $f \in \Pi_I$. Assuming a full column rank matrix $\mathbf{A}$, the reconstruction of the Fourier coefficients $\hat{\mathbf{f}} = (\hat{f}_{\mathbf{k}})_{\mathbf{k}\in I}$ from sampling values $\mathbf{f} = (f(\mathbf{x}))_{\mathbf{x}\in\Lambda(\mathbf{r},M)}$ can be realized by solving $\mathbf{A}^*\mathbf{A}\hat{\mathbf{f}} = \mathbf{A}^*\mathbf{f}$ using a standard conjugate gradient method, see [1, Ch. 11]. In particular, we aim to find generated sets $\Lambda(\mathbf{r},M)$ that even allow for a stable reconstruction of the Fourier coefficients of specific trigonometric polynomials.

For that reason we consider the spectral condition number of the matrix $\mathbf{B} = M^{-1}\mathbf{A}^*\mathbf{A}$, which is defined as

$$\mathrm{cond}_2(\mathbf{B}) = \frac{\lambda_{\max}}{\lambda_{\min}},$$

where $\lambda_{\max}$ and $\lambda_{\min}$ are the largest and smallest eigenvalues of $\mathbf{B}$, respectively. Note that $\mathbf{B}$ is a symmetric, positive semidefinite matrix with eigenvalues $0 \leq \lambda_{\min} \leq \lambda_{\max}$. In particular, the condition number of $\mathbf{B}$ is bounded below by one.

Besides the stability, the considered condition number measures the speed of convergence of the conjugate gradient method used to reconstruct the trigonometric polynomial f, cf. [1, Ch. 13]. The lower the condition number the faster our reconstruction algorithm converges.

Of course, one can consider the condition number as a function of different variables. Our approach fixes the frequency index set I, which results in a functional

$$\kappa(\mathbf{r}, M) := \mathrm{cond}_2(\mathbf{B}(\mathbf{r}, M))$$

depending on the generating vector $\mathbf{r}$ and the number of samples M, where $\mathbf{B}(\mathbf{r}, M) = M^{-1}\left(\mathbf{A}(\mathbf{r}, M)\right)^* \mathbf{A}(\mathbf{r}, M)$ and $\mathbf{A}(\mathbf{r}, M) = \left(\mathrm{e}^{2\pi \mathrm{i}\mathbf{k}\cdot\mathbf{x}}\right)_{\mathbf{x}\in\Lambda(\mathbf{r},M),\mathbf{k}\in I}$. Now we are interested in a generating vector $\mathbf{r}$ which minimizes the functional κ for fixed M. For relatively small cardinalities $|I|$ one can evaluate this condition number exactly. Thus, we can minimize the functional κ using nonlinear optimization techniques such as nonlinear simplex methods. The vectors (1) in the following example were determined in this way.

Example 1. We consider the weighted hyperbolic cross $H^{d,\boldsymbol{\gamma}}_{256}$ with $\boldsymbol{\gamma} = \left(4^{1-s}\right)_{s\in\mathbb{N}}$ and fix the number of sampling points $M = 16\,381 < 16\,384 = \lfloor \gamma_1 N \rfloor \lfloor \gamma_2 N \rfloor$. Hence, for $d \geq 2$ Lemma 2.1 in [6] yields that there does not exist any sampling scheme of $M = 16\,381$ nodes that allows for a perfectly stable reconstruction, i.e. it is proven that $\mathrm{cond}_2(\mathbf{B}) > 1$.

Nevertheless, we ask for a sampling scheme of cardinality $M = 16\,381$ with a stable Fourier matrix $\mathbf{A}$. Generated sets are our first choice because of the easy possibility of the fast evaluation and reconstruction. In fact, the vectors

$$\mathbf{r}_2 = \begin{pmatrix} 0.508425953824 \\ 0.058509185871 \end{pmatrix} \qquad \text{and} \qquad \mathbf{r}_5 = \begin{pmatrix} 0.075119519237 \\ 0.285056619170 \\ 0.500703041738 \\ 0.970811563102 \\ 0.568203958723 \end{pmatrix} \tag{1}$$

generate the two-dimensional set $\Lambda_2 = \Lambda(\mathbf{r}_2, 16\,381)$ and the five-dimensional set $\Lambda_5 = \Lambda(\mathbf{r}_5, 16\,381)$. The corresponding condition numbers $\mathrm{cond}_2(\mathbf{B}_s)$ with $\mathbf{B}_s = M^{-1}\left(\mathbf{A}_s\right)^* \mathbf{A}_s$ and Fourier matrices $\mathbf{A}_s = \left(\mathrm{e}^{2\pi \mathrm{i}\mathbf{k}\cdot\mathbf{x}}\right)_{\mathbf{x}\in\Lambda_s,\mathbf{k}\in H^{s,\boldsymbol{\gamma}}_N}$, $s = 2, 5$, are

$$\mathrm{cond}_2(\mathbf{B}_s) \approx \begin{cases} 3.9177, & \text{for } s = 2, \\ 11.934, & \text{for } s = 5. \end{cases}$$

Note that the corresponding matrices are square matrices with $\mathbf{B}_2 \in \mathbb{C}^{1761\times 1761}$ and $\mathbf{B}_5 \in \mathbb{C}^{2187\times 2187}$, respectively. □

Considering frequency index sets and corresponding generated sets of larger cardinalities, we cannot compute exact condition numbers efficiently. For that reason, we want to estimate the condition numbers from above.

3 Gerschgorin Circle Theorem and Generated Sets

In the following, we consider the Fourier matrix $\mathbf{A}(\mathbf{r},M)$ and its adjoint $\mathbf{A}^*(\mathbf{r},M)$ like above and apply the Gerschgorin circle theorem to the matrix $\mathbf{B}(\mathbf{r},M)$. Let us consider the elements

$$(\mathbf{B}(\mathbf{r},M))_{\mathbf{h},\mathbf{k}} = \frac{1}{M}\sum_{j=0}^{M-1} \mathrm{e}^{2\pi \mathrm{i} j(\mathbf{k}-\mathbf{h})\cdot\mathbf{r}} = \frac{1}{M}\sum_{j=0}^{M-1} \mathrm{e}^{2\pi \mathrm{i} j(y_{\mathbf{k}}-y_{\mathbf{h}})} =: K_M(y_{\mathbf{k}}-y_{\mathbf{h}}) \qquad (2)$$

of the matrix $\mathbf{B}(\mathbf{r},M)$. We define $y_{\mathbf{h}} = \mathbf{h}\cdot\mathbf{r} \bmod 1$ for all $\mathbf{h}\in I$ and therefore we can regard K_M as a univariate trigonometric kernel, which obviously is a Dirichlet kernel. Now we adapt some results from [9, Theorem 4.1] and formulate the following

Theorem 1. *We fix $\mathbf{r}\in\mathbb{R}^d$ and $I\in\mathbb{Z}^d$. Let $y_{\mathbf{h}} = \mathbf{h}\cdot\mathbf{r} \bmod 1$ for all $\mathbf{h}\in I$. Moreover, let us assume that we have sorted the sequence of $y_{\mathbf{h}}$'s in ascending order, i.e. $0 \le y_{\mathbf{h}_1} \le y_{\mathbf{h}_2} \le \ldots \le y_{\mathbf{h}_{|I|}} < 1$. In addition, we define the sequence of gaps $\mathbf{g}$*

$$g_j = \begin{cases} 1 + y_{\mathbf{h}_1} - y_{\mathbf{h}_{|I|}}, & \text{for } j = 1, \\ y_{\mathbf{h}_j} - y_{\mathbf{h}_{j-1}}, & \text{for } j = 2,\ldots,|I|. \end{cases}$$

Then, for $M\in\mathbb{N}$ the interval $\left[1 - M^{-1}\Delta(\mathbf{r}), 1 + M^{-1}\Delta(\mathbf{r})\right]$ with

$$\Delta\colon \mathbb{R}^d \to \mathbb{R}, \quad \mathbf{r} \mapsto \Delta(\mathbf{r}) := \sum_{k=1}^{\left\lfloor \frac{|I|}{2} \right\rfloor} \left(\sum_{t=1}^{k} g_{\pi(t)} \right)^{-1}, \qquad (3)$$

and π being a permutation of $\{1,\ldots,|I|\}$ ordering the gaps $0 \le g_{\pi(1)} \le g_{\pi(2)} \le \ldots \le g_{\pi(|I|)}$ contains all eigenvalues of the matrix

$$(\mathbf{B}(\mathbf{r},M))_{\mathbf{h},\mathbf{k}\in I} = (K_M(y_{\mathbf{k}}-y_{\mathbf{h}}))_{\mathbf{h},\mathbf{k}\in I}\,.$$

Proof. We consider the sequence $\left(g_{\pi(t)}\right)_{t=1}^{|I|}$. For $g_{\pi(1)} = 0$ we obtain at least one pair $\mathbf{k},\mathbf{h}\in I$, $\mathbf{h}\neq\mathbf{k}$ with $y_{\mathbf{h}} \equiv y_{\mathbf{k}} \pmod 1$. Accordingly, the matrix $\mathbf{B}(\mathbf{r},M)$ contains at least two identical columns and thus is not of full rank. So, a unique solution of $\mathbf{B}(\mathbf{r},M)\mathbf{x} = \mathbf{b}$ is not guaranteed. The smallest eigenvalue of the matrix $\mathbf{B}(\mathbf{r},M)$ is zero. On the other hand, the corresponding upper bound $M^{-1}\Delta(\mathbf{r})$ of the Ger-

schgorin circle radius of $\mathbf{B}(\mathbf{r},M)$ is infinite. Certainly, the interval $[-\infty,\infty]$ contains all eigenvalues of $\mathbf{B}(\mathbf{r},M)$.

Now let us assume $g_{\pi(1)} > 0$. Obviously, the diagonal elements of the considered matrices $\mathbf{B}(\mathbf{r},M)$ are all ones. Let λ_* be an arbitrary eigenvalue of $\mathbf{B}(\mathbf{r},M)$. Following the Gerschgorin circle theorem, there exists at least one index $j \in \{1,\dots,|I|\}$ with

$$|\lambda_* - 1| \leq \sum_{l=1;l\neq j}^{|I|} |K_M(y_{\mathbf{h}_j} - y_{\mathbf{h}_l})|.$$

For $x \in \mathbb{R}\setminus\mathbb{Z}$ we obtain

$$K_M(x) = \frac{1}{M}\sum_{j=0}^{M-1} \mathrm{e}^{2\pi \mathrm{i} jx} = \frac{1}{M}\frac{\mathrm{e}^{2\pi \mathrm{i} Mx}-1}{\mathrm{e}^{2\pi \mathrm{i} x}-1} = \frac{\mathrm{e}^{\pi \mathrm{i} Mx}}{\mathrm{e}^{\pi \mathrm{i} x}}\frac{\sin \pi Mx}{M\sin \pi x}. \tag{4}$$

Due to $2x \leq \sin \pi x$ for $x \in (0,1/2]$ we estimate

$$|K_M(x)| = \left|\frac{\sin \pi Mx}{M\sin \pi x}\right| \leq \frac{1}{|M\sin \pi x|} \leq \frac{1}{2Mx}$$

for all $x \in (0,1/2]$. Moreover, we have $|K_M(x)| = |K_M(-x)|$ for $x \in \mathbb{R}$.

We split the index set $J = \{1,\dots,|I|\}\setminus\{j\}$ in the following two subsets

$$J_1 = \{l \in J:\ 0 < y_{\mathbf{h}_j} - y_{\mathbf{h}_l} \bmod 1 \leq \frac{1}{2}\}$$
$$\text{and}\quad J_2 = \{l \in J:\ \frac{1}{2} < y_{\mathbf{h}_j} - y_{\mathbf{h}_l} \bmod 1 < 1\}.$$

This yields

$$\begin{aligned}\sum_{l=1;l\neq j}^{|I|} |K_M(y_{\mathbf{h}_j} - y_{\mathbf{h}_l})| &= \sum_{l\in J_1} |K_M(y_{\mathbf{h}_j} - y_{\mathbf{h}_l} \bmod 1)| + \sum_{l\in J_2} |K_M(-y_{\mathbf{h}_j} + y_{\mathbf{h}_l} \bmod 1)| \\ &\leq \frac{1}{2M}\sum_{l\in J_1}\frac{1}{y_{\mathbf{h}_j} - y_{\mathbf{h}_l} \bmod 1} + \frac{1}{2M}\sum_{l\in J_2}\frac{1}{y_{\mathbf{h}_l} - y_{\mathbf{h}_j} \bmod 1}.\end{aligned}$$

Now, we estimate the differences $y_{\mathbf{h}_j} - y_{\mathbf{h}_l} \bmod 1$. In principle, we interpret the index set J_1 as the indices of the left neighbors of $y_{\mathbf{h}_j}$. So, the distance of the nearest neighbor on the left hand side to $y_{\mathbf{h}_j}$ is at least $g_{\pi(1)}$. Clearly the second nearest neighbor at the left hand side brings a distance of at least $g_{\pi(1)} + g_{\pi(2)}$. In general the k-th nearest neighbor to the left of $y_{\mathbf{h}_j}$ has a distance not less than $\sum_{t=1}^{k} g_{\pi(t)}$ to $y_{\mathbf{h}_j}$. The index set J_2 can be interpreted as the index set of the right neighbors of $y_{\mathbf{h}_j}$ and we determine the lower bounds on the distances in the same way as done for the left neighbors. We obtain

$$\sum_{l=1;l\neq j}^{|I|} |K_M(y_{\mathbf{h}_j}-y_{\mathbf{h}_l})| \leq \frac{1}{2M}\sum_{k=1}^{|J_1|}\left(\sum_{t=1}^{k} g_{\pi(t)}\right)^{-1} + \frac{1}{2M}\sum_{k=1}^{|J_2|}\left(\sum_{t=1}^{k} g_{\pi(t)}\right)^{-1}.$$

Using $\sum_{t=1}^{k} g_{\pi(t)} \leq \sum_{t=1}^{r} g_{\pi(t)}$ for $k \leq r$ we balance the two sums and hence

$$\sum_{l=1;l\neq j}^{|I|} |K_M(y_{\mathbf{h}_j}-y_{\mathbf{h}_l})| \leq \frac{1}{M}\sum_{k=1}^{\left\lfloor\frac{|I|}{2}\right\rfloor}\left(\sum_{t=1}^{k} g_{\pi(t)}\right)^{-1},$$

which proves the theorem. □

Remark 1. In order to obtain the upper bound of the radii of all Gerschgorin circles in Theorem 1, we estimated the absolute value of the kernel K_M by a monotonically non-increasing upper bound $|2Mx|^{-1}$ in $\left[0,\frac{1}{2}\right]$. Due to $|K_M(\frac{t}{M})| = 0 < \frac{1}{|2t|} = |2M\frac{t}{M}|^{-1}$, for $t \in \mathbb{Z}\setminus M\mathbb{Z}$, the upper bound and the absolute value of the kernel K_M possibly differ widely. In addition, we sorted the pairwise distances of the sorted sequence $\left(y_{\mathbf{h}_j}\right)_{j=1,\ldots,|I|}$ in a worst case scenario. Thus, we also have to expect some differences between the estimation and the exact maximum Gerschgorin radius. Altogether, we obtain an estimation of the maximum Gerschgorin radius which eventually is much larger than the exact maximum Gerschgorin circle radius.

Corollary 1. *With the notation from Theorem 1, $\Delta(\mathbf{r}) < \infty$, and $C > 1$, we determine*

$$M^*(C) = \left\lceil \frac{C+1}{C-1}\Delta(\mathbf{r}) \right\rceil. \tag{5}$$

The condition number of the matrix $\mathbf{B}(\mathbf{r},M^*(C))$ is bounded by

$$1 \leq \kappa(\mathbf{r},M^*(C)) \leq C.$$

Proof. Fixing $M^*(C)$ in (5) ensures

$$C \geq \frac{1+M^*(C)^{-1}\Delta(\mathbf{r})}{1-M^*(C)^{-1}\Delta(\mathbf{r})} \geq \kappa(\mathbf{r},M^*(C)) \geq 1.$$

□

Our approach is to find generated sets $\Lambda(\mathbf{r},M)$ with small condition numbers $\kappa(\mathbf{r},M)$. Obviously, the term $\Delta(\mathbf{r})$ should be of our main interest here. The functional Δ is the important term of the upper bound $M^{-1}\Delta(\mathbf{r})$ of the radii of all Gerschgorin circles of the matrix $\mathbf{B}(\mathbf{r},M)$. Note that $\Delta(\mathbf{r})$ depends on the generating vector $\mathbf{r}$ of the generated set $\Lambda(\mathbf{r},M)$ but not on M. On the contrary, knowing $\Delta(\mathbf{r})$ one can simply determine a suitable $M^*(C)$ guaranteeing the condition number $\kappa(\mathbf{r},M^*(C)) \leq C$, see (5).

Algorithm 1 computes the value of $\Delta(\mathbf{r})$ for given I and $\mathbf{r}$ with a complexity of $\mathcal{O}(|I|(\log|I|+d))$.

Algorithm 1 Computing $\Delta(\mathbf{r})$ from (3)

Input: I frequency index set
$\mathbf{r} \in \mathbb{R}^d$ generating vector

$\Delta(\mathbf{r}) = 0$
for $j = 1, \dots, |I|$ **do**
  $y_j = \mathbf{h}_j \cdot \mathbf{r} \bmod 1$
end for
in-place sort $\mathbf{y}$ in ascending order
$g_1 = 1 + y_1 - y_{|I|}$
for $j = 2, \dots, |I|$ **do**
  $g_j = y_j - y_{j-1}$
end for
in-place sort $\mathbf{g}$ in ascending order
for $j = 1, \dots, \left\lfloor \frac{|I|}{2} \right\rfloor$ **do**
  $\Delta(\mathbf{r}) = \Delta(\mathbf{r}) + \frac{1}{g_j}$
  $g_{j+1} = g_{j+1} + g_j$
end for

Output: $\Delta(\mathbf{r})$

Another point of view is described by our approach as follows: Let us assume, that we search for a generated set $\Lambda(\mathbf{r}, M)$ such that the condition number $\kappa(\mathbf{r}, M)$ of the matrix $\mathbf{B}(\mathbf{r}, M)$ does not exceed C. We call the generating vector $\mathbf{r}$ suitable in the sense of Theorem 1 if $\Delta(\mathbf{r}) < \infty$, i.e. $|\mathscr{Y}| = |I|$. For each suitable $\mathbf{r}$, Corollary 1 specifies an $M^*_{\mathbf{r}}(C)$ guaranteeing a condition number $\kappa(\mathbf{r}, M^*_{\mathbf{r}}(C))$ not larger than C. So, minimizing the functional Δ directly reduces the cardinality $M^*_{\mathbf{r}}(C)$ of the corresponding generated set for fixed C. This means that the theoretical number of sampling nodes needed for the fast and stable reconstruction of the Fourier coefficients $(f_{\mathbf{k}})_{\mathbf{k} \in I}$ decreases.

Note that a simple lower bound on the functional Δ is given by

$$\Delta(\mathbf{r}) \geq |I| \sum_{k=1}^{\left\lfloor \frac{|I|}{2} \right\rfloor} k^{-1}, \quad \text{for all } \mathbf{r} \in \mathbb{R}^d. \tag{6}$$

We obtain equality, iff the sequence of $(y_{\mathbf{h}})_{\mathbf{h} \in I}$ is an equispaced lattice on the one-dimensional torus. In that case we can translate $y_{\mathbf{h}}$ such that $y_{\mathbf{h}_1} = 0$ and apply an equispaced FFT of length $|I|$ to reconstruct all Fourier coefficients supported on I.

Example 2. Continuing Example 1, we obtain the following rounded results by minimizing Δ using a nonlinear simplex search method:

$$\mathbf{r}_{2,\Delta} = \begin{pmatrix} 0.14266632 \\ 0.40770614 \end{pmatrix} \qquad \text{and} \qquad \mathbf{r}_{5,\Delta} = \begin{pmatrix} 0.24342553 \\ 0.42933779 \\ 0.05122878 \\ 0.88917104 \\ 0.94691925 \end{pmatrix}$$

with

$$\Delta(\mathbf{r}_{2,\Delta}) \approx 113\,324.3 \qquad \text{and} \qquad \Delta(\mathbf{r}_{5,\Delta}) \approx 161\,500.5.$$

The corresponding cardinalities $M_{2,\Delta}(10)$ and $M_{5,\Delta}(10)$ of $\Lambda(\mathbf{r}_{s,\Delta}, M_{s,\Delta}(10))$ guaranteeing a condition number $\kappa(\mathbf{r}_{s,\Delta}, M_{s,\Delta}(10)) = \mathrm{cond}_2(\mathbf{B}(\mathbf{r}_{s,\Delta}, M_{s,\Delta}))$ of at most ten are determined by

$$M_{2,\Delta}(10) = 138\,508 \qquad \text{and} \qquad M_{5,\Delta}(10) = 197\,390,$$

cf. (5). Of course, these $M_{s,\Delta}(10)$ are simply based on an upper bound of the Gerschgorin radii. We also computed the exact Gerschgorin radii numerically for the generating vectors $\mathbf{r}_{2,\Delta}$ and $\mathbf{r}_{5,\Delta}$ and different M_s and obtain

$$M_2^* = 14\,989 \qquad \text{and} \qquad M_5^* = 20\,129$$

guaranteeing condition numbers of $\mathbf{B}(\mathbf{r}_{s,\Delta}, M_s^*)$ smaller or equal ten. In fact, we get condition numbers

$$\kappa(\mathbf{r}_{s,\Delta}, M_s^*) \approx \begin{cases} 2.1847, & \text{for } s = 2, \\ 2.1037, & \text{for } s = 5. \end{cases}$$

Finally, we give the condition numbers of the problem of Example 1. We simply took the generating vectors $\mathbf{r}_{s,\Delta}$ and computed the condition numbers of $\mathbf{B}(\mathbf{r}_{s,\Delta}, M)$ for $M = 16\,381$ resulting in

$$\kappa(\mathbf{r}_{s,\Delta}, M) \approx \begin{cases} 1.7548, & \text{for } s = 2, \\ 2.9223, & \text{for } s = 5. \end{cases}$$

Obviously, these condition numbers are much smaller than those from Example 1, where we minimized the condition numbers directly. Note that the minimization of the main term Δ of the upper bound of all Gerschgorin radii is much faster than the direct minimization of the condition number κ. □

4 Numerical Examples

The numerical minimization of $\Delta(\mathbf{r})$ returns minimizers $\mathbf{r}^*$, which are vectors of rational numbers. So one can find a possibly huge $\bar{M}$, such that the generated set $\Lambda(\mathbf{r}^*, \bar{M})$ is a rank-1 lattice. With $M < \bar{M}$, one can interpret the generated set $\Lambda(\mathbf{r}^*, M)$ as the first M elements of the rank-1 lattice $\Lambda(\mathbf{r}^*, \bar{M})$. In general we obtain $M \ll \bar{M}$.

Our numerical examples use generating vectors $\mathbf{r}^*$ found by minimizing Δ, cf. (3). We used the nonlinear simplex search method `fminsearch` of the Optimization Toolbox™ of MATLAB in version 7.14.0.739 (R2012a).

Using the `rand` function, we started the minimization at a randomly chosen vector. Providing a diameter of the actual simplex smaller than 10^{-10} and differences of the function values at the corners of the simplex smaller than $10^{-8}|I|$, we terminated the minimization. Alternatively, we stopped the minimization after a fixed number of function evaluations, even if these conditions are not fulfilled. In order to compute the minimizers of Tables 1 and 2 we limited the number of function evaluations to 3 000. In Table 3, we accepted at most 30 000 function evaluations. The applied simplex search method finds only local minimizers. For each index set I we computed twenty local minimizers of Δ and took as $\mathbf{r}^*$ the local minimizer that yields the smallest value $\Delta(\mathbf{r})$ in order to avoid obtaining minimizers of relatively large local minima.

Besides the computation of $M^*(C) = \left\lceil \frac{C+1}{C-1}\Delta(\mathbf{r}^*)\right\rceil$ from (5) guaranteeing a condition number smaller or equal C we computed exact maximum Gerschgorin circle radii defined by

$$\rho(\mathbf{r}^*,M) = \max_{\mathbf{k}\in I} \sum_{\mathbf{h}\in I\setminus\{\mathbf{k}\}} |K_M(y_\mathbf{k} - y_\mathbf{h})|$$

for several M, where $K_M(y_\mathbf{k} - y_\mathbf{h})$ describes the elements of the matrix $\mathbf{B}(\mathbf{r}^*,M)$ as defined in (2). The Gerschgorin circle theorem ensures that the condition $\rho(\mathbf{r}^*,M) \leq \frac{C-1}{C+1}$ implies $\kappa(\mathbf{r}^*,M) \leq C$.

For a fixed vector $\mathbf{r}^*$ we define $M^*_{\mathsf{G}}(C)$ as the smallest power of two such that the exact maximum Gerschgorin circle radius ensures a condition number not larger than C,

$$M^*_{\mathsf{G}}(C) = \min_{n\in\mathbb{N}} \left\{ 2^n : \rho(\mathbf{r}^*,2^n) \leq \frac{C-1}{C+1} \right\}.$$

Moreover, we call $\kappa^*_{\mathsf{G}}(M) := \frac{1+\rho(\mathbf{r}^*,M)}{1-\rho(\mathbf{r}^*,M)}$ the estimation of the condition number of the matrix $\mathbf{B}(\mathbf{r}^*,M)$ based on the exact maximum Gerschgorin circle radius. Certainly, we have to assume $\rho(\mathbf{r}^*,M) < 1$ to estimate the condition number $\kappa^*(M) := \kappa(\mathbf{r}^*,M) \leq \kappa^*_{\mathsf{G}}(M)$. Otherwise, i.e. $\rho(\mathbf{r}^*,M) \geq 1$, we obtain $0 \in [1-\rho(\mathbf{r}^*,M), 1+\rho(\mathbf{r}^*,M)]$ and so zero is a candidate for the smallest eigenvalue of $\mathbf{B}(\mathbf{r}^*,M)$. Consequently, $\kappa^*_G(M) < 0$ does not bound the condition number of $\mathbf{B}(\mathbf{r}^*,M)$.

Applying (4), the computational costs for calculating $M^*_{\mathsf{G}}(C)$ is bounded by $c|I|^2\log_2(M^*(C))$, where c is independent of I and $M^*(C)$. In general, this computation is not necessary in order to obtain stable spatial discretizations, but the costs for computing $M^*_{\mathsf{G}}(C)$ can be quickly compensated using the generated set $\Lambda(\mathbf{r}^*,M^*_{\mathsf{G}}(C))$ instead of $\Lambda(\mathbf{r}^*,M^*(C))$ in practical applications. In particular for a frequently used fixed index set I and generating vector $\mathbf{r}^*$, the generated set $\Lambda(\mathbf{r}^*,M^*_{\mathsf{G}}(C))$ with cardinality $M^*_{\mathsf{G}}(C) < M^*(C)$ saves sampling and computational costs.

Table 1 Cardinalities $M^*(10)$, $M^*_{\mathsf{G}}(10)$, and $M^{**}(10)$ of generated sets generated by vectors $\mathbf{r}^*$ that are found by minimizing Δ for index sets I which are weighted hyperbolic crosses $H_{32}^{d,\boldsymbol{\gamma}}$ with weights $\boldsymbol{\gamma}_a = (2^{-1})_{s\in\mathbb{N}}$ and dimensions $d = 2,\ldots,12$; additionally, the condition numbers κ^* and upper bounds κ^*_{G} of the corresponding matrices $\mathbf{B}(\mathbf{r}^*, M^*_{\mathsf{G}}(10))$ and $\mathbf{B}(\mathbf{r}^*, M^{**}(10))$, respectively

d	$\lvert H_{32}^{d,\boldsymbol{\gamma}_a}\rvert$	$M^*(10)$	$M^*_{\mathsf{G}}(10)$	$\kappa^*_{\mathsf{G}}(M^*_{\mathsf{G}}(10))$	$\kappa^*(M^*_{\mathsf{G}}(10))$	$M^{**}(10)$	$\kappa^*_{\mathsf{G}}(M^{**}(10))$	$\kappa^*(M^{**}(10))$
2	145	2 216	1 024	2.8466	1.4540	370	11.9876	2.0705
3	441	9 709	4 096	2.8441	1.2811	1 408	-7.1782	1.8418
4	1 105	48 328	8 192	8.1211	1.9993	6 291	-15.0278	2.1016
5	2 433	151 727	32 768	3.0456	1.4690	18 119	4.5528	1.4570
6	4 865	471 958	65 536	6.1468	1.8703	52 492	4.3597	1.6842
7	9 017	1 115 494	131 072	5.2118	1.9046	116 850	7.1495	1.8891
8	15 713	2 538 107	262 144	4.3721	1.7571	252 533	4.1645	1.7939
9	26 017	6 256 440	524 288	5.5048	2.1663	595 180	3.2571	1.7159
10	41 265	15 910 747	2 097 152	3.1616	1.7769	1 454 830	4.1586	1.8770
11	63 097	29 880 128	2 097 152	5.7801	2.5378	2 637 334	2.9012	1.8379
12	93 489	46 057 959	4 194 304	4.4024	1.7782	4 065 252	4.0095	1.7426

4.1 Weighted Hyperbolic Crosses

Tables 1 and 2 shows some numerical examples for weighted hyperbolic crosses $H_N^{d,\boldsymbol{\gamma}}$ as frequency index sets I. In Table 1 we consider weights $\boldsymbol{\gamma}_a = \left(2^{-1}\right)_{s\in\mathbb{N}}$, refinement $N = 32$, and dimensions d from two up to twelve as parameters and determine suitable generated sets for reconstructing trigonometric polynomials with frequencies supported on I. Table 2 contains similar results for weights $\boldsymbol{\gamma}_b = (3^{-1})_{s\in\mathbb{N}}$, refinement $N = 48$, and dimensions d up to 27. The parameters chosen ensure $H_{48}^{d,\boldsymbol{\gamma}_b} \subset H_{32}^{d,\boldsymbol{\gamma}_a}$. In detail, we obtain

$$H_{32}^{d,\boldsymbol{\gamma}_a} \setminus H_{48}^{d,\boldsymbol{\gamma}_b} \subset \left\{ \mathbf{k} \in H_{32}^{d,\boldsymbol{\gamma}_a} : \|\mathbf{k}\|_0 = \sum_{s=1}^{d} (1 - \delta_0(k_s)) > 1 \right\},$$

i.e. the hyperbolic cross $H_{48}^{d,\boldsymbol{\gamma}_b}$ is sparser than $H_{32}^{d,\boldsymbol{\gamma}_a}$ in mixed indices only.

The first column of these tables shows the dimension d and the second column the cardinality of the considered frequency index set $H_N^{d,\boldsymbol{\gamma}}$. We minimize Δ like described above and obtain the resulting theoretical number of sampling points $M^*(10)$ needed to ensure a condition number of $\mathbf{B}(\mathbf{r}^*, M^*(10))$ not larger than ten. $M^*(10)$ is listed in column three. Fixing I and $\mathbf{r}^*$, in column four we present the smallest power of two $M^*_{\mathsf{G}}(10)$ guaranteeing that the exact maximum Gerschgorin radius is not larger than $\frac{9}{11}$. This restriction ensures that even the condition number of $\mathbf{B}(\mathbf{r}^*, M^*_{\mathsf{G}}(10))$ is not larger than ten. In other words, sampling along the first $M^*_{\mathsf{G}}(10)$ multiples of the generating vector $\mathbf{r}^*$ already guarantees a stable reconstruction of all multivariate trigonometric polynomials with frequencies supported on $H_N^{d,\boldsymbol{\gamma}}$. We specify the corresponding estimations of the condition numbers based on the maximum Gerschgorin radius labeled with $\kappa^*_{\mathsf{G}}(M^*_{\mathsf{G}}(10))$ in column five. Column six shows the corresponding exact condition numbers $\kappa^*(M^*_{\mathsf{G}}(10))$.

Regarding both tables, one observes that the values of $M^*_{\mathsf{G}}(10)$ behave like

Table 2 Cardinalities $M^*(10)$, $M^*_{\mathrm{G}}(10)$, and $M^{**}(10)$ of generated sets generated by vectors $\mathbf{r}^*$ that are found by minimizing Δ for index sets I which are weighted hyperbolic crosses $H^{d,\boldsymbol{\gamma}}_{48}$ with weights $\boldsymbol{\gamma}_b = (3^{-1})_{s\in\mathbb{N}}$ and dimensions $d = 2,\dots,27$; additionally, the condition numbers κ^* and upper bounds κ^*_{G} of the corresponding matrices $\mathbf{B}(\mathbf{r}^*, M^*_{\mathrm{G}}(10))$ and $\mathbf{B}(\mathbf{r}^*, M^{**}(10))$, respectively

d	$\lvert H^{d,\boldsymbol{\gamma}_b}_{48}\rvert$	$M^*(10)$	$M^*_{\mathrm{G}}(10)$	$\kappa^*_{\mathrm{G}}(M^*_{\mathrm{G}}(10))$	$\kappa^*(M^*_{\mathrm{G}}(10))$	$M^{**}(10)$	$\kappa^*_{\mathrm{G}}(M^{**}(10))$	$\kappa^*(M^{**}(10))$
2	105	1 878	512	5.6271	1.8467	354	8.9423	1.7548
3	225	5 916	2 048	2.6898	1.4053	1 006	31.1502	1.9347
4	401	16 432	4 096	5.4966	1.9237	2 588	5.0743	1.4437
5	641	34 464	8 192	3.8947	1.7342	5 110	2.6683	1.2588
6	953	91 526	16 384	4.5491	1.9292	12 912	3.1832	1.6643
7	1 345	120 893	16 384	5.3483	1.8287	16 351	5.3015	1.8288
8	1 825	244 266	32 768	3.1868	1.7918	31 856	3.6431	1.7963
9	2 401	400 917	65 536	2.1847	1.5143	50 639	3.1209	1.6695
10	3 081	595 978	65 536	3.3692	1.8932	73 163	3.0289	1.7584
11	3 873	960 647	131 072	2.3670	1.5973	114 946	2.3611	1.5942
12	4 785	1 265 910	131 072	3.2820	1.9410	147 990	2.3929	1.6242
13	5 825	1 875 694	262 144	2.6817	1.7231	214 662	2.4461	1.5622
14	7 001	2 135 009	262 144	2.9406	1.7604	239 603	2.9177	1.7633
15	8 321	3 310 334	262 144	5.2203	2.8743	364 835	2.7691	1.7001
16	9 793	4 831 312	524 288	2.6954	1.6401	523 570	2.6728	1.6412
17	11 425	6 156 192	1 048 576	3.0616	1.8571	656 735	2.4556	1.7802
18	13 225	7 735 764	1 048 576	2.9312	1.8649	813 162	2.5909	1.8368
19	15 201	9 885 874	1 048 576	2.0752	1.6079	1 024 862	2.1632	1.5777
20	17 361	11 784 210	1 048 576	2.8501	2.0048	1 205 779	2.2593	1.6075
21	19 713	16 342 704	2 097 152	2.8862	1.8002	1 651 639	2.0994	1.5800
22	22 265	18 916 637	2 097 152	1.9674	1.5404	1 889 453	2.0036	1.5755
23	25 025	27 027 375	2 097 152	4.1966	2.7862	2 669 617	2.1054	1.6669
24	28 001	30 693 609	4 194 304	2.8452	1.8359	2 999 686	1.8831	1.5614
25	31 201	37 040 314	4 194 304	1.8965	1.5780	3 583 403	1.9974	1.6094
26	34 633	41 051 986	4 194 304	2.7275	1.8536	3 933 160	2.8633	1.8562
27	38 305	46 404 278	4 194 304	2.0293	1.6813	4 328 544	2.0485	1.5950

$$M^*_{\mathrm{G}}(10) \sim M^{**}(10) := \left\lfloor M^*(10) \left(\sum_{k=1}^{\left\lfloor \frac{|I|}{2} \right\rfloor} k^{-1} \right)^{-1} \right\rfloor .$$

We listed the values of $M^{**}(10)$. The equispaced case discussed in the context of (6) illustrates that this observation is being caused by the construction of the functional Δ from (3). We also computed the exact maximum Gerschgorin circle radii $\rho(\mathbf{r}^*, M^{**}(10))$, the estimator of the condition number $\kappa^*_{\mathrm{G}}(M^{**}(10))$, and the exact condition numbers $\kappa^*(M^{**}(10))$ of the corresponding matrices $\mathbf{B}(\mathbf{r}^*, M^{**}(10))$. One obtains a few exceptions only where the maximum Gerschgorin circle radii $\rho(\mathbf{r}^*, M^{**}(10))$ strongly exceeds the bound $\frac{9}{11}$ that guarantees an upper bound $\kappa^*_{\mathrm{G}}(M^{**}(10))$ of the condition number $\kappa^*(M^{**}(10))$ smaller or equal ten. Nevertheless, all exact condition numbers $\kappa^*(M^{**}(10))$ of the matrices $\mathbf{B}(\mathbf{r}^*, M^{**}(10))$ do not exceed three in Table 1 and two in Table 2, evidently.

4.2 Randomly Chosen Index Sets

As described above, our approach finds stable spatial discretizations of trigonometric polynomials with frequencies supported on arbitrary known index sets I. So, we consider index sets I randomly chosen from the d-dimensional cube $[-128,128]^d \subset \mathbb{Z}^d$ in Table 3. It presents results for several cardinalities of the index set I and dimensions d which are powers of two. The content of each column is as described above. To achieve these results we increased the maximum number of the allowed function evaluations. In higher dimensions, this seems to be necessary to suitably decrease the diameter of the simplex in the used optimization method. For comparability we chose this parameter independent on the dimension d. So we allowed at most 30 000 function evaluations to minimize $\Delta(\mathbf{r})$.

We see that in principle the cardinalities $M^*(10)$, $M^*_{\mathsf{G}}(10)$, and $M^{**}(10)$ mildly decrease with growing dimensions. In other words, an increasing number of degrees of freedom of the functional Δ results in a lower minimal value.

Furthermore, we observe a growing oversampling with increasing cardinality of the index set I. For a doubled cardinality of I, the values of $M^*(10)$, $M^*_{\mathsf{G}}(10)$, and $M^{**}(10)$ increase approximately fourfold. Thus, the cardinalities of the found generated sets grow nearly quadratical in the cardinality of the index set I. Taking into account some modifications of the results of Theorem 3.2 in [6], we also expect this behavior for rank-1 lattices which bring a full column rank of the corresponding Fourier matrix $\mathbf{A}$. Accordingly, we expect to evaluate and reconstruct the multivariate trigonometric polynomial with frequencies supported on I with a complexity of $\mathcal{O}\left(|I|^2 \log|I| + (|\log\varepsilon| + d)|I|\right)$. Precomputing the set $\mathscr{Y}$ and saving the necessarily bijective mapping $I \to [0,1) : \mathbf{h} \mapsto \mathbf{h}\cdot\mathbf{r} \pmod 1$ we reduce the complexity to $\mathcal{O}\left(|I|^2 \log|I| + |\log\varepsilon||I|\right)$, which is independent of the spatial dimension d.

5 Summary

The concept of generated sets provides mildly oversampled and stable spatial discretizations for multivariate trigonometric polynomials with frequencies supported on index sets I of reasonable cardinalities. In addition, the NFFT and some simple precomputations allow for the fast evaluation of multivariate trigonometric polynomials f at all sampling nodes of generated sets $\Lambda(\mathbf{r},M)$. Assuming the condition number $\mathrm{cond}_2(\mathbf{B}(\mathbf{r},M))$ equal or near one, the conjugate gradient method using the NFFT and its adjoint provide the fast, stable, and unique reconstruction of f from samples along the generated set $\Lambda(\mathbf{r},M)$. Our approach imposes only one important condition on the generating vector $\mathbf{r} \in \mathbb{R}^d$: Successive elements of the one-dimensional sampling scheme $\mathscr{Y} = \{\mathbf{k}\cdot\mathbf{r} \bmod 1 : \ \mathbf{k} \in I\}$ should have relatively large distances.

Acknowledgements The author thanks Daniel Potts and Stefan Kunis for numerous valuable discussions. Moreover, he thanks the referees for their very useful suggestions for improvements

Table 3 Cardinalities $M^*(10)$, $M^*_{\mathrm{G}}(10)$, and $M^{**}(10)$ of generated sets generated by vectors $\mathbf{r}^*$ that are found by minimizing Δ for index sets I of dimensions $d = 2^n$, $n = 1,\dots,8$. Elements of I taken from $[-128,128]^d \cap \mathbb{Z}^d$ uniformly at random; additionally, condition numbers κ^* and upper bounds κ^*_{G} of the corresponding matrices $\mathbf{B}(\mathbf{r}^*, M^*_{\mathrm{G}}(10))$ and $\mathbf{B}(\mathbf{r}^*, M^{**}(10))$, respectively

d	$\lvert I\rvert$	$M^*(10)$	$M^*_{\mathrm{G}}(10)$	$\kappa^*_{\mathrm{G}}(M^*_{\mathrm{G}}(10))$	$\kappa^*(M^*_{\mathrm{G}}(10))$	$M^{**}(10)$	$\kappa^*_{\mathrm{G}}(M^{**}(10))$	$\kappa^*(M^{**}(10))$
2	750	195 091	32 768	1.8286	1.4154	29 988	2.1015	1.6151
4	750	196 928	32 768	2.8119	1.9440	30 271	3.9359	2.2560
8	750	166 797	32 768	1.9991	1.5679	25 639	2.2685	1.7329
16	750	144 334	16 384	7.0620	2.8097	22 186	2.5448	1.7345
32	750	104 756	16 384	2.8085	1.6086	16 102	2.6035	1.6000
64	750	61 856	8 192	7.7925	2.0178	9 508	5.9218	1.8894
128	750	62 873	8 192	8.3523	2.1612	9 664	6.1158	1.9719
256	750	54 611	8 192	5.3793	1.8811	8 394	5.3329	1.9067
2	1 500	412 137	65 536	1.7069	1.2490	57 257	1.8188	1.3153
4	1 500	851 612	131 072	2.3944	1.8503	118 313	3.6857	2.4721
8	1 500	647 439	65 536	5.0975	3.2850	89 947	3.4072	1.8601
16	1 500	619 395	65 536	6.2502	3.2448	86 051	2.2943	1.7509
32	1 500	411 622	65 536	2.0989	1.5788	57 185	2.6901	1.7571
64	1 500	324 658	32 768	6.7930	2.6120	45 104	2.5211	1.7025
128	1 500	254 375	32 768	3.2401	1.8037	35 339	2.6962	1.8009
256	1 500	226 842	32 768	3.5078	1.8600	31 514	3.4103	1.8070
2	3 000	547 361	65 536	1.9555	1.2360	69 367	2.0029	1.2113
4	3 000	3 265 505	262 144	8.4853	6.5363	413 838	2.6677	2.4452
8	3 000	3 078 366	262 144	8.7348	4.9750	390 122	2.4379	2.0957
16	3 000	2 434 510	262 144	3.0088	2.6453	308 526	2.0252	1.8461
32	3 000	1 675 774	262 144	1.9351	1.6033	212 371	2.2124	1.6160
64	3 000	1 213 198	131 072	5.4684	2.8743	153 749	2.5438	1.7883
128	3 000	1 034 141	131 072	2.3803	1.6471	131 057	2.3964	1.6471
256	3 000	805 916	131 072	3.1876	1.8898	102 134	2.7344	1.7359
2	6 000	674 627	65 536	6.0235	1.6905	78 593	6.9360	1.5907
4	6 000	13 250 802	1 048 576	7.2950	6.1781	1 543 707	2.9009	2.6842
8	6 000	11 501 870	2 097 152	1.8113	1.5978	1 339 958	4.3485	2.9201
16	6 000	10 191 192	1 048 576	3.2384	2.8720	1 187 265	2.3771	2.1829
32	6 000	7 573 185	1 048 576	2.3761	1.7520	882 269	2.1095	1.8821
64	6 000	5 661 152	524 288	6.4483	3.8637	659 519	2.2256	1.7379
128	6 000	3 777 565	524 288	2.2653	1.6630	440 083	2.1709	1.6788
256	6 000	3 311 017	524 288	2.6142	1.8378	385 730	2.3017	1.6661

and he gratefully acknowledges support by German Research Foundation within the project KU 2557/1-1.

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