Analysis II

13. Übung – Unbestimmte Integration

---

1. Man bestimme mit Hilfe (elementarer) Zurückführung auf Grundintegrale

   (a) $\displaystyle\int \left(1-\frac{1}{x^2}\right)\sqrt{x\sqrt{x}}\,dx\,,$ (b) $\displaystyle\int \frac{\sqrt{1+x^2}+\sqrt{1-x^2}}{\sqrt{1-x^4}}\,dx\,,$ (c) $\displaystyle\int \sqrt{1-\sin 2x}\,dx\,,$

   (d) $\displaystyle\int \frac{e^{3x}+1}{e^x+1}\,dx\,,$ (e) $\displaystyle\int \tan^2 x\,dx\,,$ (f) **(HA)** $\displaystyle\int (1-x)(1-2x)(1-3x)\,dx\,,$

   (g) **(HA)** $\displaystyle\int \frac{x+1}{\sqrt{x}}\,dx\,,$ (h) **(HA)** $\displaystyle\int \frac{\sqrt{x}-2\sqrt[3]{x^2}+1}{\sqrt[4]{x}}\,dx\,,$ (i) **(HA)** $\displaystyle\int \frac{(1-x)^3}{x\sqrt[3]{x}}\,dx\,.$

2. Man bestimme mit Hilfe geeigneter Substitutionen

   (a) $\displaystyle\int \frac{x}{\sqrt{1-x^2}}\,dx\,,$ (b) $\displaystyle\int \frac{1}{x^2}\,\sin\frac{1}{x}\,dx\,,$ (c) $\displaystyle\int \frac{dx}{\sqrt{x(1+x)}}$ (d) $\displaystyle\int \frac{dx}{(x\ln x)\,\ln(\ln x)}\,,$

   (e) $\displaystyle\int \tan x\,dx\,,$ (f) $\displaystyle\int \sin^5 x\cos x\,dx\,,$ (g) $\displaystyle\int \frac{dx}{\sin x}$ (h) **(HA)** $\displaystyle\int \frac{\arctan x}{1+x^2}\,dx\,,$

   (i) **(HA)** $\displaystyle\int \frac{x\,dx}{3-2x^2}\,,$ (j) **(HA)** $\displaystyle\int \frac{e^x}{2+e^x}\,dx\,,$ (k) **(HA)** $\displaystyle\int x\,e^{-x^2}\,dx\,,$

   (l) **(HA)** $\displaystyle\int \frac{dx}{\sinh x}\,.$

3. Man bestimme mittels partieller Integration

   (a) $\displaystyle\int \left(\frac{\ln x}{x}\right)^2 dx\,,$ (b) $\displaystyle\int \sqrt{x}\ln^2 x\,dx\,,$ (c) $\displaystyle\int \arctan\sqrt{x}\,dx\,,$

   (d) $\displaystyle\int \sin x\,\ln(\tan x)\,dx\,,$ (e) $\displaystyle\int x\ln\frac{1+x}{1-x}\,dx\,,$ (f) $\displaystyle\int \sin^2 x\,dx\,,$

   (g) **(HA)** $\displaystyle\int e^{\alpha x}\sin\beta x\,dx$ $(\alpha,\beta\in\mathbb{R}\,,\ \beta\neq 0)\,,$ (h) **(HA)** $\displaystyle\int x^2e^{-2x}\,dx\,,$

   (i) **(HA)** $\displaystyle\int x^2\sin 2x\,dx\,,$ (j) **(HA)** $\displaystyle\int \arctan x\,dx\,,$ **(Z)** $\displaystyle\int x^n e^x\,dx\,.$

4. Man berechne mittels Partialbruchzerlegung

   (a) $\displaystyle\int \frac{2x+3}{(x-2)(x+5)}\,dx\,,$ (b) $\displaystyle\int \frac{x^4dx}{x^4+5x^2+4}\,,$ (c) $\displaystyle\int \frac{dx}{(x+1)(x+2)^2(x+3)^2}\,,$

   (d) $\displaystyle\int \left(\frac{x}{x^2+3x+2}\right)^2 dx\,,$ (e) $\displaystyle\int \frac{dx}{x^5+x^4-2x^3-2x^2+x+1}\,,$ (f) $\displaystyle\int \frac{dx}{x^4+1}\,,$

   (g) **(HA)** $\displaystyle\int \frac{x\,dx}{(x+1)(x+2)(x+3)}\,,$ (h) **(HA)** $\displaystyle\int \left(\frac{x}{x^2-3x+2}\right)^2 dx\,,$

   (i) **(HA)** $\displaystyle\int \frac{dx}{x^3+1}\,,$ (j) **(HA)** $\displaystyle\int \frac{x^2\,dx}{(x^2+2x+2)^2}\,.$

5. Man berechne

(a) $\displaystyle\int \frac{\sin x \cos x}{\sin^4 x + \cos^4 x}\, dx\,,$ (b) $\displaystyle\int \frac{dx}{1+2\sin^2 x}\,,$ (c) $\displaystyle\int \frac{dx}{(x^2-1)(x^2+1)}\,,$

(d) $\displaystyle\int \frac{1+x}{1-x}\, dx\,,$ (e) $\displaystyle\int \frac{2x+5}{x^2+4x+6}\, dx\,,$ (f) $\displaystyle\int \frac{x^2}{(x-1)^{100}}\, dx\,,$ (l) **(HA)** $\displaystyle\int \frac{dx}{2\sin 2x}\,,$

(g) $\displaystyle\int \frac{dx}{(x^2-4x+4)(x^2-4x+5)}\,,$ (h) $\displaystyle\int \sqrt{\frac{a+x}{a-x}}dx\,,$ (i) **(HA)** $\displaystyle\int (\arcsin x)^2\, dx\,,$

(j) **(HA)** $\displaystyle\int x\,(\arctan x)^2\, dx\,,$ (k) **(HA)** $\displaystyle\int \frac{x\, dx}{\sqrt{1+x^2+\sqrt{(1+x^2)^3}}}\,.$

6. Entwickeln Sie eine Rekursionsformel zur Berechnung von

(a) $\displaystyle I_n(x) = \int \frac{dx}{\cos^n x}\,,$ (b) **(HA)** $\displaystyle S_n(x) = \int \sin^n x dx\,.$

Zusatz:

Berechnen Sie $\displaystyle\int \left(\frac{\ln x}{x}\right)^n dx\,,\ n \in \mathbb{N}\setminus\{1\}$, mit Hilfe eines Polynomansatzes

$$\frac{P_n(\ln x)}{Q_{n-1}(x)}\,,$$

wobei P_k und Q_k Polynome k-ten Grades bezeichnen.

13. Hausaufgabe

---

1. Bestimmen Sie die Stammfunktionen folgender Funktionen:

(a) $\displaystyle f(x) = \frac{1}{x^3-2x^2+3x},$ (b) $f(x) = x^n \ln x\ (n\in\mathbb{N}),$ (c) $f(x) = x\, e^{x^2},$

(d) $\displaystyle f(x) = \frac{1}{3^x+1}.$

Überprüfen Sie Ihre Ergebnisse!

2. Für welche $a, b, c \in \mathbb{R}$ ist das Integral $\displaystyle\int \frac{ax^2+bx+c}{x^3(x-1)^2}\, dx$ eine rationale Funktion?

3. Lösen Sie die mit **(HA)** gekennzeichneten Aufgaben der 13. Übung.

Zusatz: Berechnen Sie

(a) $\displaystyle\int \frac{\cos x}{\sqrt{\cos 2x}}\, dx\,,$ (b) $\displaystyle\int \frac{x \arctan x}{\sqrt{1-x^2}}\, dx\,,$ (c) $\displaystyle\int (\tan x)\, \tan(x+a)\, dx\,,$

(d) $\displaystyle\int x f''(x)\, dx\,,$ (e) $\displaystyle\int \frac{(9\sin^2 x - 3\sin^3 x)\cos x - 5\sin 2x + 10\cos x}{\sin^4 x - 2\sin^3 x + 2\sin x - 1}\, dx\,,$

(f) $\displaystyle\int \frac{\cos^2 x}{1+\sin^2 x}\, dx\,,$ (g) $\displaystyle\int \frac{dx}{1+\sin x+\cos x}\,.$