Seminar: Abelian Varieties (Winter 26/27)

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Time: Fri 10:15 – 11:45

Venue: MIN-F (Bu 56a) – SemRm 3.3

Abelian varieties play a fundamental role in modern algebraic and arithmetic geometry. Over the complex numbers, they are those complex tori which can be embedded as an algebraic subvariety of a projective space. As such they are at the same time flat manifolds and smooth projective varieties, and they come with a natural group structure. It is this interplay of different structures which makes them particularly interesting: On the one hand, they naturally arise from the study of path integrals on complex manifolds, and their presentation as a quotient of a complex vector space by a lattice allows to make many general notions from higher-dimensional complex geometry explicitly accessible. On the other hand, as subvarieties of projective space they carry a rich algebraic structure which lies at the heart of many applications in moduli theory and arithmetic geometry.

The goal of the seminar is to give a self-contained introduction to the theory of complex abelian varieties. We will only assume some basic complex analysis and the notion of a complex manifold, so the seminar may also serve as a first introduction to complex geometry. After some preliminaries about complex tori, we will start with a discussion of 1-dimensional abelian varieties (i.e. elliptic curves) which will motivate most of what follows. We will then turn to the higher-dimensional case: After a brief introduction to differential forms and de Rham cohomology, we will discuss conditions under which a complex manifold can be embedded in projective space. Making these conditions explicit for complex tori will lead us to the notion of theta functions. Conceptually these can be seen as sections of line bundles, one of the most important tools in modern algebraic geometry. Finally, by putting everything together, we will develop the basic theory of abelian varieties and their projective embeddings. If time permits, we may conclude with a brief outlook on further topics such as moduli spaces of abelian varieties.

We will mostly follow the first six chapters in the beautiful book by Debarre [1], with each chapter taking 1-2 talks as indicated below. There are many other excellent references for further reading: In particular, Mumford's book [2] is a classic for abelian varieties over arbitrary fields which includes a very readable chapter about the complex theory as well, and the book by Birkenhake and Lange [3] provides an extensive discussion of many further topics about complex abelian varieties. The following schedule is tentative and can be adapted depending on the audience; if you would like to participate in the seminar, please send me an emal to thomas.kraemer@mathematik.tu-chemnitz.de with a brief indication of your background and the talk(s) you would prefer / could imagine to give.

1. From complex tori to elliptic curves

Introduce lattices and complex tori [1, 1.1 - 1.2]. Discuss the Weierstraß $\wp$-function and explain how this function and its derivative can be used to identify complex tori with cubic plane curves [1, 2.1]; it may be good to do this first away from the set of poles, and then make a brief digression about projective spaces [1, 1.3] before passing to the associated projective curve.

2. Theta functions and Riemann-Roch for elliptic curves

In constructing embeddings to projective space, one may replace the role of the periodic meromorphic functions $\wp$ and $\wp'$ by so-called theta functions which are only almost periodic, but holomorphic. Explain this construction as in [1, 2.2], and then discuss divisors and the Riemann-Roch theorem [1, 2.3].

3. Differential Forms and de Rham cohomology

Explain the notion of differential forms, their integrals, and de Rham cohomology [1, 3.1 - 3.3]. Then illustrate these notions by (1) the example of complex tori [1, 3.4], and (2) the example of projective space [1, 3.5]. The latter in particular leads to a necessary condition for a complex manifold to be a submanifold of projective space [1, th. 3.12] (in fact this criterion is also sufficient).

4. Theta functions and divisors

The main goal of this talk is to show that every embedding of a complex torus into a projective space is given by theta functions, see cor. 2 in [1, 4.3, cor. 2]. This in particular leads to a more explicit view on the projectivity criterion from the previous talk, see cor. 3. To explain this on higher-dimensional tori, you first need to introduce the general notion of theta functions and divisors following [1, chapt. 4].

5. Line bundles, sheaves and sheaf cohomology

Introduce the notion of line bundles and, more generally, sheaves [1, 5.1 and 5.3]. Then explain by an example that for a short exact notion of sheaves, the associated sequence of global sections is usually not exact on the right, and explain how to remedy this defect by introducing sheaf cohomology [1, 5.4].

6. The Picard group and line bundles on complex tori

Introduce the first Chern class of a line bundle as in [1, 5.5]. Then explain the structure of the Picard group in the case of complex tori: First give the construction of line bundles on complex tori in [1, 5.2], then discuss the description of the first Chern class in these terms [1, example 5.15(2)], and prove the Appell-Humbert theorem [1, th. 5.17] by which *all* line bundles on a complex torus arise in this way.

7. Abelian Varieties

The main goal of this talk is the Lefschetz theorem which gives a complete answer to the question which complex tori can be embedded in projective space [1, th. 6.8]. For this you will need to discuss the Riemann conditions and the Riemann-Roch theorem on higher-dimensional complex tori [1, 6.1 - 6.2]; at some point you should also give the definition of an abelian variety [1, def. 6.16].

Depending on how much time remains, you may then give an outlook on the basic structure theory of abelian varieties as outlined in the other parts of this section, e.g. polarizations, Poincaré's reducibility theorem, endomorphisms etc. This might also be a separate talk.

References

[1] O. Debarre, Complex Tori and Abelian Varieties. SMF/AMS (1999)

[2] D. Mumford, Abelian Varieties. Oxford University Press / TIFR (1974)

[3] C. Birkenhake and H. Lange, Complex Abelian Varieties. Springer (2004).