Approximation of control systems via ANOVA methods

Melanie Kircheis

Chemnitz University of Technology
Faculty of Mathematics

joint work with
Daniel Potts, Manuel Schaller and Karl Worthmann

Control Systems Seminar
Hamburg, December 1st, 2025

Motivation

Unknown function (control system):

$$H\colon \mathbb{R}^d \times \mathbb{R}^m \to \mathbb{R}^d,$$
$$H(\boldsymbol{x}, \boldsymbol{u}) := F(\boldsymbol{x}) + G(\boldsymbol{x})\boldsymbol{u},$$

with

- $F\colon \mathbb{R}^d \to \mathbb{R}^d$, $G\colon \mathbb{R}^d \to \mathbb{R}^{d\times m}$,
- $d, m \in \mathbb{N}$, $m \ll d$.

Motivation

Unknown function (control system):

$$H\colon \mathbb{R}^d \times \mathbb{R}^m \to \mathbb{R}^d,$$
$$H(\boldsymbol{x}, \boldsymbol{u}) \coloneqq F(\boldsymbol{x}) + G(\boldsymbol{x})\boldsymbol{u},$$

with

- $F\colon \mathbb{R}^d \to \mathbb{R}^d$, $G\colon \mathbb{R}^d \to \mathbb{R}^{d\times m}$,
- $d, m \in \mathbb{N}$, $m \ll d$.

Given: samples $H(\boldsymbol{x}^i, \boldsymbol{u}^i)$ for $i = 1, \dots, M$

Motivation

Unknown function (control system):

$$H\colon \mathbb{R}^d \times \mathbb{R}^m \to \mathbb{R}^d,$$
$$H(\boldsymbol{x}, \boldsymbol{u}) := F(\boldsymbol{x}) + G(\boldsymbol{x})\boldsymbol{u},$$

with

- $F\colon \mathbb{R}^d \to \mathbb{R}^d$, $G\colon \mathbb{R}^d \to \mathbb{R}^{d\times m}$,
- $d, m \in \mathbb{N}$, $m \ll d$.

Given: samples $H(\boldsymbol{x}^i, \boldsymbol{u}^i)$ for $i = 1, \dots, M$

Goal: find $\tilde{H} \approx H$

Motivation

Unknown function (control system):

$$H\colon \mathbb{R}^d \times \mathbb{R}^m \to \mathbb{R}^d,$$
$$H(\boldsymbol{x}, \boldsymbol{u}) := F(\boldsymbol{x}) + G(\boldsymbol{x})\boldsymbol{u},$$

with

- $F\colon \mathbb{R}^d \to \mathbb{R}^d$, $G\colon \mathbb{R}^d \to \mathbb{R}^{d\times m}$,
- $d, m \in \mathbb{N}$, $m \ll d$.

Given: samples $H(\boldsymbol{x}^i, \boldsymbol{u}^i)$ for $i = 1, \ldots, M$

Goal: find $\tilde{H} \approx H$

Problem: curse of dimensionality

Motivation

Unknown function (control system):

$$H\colon \mathbb{R}^d \times \mathbb{R}^m \to \mathbb{R}^d,$$
$$H(\boldsymbol{x}, \boldsymbol{u}) := F(\boldsymbol{x}) + G(\boldsymbol{x})\boldsymbol{u},$$

with

- $F\colon \mathbb{R}^d \to \mathbb{R}^d$, $G\colon \mathbb{R}^d \to \mathbb{R}^{d\times m}$,
- $d, m \in \mathbb{N}$, $m \ll d$.

Given: samples $H(\boldsymbol{x}^i, \boldsymbol{u}^i)$ for $i = 1, \dots, M$

Goal: find $\tilde{H} \approx H$

Problem: curse of dimensionality

Assumption: F and G of low-dimensional structure

Motivation

Unknown function (control system):

$$H\colon \mathbb{R}^d \times \mathbb{R}^m \to \mathbb{R}^d,$$
$$H(\boldsymbol{x},\boldsymbol{u}) := F(\boldsymbol{x}) + G(\boldsymbol{x})\boldsymbol{u},$$

with

- $F\colon \mathbb{R}^d \to \mathbb{R}^d$, $G\colon \mathbb{R}^d \to \mathbb{R}^{d\times m}$,
- $d, m \in \mathbb{N}$, $m \ll d$.

Given: samples $H(\boldsymbol{x}^i, \boldsymbol{u}^i)$ for $i = 1, \ldots, M$

Goal: find $\tilde{H} \approx H$

Problem: curse of dimensionality

Assumption: F and G of low-dimensional structure

Example: Euler approximation of a controlled Duffing oscillator

$$\begin{aligned} H(\boldsymbol{x},\boldsymbol{u}) &= \boldsymbol{x} + \Delta t \begin{pmatrix} x_2 \\ x_1 - 3x_1^3 u \end{pmatrix} \\ &= \begin{pmatrix} x_1 + \Delta t x_2 \\ x_2 + \Delta t x_1 \end{pmatrix} + \begin{pmatrix} 0 \\ -3\Delta t x_1^3 \end{pmatrix} u \\ &= F(\boldsymbol{x}) + G(\boldsymbol{x}) u \end{aligned}$$

i. e.,

- $d = 2$,
- $m = 1$,
- only 1D terms $\rightsquigarrow$ low-dimensional

Overview

1. ANOVA approximation for scalar-valued functions
2. ANOVA approximation for control systems
3. Numerical Example

ANOVA approximation

ANalysis **O**f **VA**riance (ANOVA) decomposition

[Caflish, Morokoff, Owen 97], [Rabitz, Alis 99], [Liu, Owen 06], [Kuo, Sloan, Wasilkowski, Wozniakowski 10], ...

Decompose a d-dimensional function f into

$$
\begin{aligned}
f(\boldsymbol{x}) = f_\varnothing & & \text{constant}\\
&+ f_1(x_1) + \ldots + f_d(x_d) & \text{one-dimensional terms}\\
&+ f_{1,2}(x_1, x_2) + f_{1,3}(x_1, x_3) + \ldots + f_{d-1,d}(x_{d-1}, x_d) & \text{two-dimensional terms}\\
&+ f_{1,2,3}(x_1, x_2, x_3) + \ldots + f_{d-2,d-1,d}(x_{d-2}, x_{d-1}, x_d) & \text{three-dimensional terms}\\
&\quad \vdots & \\
&+ f_{[d]}(\boldsymbol{x}) & d\text{-dimensional term}
\end{aligned}
$$

$[d] := \{1, \ldots, d\}$

ANalysis Of VAriance (ANOVA) decomposition

[Caflish, Morokoff, Owen 97], [Rabitz, Alis 99], [Liu, Owen 06], [Kuo, Sloan, Wasilkowski, Wozniakowski 10], ...

Decompose a d-dimensional function f into

$$\begin{aligned} f(\boldsymbol{x}) = f_\varnothing & & \text{constant} \\ & + f_1(x_1) + \ldots + f_d(x_d) & \text{one-dimensional terms} \\ & + f_{1,2}(x_1, x_2) + f_{1,3}(x_1, x_3) + \ldots + f_{d-1,d}(x_{d-1}, x_d) & \text{two-dimensional terms} \\ & + f_{1,2,3}(x_1, x_2, x_3) + \ldots + f_{d-2,d-1,d}(x_{d-2}, x_{d-1}, x_d) & \text{three-dimensional terms} \\ & \quad \vdots & \\ & + f_{[d]}(\boldsymbol{x}) & d\text{-dimensional term} \\ = & \sum_{\boldsymbol{v} \subseteq [d]} f_{\boldsymbol{v}}(\boldsymbol{x}_{\boldsymbol{v}}) & \end{aligned}$$

$$[d] := \{1, \ldots, d\}$$

ANalysis Of VAriance (ANOVA) decomposition

[Caflish, Morokoff, Owen 97], [Rabitz, Alis 99], [Liu, Owen 06], [Kuo, Sloan, Wasilkowski, Wozniakowski 10], . . .

Decompose a d-dimensional function f into

$$\begin{aligned} f(\boldsymbol{x}) = f_\varnothing & & \text{constant} \\ + f_1(x_1) + \ldots + f_d(x_d) & & \text{one-dimensional terms} \\ + f_{1,2}(x_1,x_2) + f_{1,3}(x_1,x_3) + \ldots + f_{d-1,d}(x_{d-1},x_d) & & \text{two-dimensional terms} \\ + f_{1,2,3}(x_1,x_2,x_3) + \ldots + f_{d-2,d-1,d}(x_{d-2},x_{d-1},x_d) & & \text{three-dimensional terms} \\ \vdots & & \\ + f_{[d]}(\boldsymbol{x}) & & d\text{-dimensional term} \\ = \sum_{\boldsymbol{v}\subseteq[d]} f_{\boldsymbol{v}}(\boldsymbol{x}_{\boldsymbol{v}}) & & \end{aligned}$$

$[d] := \{1,\ldots,d\}$

- $\rightarrow$ in general multiple representations possible
- $\rightarrow$ conditions for uniqueness?

Definition

Let

$$\langle f, g\rangle_{L_2(\Omega,\omega)} := \int_\Omega f(\boldsymbol{x})\, g(\boldsymbol{x})\, \omega(\boldsymbol{x})\, \mathrm{d}\boldsymbol{x}$$

Definition

Let

$$\langle f, g \rangle_{L_2(\Omega,\omega)} := \int_\Omega f(\boldsymbol{x})\, g(\boldsymbol{x})\, \omega(\boldsymbol{x})\, \mathrm{d}\boldsymbol{x}$$

$\Rightarrow$ unique decomposition satisfying:

$$\boxed{0 = \langle f_{\boldsymbol{v}}, f_{\boldsymbol{z}} \rangle_{L_2(\Omega,\omega)} \text{ for all } \boldsymbol{v} \neq \boldsymbol{z} \subseteq [d]}$$

Definition

Let

$$\langle f, g \rangle_{L_2(\Omega,\omega)} := \int_\Omega f(\boldsymbol{x})\, g(\boldsymbol{x})\, \omega(\boldsymbol{x})\, \mathrm{d}\boldsymbol{x}$$

$\Rightarrow$ unique decomposition satisfying:

$0 = \langle f_{\boldsymbol{v}}, f_{\boldsymbol{z}} \rangle_{L_2(\Omega,\omega)}$ for all $\boldsymbol{v} \neq \boldsymbol{z} \subseteq [d]$

Basis representation: For orthonormal basis $\{\varphi_{\boldsymbol{k}}\}_{\boldsymbol{k}\in\mathbb{N}_0^d}$ of $L_2(\Omega,\omega)$ we have

$$f(\boldsymbol{x}) = \sum_{\boldsymbol{k}\in\mathbb{N}_0^d} c_{\boldsymbol{k}}(f)\, \varphi_{\boldsymbol{k}}(\boldsymbol{x}), \qquad c_{\boldsymbol{k}}(f) := \langle f, \varphi_{\boldsymbol{k}} \rangle_{L_2(\Omega,\omega)}.$$

Definition

Let

$$\langle f, g\rangle_{L_2(\Omega,\omega)} := \int_\Omega f(\boldsymbol{x})\, g(\boldsymbol{x})\, \omega(\boldsymbol{x})\, \mathrm{d}\boldsymbol{x}$$

$\Rightarrow$ unique decomposition satisfying:

$$\boxed{0 = \langle f_{\boldsymbol{v}}, f_{\boldsymbol{z}}\rangle_{L_2(\Omega,\omega)} \text{ for all } \boldsymbol{v} \neq \boldsymbol{z} \subseteq [d]}$$

Basis representation: For orthonormal basis $\{\varphi_{\boldsymbol{k}}\}_{\boldsymbol{k}\in\mathbb{N}_0^d}$ of $L_2(\Omega,\omega)$ we have

$$f(\boldsymbol{x}) = \sum_{\boldsymbol{k}\in\mathbb{N}_0^d} c_{\boldsymbol{k}}(f)\, \varphi_{\boldsymbol{k}}(\boldsymbol{x}), \qquad c_{\boldsymbol{k}}(f) := \langle f, \varphi_{\boldsymbol{k}}\rangle_{L_2(\Omega,\omega)}.$$

[Potts, Schmischke 21]:

$$f_{\boldsymbol{v}}(\boldsymbol{x}_{\boldsymbol{v}}) = \sum_{\substack{\boldsymbol{k}\in\mathbb{N}_0^d \\ \operatorname{supp} \boldsymbol{k}=\boldsymbol{v}}} c_{\boldsymbol{k}}(f)\, \varphi_{\boldsymbol{k}}(\boldsymbol{x}_{\boldsymbol{v}})$$

Definition

Let

$$\langle f, g\rangle_{L_2(\Omega,\omega)} := \int_\Omega f(\boldsymbol{x})\, g(\boldsymbol{x})\, \omega(\boldsymbol{x})\, \mathrm{d}\boldsymbol{x}$$

$\Rightarrow$ unique decomposition satisfying:

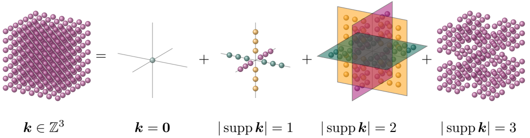
$$\boxed{0 = \langle f_{\boldsymbol{v}}, f_{\boldsymbol{z}}\rangle_{L_2(\Omega,\omega)} \text{ for all } \boldsymbol{v} \neq \boldsymbol{z} \subseteq [d]}$$

Basis representation: For orthonormal basis $\{\varphi_{\boldsymbol{k}}\}_{\boldsymbol{k}\in\mathbb{N}_0^d}$ of $L_2(\Omega,\omega)$ we have

$$f(\boldsymbol{x}) = \sum_{\boldsymbol{k}\in\mathbb{N}_0^d} c_{\boldsymbol{k}}(f)\, \varphi_{\boldsymbol{k}}(\boldsymbol{x}), \qquad c_{\boldsymbol{k}}(f) := \langle f, \varphi_{\boldsymbol{k}}\rangle_{L_2(\Omega,\omega)}.$$

[Potts, Schmischke 21]:

$$f_{\boldsymbol{v}}(\boldsymbol{x}_{\boldsymbol{v}}) = \sum_{\substack{\boldsymbol{k}\in\mathbb{N}_0^d \\ \operatorname{supp}\boldsymbol{k}=\boldsymbol{v}}} c_{\boldsymbol{k}}(f)\, \varphi_{\boldsymbol{k}}(\boldsymbol{x}_{\boldsymbol{v}})$$



[image credits: Laura Weidensager]

$\Rightarrow$ decomposition in the space of basis coefficients

Numerical realization – ANOVA approximation

[Potts, Schmischke 21]

$$f = \sum_{\boldsymbol{v} \subseteq [d]} f_{\boldsymbol{v}}, \qquad f_{\boldsymbol{v}} = \sum_{\substack{\boldsymbol{k} \in \mathbb{Z}^d \\ \operatorname{supp} \boldsymbol{k} = \boldsymbol{v}}} c_{\boldsymbol{k}}(f)\, \varphi_{\boldsymbol{k}}$$

Numerical realization – ANOVA approximation

[Potts, Schmischke 21]

$$f = \sum_{\boldsymbol{v} \subseteq [d]} f_{\boldsymbol{v}}, \qquad f_{\boldsymbol{v}} = \sum_{\substack{\boldsymbol{k} \in \mathbb{Z}^d \\ \operatorname{supp} \boldsymbol{k} = \boldsymbol{v}}} c_{\boldsymbol{k}}(f)\, \varphi_{\boldsymbol{k}}$$

Truncation

reduce number of ANOVA terms

$$\mathcal{T}_V f = \sum_{\substack{\boldsymbol{v} \subseteq [d] \\ \boldsymbol{v} \in V}} f_{\boldsymbol{v}}$$

$$f_{\boldsymbol{v}} = \sum_{\substack{\boldsymbol{k} \in \mathbb{Z}^d \\ \operatorname{supp} \boldsymbol{k} = \boldsymbol{v}}} c_{\boldsymbol{k}}(f)\, \varphi_{\boldsymbol{k}}$$

Numerical realization – ANOVA approximation

[Potts, Schmischke 21]

$$f = \sum_{\boldsymbol{v} \subseteq [d]} f_{\boldsymbol{v}}, \qquad f_{\boldsymbol{v}} = \sum_{\substack{\boldsymbol{k} \in \mathbb{Z}^d \\ \operatorname{supp} \boldsymbol{k} = \boldsymbol{v}}} c_{\boldsymbol{k}}(f)\, \varphi_{\boldsymbol{k}}$$

Truncation

reduce number of ANOVA terms

$$\mathcal{T}_V f = \sum_{\substack{\boldsymbol{v} \subseteq [d] \\ \boldsymbol{v} \in V}} f_{\boldsymbol{v}}$$

$$f_{\boldsymbol{v}} = \sum_{\substack{\boldsymbol{k} \in \mathbb{Z}^d \\ \operatorname{supp} \boldsymbol{k} = \boldsymbol{v}}} c_{\boldsymbol{k}}(f)\, \varphi_{\boldsymbol{k}}$$

Projection

choose finite number of basis functions $\{\varphi_{\boldsymbol{k}}\}_{\boldsymbol{k} \in \mathbb{Z}^d}$

$$P_{\boldsymbol{N}} f = \sum_{\substack{\boldsymbol{v} \subseteq [d] \\ \boldsymbol{v} \in V}} \tilde{f}_{\boldsymbol{v}}$$

$$\tilde{f}_{\boldsymbol{v}} = \sum_{\boldsymbol{k} \in \mathcal{I}_{\boldsymbol{N}}^{\boldsymbol{v}}} c_{\boldsymbol{k}}(f)\, \varphi_{\boldsymbol{k}}$$

Numerical realization – ANOVA approximation

[Potts, Schmischke 21]

$$f = \sum_{\boldsymbol{v} \subseteq [d]} f_{\boldsymbol{v}}, \qquad f_{\boldsymbol{v}} = \sum_{\substack{\boldsymbol{k} \in \mathbb{Z}^d \\ \operatorname{supp} \boldsymbol{k} = \boldsymbol{v}}} c_{\boldsymbol{k}}(f)\, \varphi_{\boldsymbol{k}}$$

Truncation

reduce number of ANOVA terms

$$\mathcal{T}_V f = \sum_{\substack{\boldsymbol{v} \subseteq [d] \\ \boldsymbol{v} \in V}} f_{\boldsymbol{v}}$$

$$f_{\boldsymbol{v}} = \sum_{\substack{\boldsymbol{k} \in \mathbb{Z}^d \\ \operatorname{supp} \boldsymbol{k} = \boldsymbol{v}}} c_{\boldsymbol{k}}(f)\, \varphi_{\boldsymbol{k}}$$

Projection

choose finite number of basis functions $\{\varphi_{\boldsymbol{k}}\}_{\boldsymbol{k} \in \mathbb{Z}^d}$

$$P_{\boldsymbol{N}} f = \sum_{\substack{\boldsymbol{v} \subseteq [d] \\ \boldsymbol{v} \in V}} \tilde{f}_{\boldsymbol{v}}$$

$$\tilde{f}_{\boldsymbol{v}} = \sum_{\boldsymbol{k} \in \mathcal{I}_{\boldsymbol{N}}^{\boldsymbol{v}}} c_{\boldsymbol{k}}(f)\, \varphi_{\boldsymbol{k}}$$

Regression

compute coefficients $c_{\boldsymbol{k}}^\star$ from samples

$$f^\star = \sum_{\substack{\boldsymbol{v} \subseteq [d] \\ \boldsymbol{v} \in V}} f_{\boldsymbol{v}}^\star$$

$$f_{\boldsymbol{v}}^\star = \sum_{\boldsymbol{k} \in \mathcal{I}_{\boldsymbol{N}}^{\boldsymbol{v}}} c_{\boldsymbol{k}}^\star\, \varphi_{\boldsymbol{k}}$$

Truncation

$$
\begin{aligned}
f(\boldsymbol{x}) = f_\varnothing & \qquad \text{constant} \\
+ f_1(x_1) + f_2(x_2) + \ldots + f_d(x_d) & \qquad \text{one-dimensional terms} \\
+ f_{1,2}(x_1, x_2) + f_{1,3}(x_1, x_3) + \ldots + f_{d-1,d}(x_{d-1}, x_d) & \qquad \text{two-dimensional terms} \\
+ f_{1,2,3}(x_1, x_2, x_3) + \ldots + f_{d-2,d-1,d}(x_{d-2}, x_{d-1}, x_d) & \qquad \text{three-dimensional terms} \\
\vdots & \\
+ f_{[d]}(x) & \qquad \text{d-dimensional term} \\
= \sum_{\boldsymbol{v} \subseteq [d]} f_{\boldsymbol{v}}(\boldsymbol{x}_{\boldsymbol{v}}) &
\end{aligned}
$$

Truncation

$$
\begin{aligned}
f(\boldsymbol{x}) = f_\varnothing & & \text{constant}\\
&+ f_1(x_1) + f_2(x_2) + \ldots + f_d(x_d) & \text{one-dimensional terms}\\
&+ f_{1,2}(x_1, x_2) + f_{1,3}(x_1, x_3) + \ldots + f_{d-1,d}(x_{d-1}, x_d) & \text{two-dimensional terms}\\
&+ f_{1,2,3}(x_1, x_2, x_3) + \ldots + f_{d-2,d-1,d}(x_{d-2}, x_{d-1}, x_d) & \text{three-dimensional terms}\\
&\vdots & \\
&+ f_{[d]}(x) & \text{d-dimensional term}\\
&= \sum_{\boldsymbol{v} \subseteq [d]} f_{\boldsymbol{v}}(\boldsymbol{x}_{\boldsymbol{v}})
\end{aligned}
$$

Problem: 2^d many terms (curse of dimensionality)

Truncation

$$
\begin{aligned}
f(\boldsymbol{x}) = f_{\varnothing} & && \text{constant}\\
& + f_1(x_1) + f_2(x_2) + \ldots + f_d(x_d) && \text{one-dimensional terms}\\
& + f_{1,2}(x_1, x_2) + f_{1,3}(x_1, x_3) + \ldots + f_{d-1,d}(x_{d-1}, x_d) && \text{two-dimensional terms}\\
& \cancel{+ f_{1,2,3}(x_1, x_2, x_3) + \ldots + f_{d-2,d-1,d}(x_{d-2}, x_{d-1}, x_d)} && \cancel{\text{three-dimensional terms}}\\
& \quad \vdots\\
& \cancel{+ f_{[d]}(x)} && \cancel{\text{d-dimensional term}}\\
\approx & \sum_{\substack{\boldsymbol{v} \subseteq [d] \\ |\boldsymbol{v}| \leq q}} f_{\boldsymbol{v}}(\boldsymbol{x}_{\boldsymbol{v}})
\end{aligned}
$$

Problem: 2^d many terms (curse of dimensionality) $\Rightarrow$ introduce $q \in \mathbb{N}, q < d$ (superposition dimension)

Truncation

$$
\begin{aligned}
f(\boldsymbol{x}) = {} & f_\varnothing && \text{constant} \\
& + \cancel{f_1(x_1)} + f_2(x_2) + \ldots + f_d(x_d) && \text{one-dimensional terms} \\
& + f_{1,2}(x_1, x_2) + \cancel{f_{1,3}(x_1, x_3)} + \ldots + \cancel{f_{d-1,d}(x_{d-1}, x_d)} && \text{two-dimensional terms} \\
& \cancel{+ f_{1,2,3}(x_1, x_2, x_3) + \ldots + f_{d-2,d-1,d}(x_{d-2}, x_{d-1}, x_d)} && \cancel{\text{three-dimensional terms}} \\
& \quad \vdots \\
& \cancel{+ f_{[d]}(x)} && \cancel{\text{d-dimensional term}} \\
\approx {} & \sum_{\substack{\boldsymbol{v} \subseteq [d] \\ \boldsymbol{v} \in V}} f_{\boldsymbol{v}}(\boldsymbol{x}_{\boldsymbol{v}}) = \mathcal{T}_V f
\end{aligned}
$$

$\boxed{\text{set } V \text{ according to sparsity}}$

Projection

Recap: reduced the number of ANOVA terms

$$f_{\boldsymbol{v}}(\boldsymbol{x}_{\boldsymbol{v}}) = \sum_{\substack{\boldsymbol{k} \in \mathbb{Z}^d \\ \operatorname{supp} \boldsymbol{k} = \boldsymbol{v}}} c_{\boldsymbol{k}}(f)\, \varphi_{\boldsymbol{k}}(\boldsymbol{x}_{\boldsymbol{v}})$$

Projection

Recap: reduced the number of ANOVA terms

$$f_{\boldsymbol{v}}(\boldsymbol{x}_{\boldsymbol{v}}) = \sum_{\substack{\boldsymbol{k}\in\mathbb{Z}^d \\ \operatorname{supp}\boldsymbol{k}=\boldsymbol{v}}} c_{\boldsymbol{k}}(f)\,\varphi_{\boldsymbol{k}}(\boldsymbol{x}_{\boldsymbol{v}})$$

Problem: infinitely many coefficients needed $\Rightarrow$ impossible in practice

Projection

Recap: reduced the number of ANOVA terms

$$f_{\boldsymbol{v}}(\boldsymbol{x}_{\boldsymbol{v}}) = \sum_{\substack{\boldsymbol{k}\in\mathbb{Z}^d \\ \operatorname{supp}\boldsymbol{k}=\boldsymbol{v}}} c_{\boldsymbol{k}}(f)\,\varphi_{\boldsymbol{k}}(\boldsymbol{x}_{\boldsymbol{v}})$$

Problem: infinitely many coefficients needed $\Rightarrow$ impossible in practice

$\rightsquigarrow$ introduce finite index sets $\mathcal{I}_{\boldsymbol{N}^{\boldsymbol{v}}}$ and approximate by

$$f_{\boldsymbol{v}}(\boldsymbol{x}_{\boldsymbol{v}}) \approx \sum_{\boldsymbol{k}\in\mathcal{I}_{\boldsymbol{N}^{\boldsymbol{v}}}} c_{\boldsymbol{k}}(f)\,\varphi_{\boldsymbol{k}}(\boldsymbol{x}_{\boldsymbol{v}}), \quad \boldsymbol{v}\subseteq[d]$$

Projection

Recap: reduced the number of ANOVA terms

$$f_{\boldsymbol{v}}(\boldsymbol{x}_{\boldsymbol{v}}) = \sum_{\substack{\boldsymbol{k}\in\mathbb{Z}^d \\ \operatorname{supp}\boldsymbol{k}=\boldsymbol{v}}} c_{\boldsymbol{k}}(f)\,\varphi_{\boldsymbol{k}}(\boldsymbol{x}_{\boldsymbol{v}})$$

Problem: infinitely many coefficients needed $\Rightarrow$ impossible in practice

$\rightsquigarrow$ introduce finite index sets $\mathcal{I}_{\boldsymbol{N}^{\boldsymbol{v}}}$ and approximate by

$$f_{\boldsymbol{v}}(\boldsymbol{x}_{\boldsymbol{v}}) \approx \sum_{\boldsymbol{k}\in\mathcal{I}_{\boldsymbol{N}^{\boldsymbol{v}}}} c_{\boldsymbol{k}}(f)\,\varphi_{\boldsymbol{k}}(\boldsymbol{x}_{\boldsymbol{v}}), \quad \boldsymbol{v}\subseteq[d]$$

and thus

$$f(\boldsymbol{x}) \approx \sum_{\boldsymbol{k}\in\mathcal{I}_{\boldsymbol{N}}} c_{\boldsymbol{k}}(f)\,\varphi_{\boldsymbol{k}}(\boldsymbol{x}), \qquad \mathcal{I}_{\boldsymbol{N}} := \bigcup_{|\boldsymbol{v}|\le q} \mathcal{I}_{\boldsymbol{N}^{\boldsymbol{v}}}$$

Projection

Recap: reduced the number of ANOVA terms

$$f_{\boldsymbol{v}}(\boldsymbol{x}_{\boldsymbol{v}}) = \sum_{\substack{\boldsymbol{k}\in\mathbb{Z}^d \\ \operatorname{supp}\boldsymbol{k}=\boldsymbol{v}}} c_{\boldsymbol{k}}(f)\,\varphi_{\boldsymbol{k}}(\boldsymbol{x}_{\boldsymbol{v}})$$

Problem: infinitely many coefficients needed $\Rightarrow$ impossible in practice

$\rightsquigarrow$ introduce finite index sets $\mathcal{I}_{\boldsymbol{N}^{\boldsymbol{v}}}$ and approximate by

$$f_{\boldsymbol{v}}(\boldsymbol{x}_{\boldsymbol{v}}) \approx \sum_{\boldsymbol{k}\in\mathcal{I}_{\boldsymbol{N}^{\boldsymbol{v}}}} c_{\boldsymbol{k}}(f)\,\varphi_{\boldsymbol{k}}(\boldsymbol{x}_{\boldsymbol{v}}), \quad \boldsymbol{v}\subseteq[d]$$

and thus

$$f(\boldsymbol{x}) \approx \sum_{\boldsymbol{k}\in\mathcal{I}_{\boldsymbol{N}}} c_{\boldsymbol{k}}(f)\,\varphi_{\boldsymbol{k}}(\boldsymbol{x}), \qquad \mathcal{I}_{\boldsymbol{N}} := \bigcup_{|\boldsymbol{v}|\le q} \mathcal{I}_{\boldsymbol{N}^{\boldsymbol{v}}}$$

$\hat{=}$ projection using a finite dictionary $\{\varphi_{\boldsymbol{k}}\}_{\boldsymbol{k}\in\mathcal{I}_{\boldsymbol{N}}} \subset \{\varphi_{\boldsymbol{k}}\}_{\boldsymbol{k}\in\mathbb{N}_0^d}$

Projection

Recap: reduced the number of ANOVA terms

$$f_{\boldsymbol{v}}(\boldsymbol{x}_{\boldsymbol{v}}) = \sum_{\substack{\boldsymbol{k}\in\mathbb{Z}^d \\ \operatorname{supp}\boldsymbol{k}=\boldsymbol{v}}} c_{\boldsymbol{k}}(f)\,\varphi_{\boldsymbol{k}}(\boldsymbol{x}_{\boldsymbol{v}})$$

Problem: infinitely many coefficients needed $\Rightarrow$ impossible in practice

$\rightsquigarrow$ introduce finite index sets $\mathcal{I}_{\boldsymbol{N}^{\boldsymbol{v}}}$ and approximate by

$$f_{\boldsymbol{v}}(\boldsymbol{x}_{\boldsymbol{v}}) \approx \sum_{\boldsymbol{k}\in\mathcal{I}_{\boldsymbol{N}^{\boldsymbol{v}}}} c_{\boldsymbol{k}}(f)\,\varphi_{\boldsymbol{k}}(\boldsymbol{x}_{\boldsymbol{v}}), \quad \boldsymbol{v}\subseteq[d]$$

and thus

$$f(\boldsymbol{x}) \approx \sum_{\boldsymbol{k}\in\mathcal{I}_{\boldsymbol{N}}} c_{\boldsymbol{k}}(f)\,\varphi_{\boldsymbol{k}}(\boldsymbol{x}), \qquad \mathcal{I}_{\boldsymbol{N}} := \bigcup_{|\boldsymbol{v}|\leq q} \mathcal{I}_{\boldsymbol{N}^{\boldsymbol{v}}}$$

$\widehat{=}$ projection using a finite dictionary $\{\varphi_{\boldsymbol{k}}\}_{\boldsymbol{k}\in\mathcal{I}_{\boldsymbol{N}}} \subset \{\varphi_{\boldsymbol{k}}\}_{\boldsymbol{k}\in\mathbb{N}_0^d}$

Still: computation of the integrals $c_{\boldsymbol{k}}(f) = \langle f, \varphi_{\boldsymbol{k}}\rangle_{L_2(\Omega,\omega)}, \boldsymbol{k}\in\mathcal{I}_{\boldsymbol{N}}$

Projection

Recap: reduced the number of ANOVA terms

$$f_{\boldsymbol{v}}(\boldsymbol{x}_{\boldsymbol{v}}) = \sum_{\substack{\boldsymbol{k}\in\mathbb{Z}^d \\ \operatorname{supp}\boldsymbol{k}=\boldsymbol{v}}} c_{\boldsymbol{k}}(f)\,\varphi_{\boldsymbol{k}}(\boldsymbol{x}_{\boldsymbol{v}})$$

Problem: infinitely many coefficients needed $\Rightarrow$ impossible in practice

$\rightsquigarrow$ introduce finite index sets $\mathcal{I}_{\boldsymbol{N}^{\boldsymbol{v}}}$ and approximate by

$$f_{\boldsymbol{v}}(\boldsymbol{x}_{\boldsymbol{v}}) \approx \sum_{\boldsymbol{k}\in\mathcal{I}_{\boldsymbol{N}^{\boldsymbol{v}}}} c_{\boldsymbol{k}}(f)\,\varphi_{\boldsymbol{k}}(\boldsymbol{x}_{\boldsymbol{v}}), \quad \boldsymbol{v}\subseteq[d]$$

and thus

$$f(\boldsymbol{x}) \approx \sum_{\boldsymbol{k}\in\mathcal{I}_{\boldsymbol{N}}} c_{\boldsymbol{k}}(f)\,\varphi_{\boldsymbol{k}}(\boldsymbol{x}), \qquad \mathcal{I}_{\boldsymbol{N}} := \bigcup_{|\boldsymbol{v}|\le q} \mathcal{I}_{\boldsymbol{N}^{\boldsymbol{v}}}$$

$\hat{=}$ projection using a finite dictionary $\{\varphi_{\boldsymbol{k}}\}_{\boldsymbol{k}\in\mathcal{I}_{\boldsymbol{N}}} \subset \{\varphi_{\boldsymbol{k}}\}_{\boldsymbol{k}\in\mathbb{N}_0^d}$

Still: computation of the integrals $c_{\boldsymbol{k}}(f) = \langle f, \varphi_{\boldsymbol{k}}\rangle_{L_2(\Omega,\omega)}, \boldsymbol{k}\in\mathcal{I}_{\boldsymbol{N}}$ $\rightsquigarrow$ data-driven approximation

Regression

Goal: approximate coefficients $c_{\boldsymbol{k}}(f)$ in $f(\boldsymbol{x}) \approx \sum_{\boldsymbol{k} \in \mathcal{I}_{\boldsymbol{N}}} c_{\boldsymbol{k}}(f)\, \varphi_{\boldsymbol{k}}(\boldsymbol{x})$

- ▷ from samples of the function f
- ▷ at points $\{\boldsymbol{x}^1, \dots, \boldsymbol{x}^M\}$ i.i.d. random according to the density ω

Regression

Goal: approximate coefficients $c_{\boldsymbol{k}}(f)$ in $f(\boldsymbol{x}) \approx \sum_{\boldsymbol{k}\in\mathcal{I}_{\boldsymbol{N}}} c_{\boldsymbol{k}}(f)\,\varphi_{\boldsymbol{k}}(\boldsymbol{x})$

- from samples of the function f
- at points $\{\boldsymbol{x}^1,\dots,\boldsymbol{x}^M\}$ i.i.d. random according to the density ω,

i. e.,

$$\underbrace{\begin{pmatrix} \varphi_{\boldsymbol{k}_1}(\boldsymbol{x}^1) & \cdots & \varphi_{\boldsymbol{k}_N}(\boldsymbol{x}^1) \\ \vdots & & \vdots \\ \varphi_{\boldsymbol{k}_1}(\boldsymbol{x}^M) & \cdots & \varphi_{\boldsymbol{k}_N}(\boldsymbol{x}^M) \end{pmatrix}}_{\boldsymbol{A}\in\mathbb{R}^{M\times|\mathcal{I}_{\boldsymbol{N}}|}} \underbrace{\begin{pmatrix} c_{\boldsymbol{k}_1} \\ \vdots \\ c_{\boldsymbol{k}_N} \end{pmatrix}}_{\boldsymbol{c}} \approx \underbrace{\begin{pmatrix} f(\boldsymbol{x}^1) \\ \vdots \\ f(\boldsymbol{x}^M) \end{pmatrix}}_{\boldsymbol{f}}$$

Regression

Goal: approximate coefficients $c_{\boldsymbol{k}}(f)$ in $f(\boldsymbol{x}) \approx \sum_{\boldsymbol{k}\in\mathcal{I}_{\boldsymbol{N}}} c_{\boldsymbol{k}}(f)\,\varphi_{\boldsymbol{k}}(\boldsymbol{x})$

- ▷ from samples of the function f
- ▷ at points $\{\boldsymbol{x}^1,\dots,\boldsymbol{x}^M\}$ i.i.d. random according to the density ω,

i. e.,

$$\underbrace{\begin{pmatrix} \varphi_{\boldsymbol{k}_1}(\boldsymbol{x}^1) & \cdots & \varphi_{\boldsymbol{k}_N}(\boldsymbol{x}^1) \\ \vdots & & \vdots \\ \varphi_{\boldsymbol{k}_1}(\boldsymbol{x}^M) & \cdots & \varphi_{\boldsymbol{k}_N}(\boldsymbol{x}^M) \end{pmatrix}}_{\boldsymbol{A}\in\mathbb{R}^{M\times|\mathcal{I}_{\boldsymbol{N}}|}} \underbrace{\begin{pmatrix} c_{\boldsymbol{k}_1} \\ \vdots \\ c_{\boldsymbol{k}_N} \end{pmatrix}}_{\boldsymbol{c}} \approx \underbrace{\begin{pmatrix} f(\boldsymbol{x}^1) \\ \vdots \\ f(\boldsymbol{x}^M) \end{pmatrix}}_{\boldsymbol{f}}$$

$\rightsquigarrow$ minimize $\|\boldsymbol{A}\boldsymbol{c}-\boldsymbol{f}\|_2^2$

Regression

Goal: approximate coefficients $c_{\boldsymbol{k}}(f)$ in $f(\boldsymbol{x}) \approx \sum_{\boldsymbol{k}\in\mathcal{I}_{\boldsymbol{N}}} c_{\boldsymbol{k}}(f)\, \varphi_{\boldsymbol{k}}(\boldsymbol{x})$

- ▷ from samples of the function f
- ▷ at points $\{\boldsymbol{x}^1, \dots, \boldsymbol{x}^M\}$ i.i.d. random according to the density ω,

i. e.,

$$\underbrace{\begin{pmatrix} \varphi_{\boldsymbol{k}_1}(\boldsymbol{x}^1) & \cdots & \varphi_{\boldsymbol{k}_N}(\boldsymbol{x}^1) \\ \vdots & & \vdots \\ \varphi_{\boldsymbol{k}_1}(\boldsymbol{x}^M) & \cdots & \varphi_{\boldsymbol{k}_N}(\boldsymbol{x}^M) \end{pmatrix}}_{\boldsymbol{A}\in\mathbb{R}^{M\times|\mathcal{I}_{\boldsymbol{N}}|}} \underbrace{\begin{pmatrix} c_{\boldsymbol{k}_1} \\ \vdots \\ c_{\boldsymbol{k}_N} \end{pmatrix}}_{\boldsymbol{c}} \approx \underbrace{\begin{pmatrix} f(\boldsymbol{x}^1) \\ \vdots \\ f(\boldsymbol{x}^M) \end{pmatrix}}_{\boldsymbol{f}} \qquad \rightsquigarrow \text{minimize } \|\boldsymbol{A}\boldsymbol{c} - \boldsymbol{f}\|_2^2$$

- least squares solution $\boldsymbol{c}^\star = (\boldsymbol{A}^\top \boldsymbol{A})^{-1} \boldsymbol{A}^\top \boldsymbol{f}$
  - [Kämmerer, Ullrich, Volkmer 21]: good condition number with high probability, if $|\mathcal{I}_{\boldsymbol{N}}| < \frac{M}{\log M}$
  - [Bartel, Potts, Schmischke 22]: can be computed efficiently (LSQR + fast multiplication)

Regression

Goal: approximate coefficients $c_{\boldsymbol{k}}(f)$ in $f(\boldsymbol{x}) \approx \sum_{\boldsymbol{k}\in\mathcal{I}_{\boldsymbol{N}}} c_{\boldsymbol{k}}(f)\,\varphi_{\boldsymbol{k}}(\boldsymbol{x})$

- ▷ from samples of the function f
- ▷ at points $\{\boldsymbol{x}^1, \dots, \boldsymbol{x}^M\}$ i.i.d. random according to the density ω,

i. e.,

$$\underbrace{\begin{pmatrix} \varphi_{\boldsymbol{k}_1}(\boldsymbol{x}^1) & \cdots & \varphi_{\boldsymbol{k}_N}(\boldsymbol{x}^1) \\ \vdots & & \vdots \\ \varphi_{\boldsymbol{k}_1}(\boldsymbol{x}^M) & \cdots & \varphi_{\boldsymbol{k}_N}(\boldsymbol{x}^M) \end{pmatrix}}_{\boldsymbol{A}\in\mathbb{R}^{M\times|\mathcal{I}_{\boldsymbol{N}}|}} \underbrace{\begin{pmatrix} c_{\boldsymbol{k}_1} \\ \vdots \\ c_{\boldsymbol{k}_N} \end{pmatrix}}_{\boldsymbol{c}} \approx \underbrace{\begin{pmatrix} f(\boldsymbol{x}^1) \\ \vdots \\ f(\boldsymbol{x}^M) \end{pmatrix}}_{\boldsymbol{f}} \qquad \rightsquigarrow \text{minimize } \|\boldsymbol{A}\boldsymbol{c} - \boldsymbol{f}\|_2^2$$

- least squares solution $\boldsymbol{c}^\star = (\boldsymbol{A}^\top\boldsymbol{A})^{-1}\boldsymbol{A}^\top\boldsymbol{f}$
  - [Kämmerer, Ullrich, Volkmer 21]: good condition number with high probability, if $|\mathcal{I}_{\boldsymbol{N}}| < \frac{M}{\log M}$
  - [Bartel, Potts, Schmischke 22]: can be computed efficiently (LSQR + fast multiplication)
- final approximation

$$f^\star(\boldsymbol{x}) := \sum_{\boldsymbol{k}\in\mathcal{I}_{\boldsymbol{N}}} c^\star_{\boldsymbol{k}}\,\varphi_{\boldsymbol{k}}(\boldsymbol{x})$$

ANOVA approximation for control systems

Setting

Unknown function:

$$H\colon \mathbb{R}^d \times \mathbb{R}^m \to \mathbb{R}^d,$$
$$H(\boldsymbol{x}, \boldsymbol{u}) := F(\boldsymbol{x}) + G(\boldsymbol{x})\boldsymbol{u},$$

with

- $F\colon \mathbb{R}^d \to \mathbb{R}^d$,
- $G\colon \mathbb{R}^d \to \mathbb{R}^{d \times m}$,
- $d, m \in \mathbb{N}$, $m \ll d$,

Setting

Unknown function:

$$H \colon \mathbb{R}^d \times \mathbb{R}^m \to \mathbb{R}^d,$$
$$H(\boldsymbol{x}, \boldsymbol{u}) \coloneqq F(\boldsymbol{x}) + G(\boldsymbol{x})\boldsymbol{u},$$

with

- $F \colon \mathbb{R}^d \to \mathbb{R}^d$,
- $G \colon \mathbb{R}^d \to \mathbb{R}^{d \times m}$,
- $d, m \in \mathbb{N}, m \ll d$,

Assumption: F and G of low-dimensional structure

Setting

Unknown function:

$$H\colon \mathbb{R}^d \times \mathbb{R}^m \to \mathbb{R}^d,$$
$$H(\boldsymbol{x}, \boldsymbol{u}) \coloneqq F(\boldsymbol{x}) + G(\boldsymbol{x})\boldsymbol{u},$$

with

- $F\colon \mathbb{R}^d \to \mathbb{R}^d$,
- $G\colon \mathbb{R}^d \to \mathbb{R}^{d\times m}$,
- $d, m \in \mathbb{N}, m \ll d$,

Assumption: F and G of low-dimensional structure

Given:

- sampling points $(\boldsymbol{x}^i, \boldsymbol{u}^i)$, $i = 1, \dots, M$
- samples $H(\boldsymbol{x}^i, \boldsymbol{u}^i)$

Setting

Unknown function:

$$H\colon \mathbb{R}^d \times \mathbb{R}^m \to \mathbb{R}^d,$$
$$H(\boldsymbol{x}, \boldsymbol{u}) \coloneqq F(\boldsymbol{x}) + G(\boldsymbol{x})\boldsymbol{u},$$

with

- $F\colon \mathbb{R}^d \to \mathbb{R}^d,$
- $G\colon \mathbb{R}^d \to \mathbb{R}^{d\times m},$
- $d, m \in \mathbb{N}, m \ll d,$

Assumption: F and G of low-dimensional structure

Given:

- sampling points $(\boldsymbol{x}^i, \boldsymbol{u}^i), i = 1, \dots, M$
- samples $H(\boldsymbol{x}^i, \boldsymbol{u}^i)$

Goal: find $\tilde{H} \approx H$

Setting

Unknown function:

$$H : \mathbb{R}^d \times \mathbb{R}^m \to \mathbb{R}^d,$$
$$H(\boldsymbol{x}, \boldsymbol{u}) \coloneqq F(\boldsymbol{x}) + G(\boldsymbol{x})\boldsymbol{u},$$

with

- $F : \mathbb{R}^d \to \mathbb{R}^d,$
- $G : \mathbb{R}^d \to \mathbb{R}^{d \times m},$
- $d, m \in \mathbb{N}, m \ll d,$

Assumption: F and G of low-dimensional structure

Given:

- sampling points $(\boldsymbol{x}^i, \boldsymbol{u}^i), i = 1, \dots, M$
- samples $H(\boldsymbol{x}^i, \boldsymbol{u}^i)$

Goal: find $\tilde{H} \approx H$

Approach: ANOVA approximation

Setting

Unknown function:

$$H\colon \mathbb{R}^d \times \mathbb{R}^m \to \mathbb{R}^d,$$
$$H(\boldsymbol{x}, \boldsymbol{u}) \coloneqq F(\boldsymbol{x}) + G(\boldsymbol{x})\boldsymbol{u},$$

with

- $F\colon \mathbb{R}^d \to \mathbb{R}^d$,
- $G\colon \mathbb{R}^d \to \mathbb{R}^{d\times m}$,
- $d, m \in \mathbb{N}$, $m \ll d$,

Assumption: F and G of low-dimensional structure

Given:

- sampling points $(\boldsymbol{x}^i, \boldsymbol{u}^i)$, $i = 1, \dots, M$
- samples $H(\boldsymbol{x}^i, \boldsymbol{u}^i)$

Goal: find $\tilde{H} \approx H$

Approach: ANOVA approximation

Problem: F and G cannot be sampled separately

Setting

Unknown function:

$$H\colon \mathbb{R}^d \times \mathbb{R}^m \to \mathbb{R}^d,$$
$$H(\boldsymbol{x},\boldsymbol{u}) \coloneqq F(\boldsymbol{x}) + G(\boldsymbol{x})\boldsymbol{u},$$

with

- $F\colon \mathbb{R}^d \to \mathbb{R}^d$,
- $G\colon \mathbb{R}^d \to \mathbb{R}^{d\times m}$,
- $d, m \in \mathbb{N}, m \ll d$,

Assumption: F and G of low-dimensional structure

Given:

- sampling points $(\boldsymbol{x}^i, \boldsymbol{u}^i), i = 1, \dots, M$
- samples $H(\boldsymbol{x}^i, \boldsymbol{u}^i)$

Goal: find $\tilde{H} \approx H$

Approach: ANOVA approximation

Problem: F and G cannot be sampled separately

$\rightsquigarrow$ individual approximation not possible

Setting

Unknown function:

$$H\colon \mathbb{R}^d \times \mathbb{R}^m \to \mathbb{R}^d,$$
$$H(\boldsymbol{x},\boldsymbol{u}) \coloneqq F(\boldsymbol{x}) + G(\boldsymbol{x})\boldsymbol{u},$$

with

- $F\colon \mathbb{R}^d \to \mathbb{R}^d$,
- $G\colon \mathbb{R}^d \to \mathbb{R}^{d\times m}$,
- $d, m \in \mathbb{N}, m \ll d$,

Assumption: F and G of low-dimensional structure

Given:

- sampling points $(\boldsymbol{x}^i, \boldsymbol{u}^i)$, $i = 1, \ldots, M$
- samples $H(\boldsymbol{x}^i, \boldsymbol{u}^i)$

Goal: find $\tilde{H} \approx H$

Approach: ANOVA approximation

Problem: F and G cannot be sampled separately

$\rightsquigarrow$ individual approximation not possible

$\rightsquigarrow$ coupled reconstruction

Setting

Unknown function:

$$H : \mathbb{R}^d \times \mathbb{R}^m \to \mathbb{R}^d,$$
$$H(\boldsymbol{x}, \boldsymbol{u}) := F(\boldsymbol{x}) + G(\boldsymbol{x})\boldsymbol{u},$$

with

- $F : \mathbb{R}^d \to \mathbb{R}^d,$
- $G : \mathbb{R}^d \to \mathbb{R}^{d \times m},$
- $d, m \in \mathbb{N}, m \ll d,$

Assumption: F and G of low-dimensional structure

Given:

- sampling points $(\boldsymbol{x}^i, \boldsymbol{u}^i), i = 1, \ldots, M$
- samples $H(\boldsymbol{x}^i, \boldsymbol{u}^i)$

Goal: find $\tilde{H} \approx H$

Approach: ANOVA approximation

Problem: F and G cannot be sampled separately

$\rightsquigarrow$ individual approximation not possible

$\rightsquigarrow$ coupled reconstruction

Important: avoid ANOVA approximation of H in $\boldsymbol{z} = (\boldsymbol{x}, \boldsymbol{u})$
for $m > 1$ introduces many additional terms – nonexistent interactions of the components of $\boldsymbol{u}$
$\rightsquigarrow$ unnecessary costs + error

Setting

Unknown function:

$$H\colon \mathbb{R}^d \times \mathbb{R}^m \to \mathbb{R}^d,$$
$$H(\boldsymbol{x},\boldsymbol{u}) \coloneqq F(\boldsymbol{x}) + G(\boldsymbol{x})\boldsymbol{u},$$

with

- $F\colon \mathbb{R}^d \to \mathbb{R}^d$,
- $G\colon \mathbb{R}^d \to \mathbb{R}^{d\times m}$,
- $d, m \in \mathbb{N}, m \ll d$,

Assumption: F and G of low-dimensional structure

Given:

- sampling points $(\boldsymbol{x}^i, \boldsymbol{u}^i)$, $i = 1, \dots, M$
- samples $H(\boldsymbol{x}^i, \boldsymbol{u}^i)$

Goal: find $\tilde{H} \approx H$

Approach: ANOVA approximation

Problem: F and G cannot be sampled separately

$\rightsquigarrow$ individual approximation not possible

$\rightsquigarrow$ coupled reconstruction

Important: avoid ANOVA approximation of H in $\boldsymbol{z} = (\boldsymbol{x}, \boldsymbol{u})$

for $m > 1$ introduces many additional terms – nonexistent interactions of the components of $\boldsymbol{u}$

$\rightsquigarrow$ unnecessary costs + error

$\rightsquigarrow$ exploit linearity in $\boldsymbol{u}$ instead

Unit vectors as control

Simplest approach: $\mathbb{U} = \{\mathbf{0}, \boldsymbol{e}_1, \dots, \boldsymbol{e}_m\}$ with $\{\boldsymbol{e}_\ell\}_{\ell=1}^m$ unit vectors of $\mathbb{R}^m$

Unit vectors as control

Simplest approach: $\mathbb{U} = \{\mathbf{0}, \boldsymbol{e}_1, \ldots, \boldsymbol{e}_m\}$ with $\{\boldsymbol{e}_\ell\}_{\ell=1}^m$ unit vectors of $\mathbb{R}^m$

evaluation of $H(\boldsymbol{x}, \boldsymbol{u}) = F(\boldsymbol{x}) + G(\boldsymbol{x})\boldsymbol{u}$ yields

Unit vectors as control

Simplest approach: $\mathbb{U} = \{\mathbf{0}, \boldsymbol{e}_1, \ldots, \boldsymbol{e}_m\}$ with $\{\boldsymbol{e}_\ell\}_{\ell=1}^m$ unit vectors of $\mathbb{R}^m$

evaluation of $H(\boldsymbol{x}, \boldsymbol{u}) = F(\boldsymbol{x}) + G(\boldsymbol{x})\boldsymbol{u}$ yields

$$H(\boldsymbol{x}, \mathbf{0}) = F(\boldsymbol{x}),$$

Unit vectors as control

Simplest approach: $\mathbb{U} = \{\mathbf{0}, \boldsymbol{e}_1, \dots, \boldsymbol{e}_m\}$ with $\{\boldsymbol{e}_\ell\}_{\ell=1}^m$ unit vectors of $\mathbb{R}^m$

evaluation of $H(\boldsymbol{x}, \boldsymbol{u}) = F(\boldsymbol{x}) + G(\boldsymbol{x})\boldsymbol{u}$ yields

$$\begin{aligned} H(\boldsymbol{x}, \mathbf{0}) &= F(\boldsymbol{x}), \\ H(\boldsymbol{x}, \boldsymbol{e}_\ell) &= \begin{pmatrix} F_1(\boldsymbol{x}) \\ \vdots \\ F_d(\boldsymbol{x}) \end{pmatrix} + \begin{pmatrix} G_{11}(\boldsymbol{x}) & \dots & G_{1m}(\boldsymbol{x}) \\ \vdots & & \vdots \\ G_{d1}(\boldsymbol{x}) & \dots & G_{dm}(\boldsymbol{x}) \end{pmatrix} \boldsymbol{e}_\ell \\ &= F(\boldsymbol{x}) + G_\ell(\boldsymbol{x}), \quad \ell = 1, \dots, m, \end{aligned}$$

Unit vectors as control

Simplest approach: $\mathbb{U} = \{\mathbf{0}, \boldsymbol{e}_1, \dots, \boldsymbol{e}_m\}$ with $\{\boldsymbol{e}_\ell\}_{\ell=1}^m$ unit vectors of $\mathbb{R}^m$

evaluation of $H(\boldsymbol{x}, \boldsymbol{u}) = F(\boldsymbol{x}) + G(\boldsymbol{x})\boldsymbol{u}$ yields

$$\begin{aligned}
H(\boldsymbol{x}, \mathbf{0}) &= F(\boldsymbol{x}), \\
H(\boldsymbol{x}, \boldsymbol{e}_\ell) &= \begin{pmatrix} F_1(\boldsymbol{x}) \\ \vdots \\ F_d(\boldsymbol{x}) \end{pmatrix} + \begin{pmatrix} G_{11}(\boldsymbol{x}) & \dots & G_{1m}(\boldsymbol{x}) \\ \vdots & & \vdots \\ G_{d1}(\boldsymbol{x}) & \dots & G_{dm}(\boldsymbol{x}) \end{pmatrix} \boldsymbol{e}_\ell \\
&= F(\boldsymbol{x}) + G_\ell(\boldsymbol{x}), \quad \ell = 1, \dots, m,
\end{aligned}$$

i. e.,

$$\begin{aligned}
G_\ell(\boldsymbol{x}) &= H(\boldsymbol{x}, \boldsymbol{e}_\ell) - F(\boldsymbol{x}) \\
&= H(\boldsymbol{x}, \boldsymbol{e}_\ell) - H(\boldsymbol{x}, \mathbf{0}), \quad \ell = 1, \dots, m
\end{aligned}$$

Unit vectors as control

Simplest approach: $\mathbb{U} = \{\mathbf{0}, \boldsymbol{e}_1, \dots, \boldsymbol{e}_m\}$ with $\{\boldsymbol{e}_\ell\}_{\ell=1}^m$ unit vectors of $\mathbb{R}^m$

evaluation of $H(\boldsymbol{x}, \boldsymbol{u}) = F(\boldsymbol{x}) + G(\boldsymbol{x})\boldsymbol{u}$ yields

$$\begin{aligned} H(\boldsymbol{x}, \mathbf{0}) &= F(\boldsymbol{x}), \\ H(\boldsymbol{x}, \boldsymbol{e}_\ell) &= \begin{pmatrix} F_1(\boldsymbol{x}) \\ \vdots \\ F_d(\boldsymbol{x}) \end{pmatrix} + \begin{pmatrix} G_{11}(\boldsymbol{x}) & \dots & G_{1m}(\boldsymbol{x}) \\ \vdots & & \vdots \\ G_{d1}(\boldsymbol{x}) & \dots & G_{dm}(\boldsymbol{x}) \end{pmatrix} \boldsymbol{e}_\ell \\ &= F(\boldsymbol{x}) + G_\ell(\boldsymbol{x}), \quad \ell = 1, \dots, m, \end{aligned}$$

i. e.,

$$\begin{aligned} G_\ell(\boldsymbol{x}) &= H(\boldsymbol{x}, \boldsymbol{e}_\ell) - F(\boldsymbol{x}) \\ &= H(\boldsymbol{x}, \boldsymbol{e}_\ell) - H(\boldsymbol{x}, \mathbf{0}), \quad \ell = 1, \dots, m \end{aligned}$$

$\Rightarrow$ may produce samples of F and G artificially

Unit vectors as control

Simplest approach: $\mathbb{U} = \{\mathbf{0}, \boldsymbol{e}_1, \dots, \boldsymbol{e}_m\}$ with $\{\boldsymbol{e}_\ell\}_{\ell=1}^m$ unit vectors of $\mathbb{R}^m$

evaluation of $H(\boldsymbol{x}, \boldsymbol{u}) = F(\boldsymbol{x}) + G(\boldsymbol{x})\boldsymbol{u}$ yields

$$\begin{aligned} H(\boldsymbol{x}, \mathbf{0}) &= F(\boldsymbol{x}), \\ H(\boldsymbol{x}, \boldsymbol{e}_\ell) &= \begin{pmatrix} F_1(\boldsymbol{x}) \\ \vdots \\ F_d(\boldsymbol{x}) \end{pmatrix} + \begin{pmatrix} G_{11}(\boldsymbol{x}) & \dots & G_{1m}(\boldsymbol{x}) \\ \vdots & & \vdots \\ G_{d1}(\boldsymbol{x}) & \dots & G_{dm}(\boldsymbol{x}) \end{pmatrix} \boldsymbol{e}_\ell \\ &= F(\boldsymbol{x}) + G_\ell(\boldsymbol{x}), \quad \ell = 1, \dots, m, \end{aligned}$$

i. e.,

$$\begin{aligned} G_\ell(\boldsymbol{x}) &= H(\boldsymbol{x}, \boldsymbol{e}_\ell) - F(\boldsymbol{x}) \\ &= H(\boldsymbol{x}, \boldsymbol{e}_\ell) - H(\boldsymbol{x}, \mathbf{0}), \quad \ell = 1, \dots, m \end{aligned}$$

$\Rightarrow$ may produce samples of F and G artificially

Remark: ANOVA only for scalar-valued functions

$\rightsquigarrow$ proceed rowwise:

$$H_j(\boldsymbol{x}, \boldsymbol{u}) = F_j(\boldsymbol{x}) + \sum_{\ell=1}^m G_{j\ell}(\boldsymbol{x})\, u_\ell, \quad j = 1, \dots, d$$

Unit vectors as control

Simplest approach: $\mathbb{U} = \{\mathbf{0}, \boldsymbol{e}_1, \dots, \boldsymbol{e}_m\}$ with $\{\boldsymbol{e}_\ell\}_{\ell=1}^m$ unit vectors of $\mathbb{R}^m$

evaluation of $H(\boldsymbol{x}, \boldsymbol{u}) = F(\boldsymbol{x}) + G(\boldsymbol{x})\boldsymbol{u}$ yields

$$H(\boldsymbol{x}, \mathbf{0}) = F(\boldsymbol{x}),$$

$$\begin{aligned} H(\boldsymbol{x}, \boldsymbol{e}_\ell) &= \begin{pmatrix} F_1(\boldsymbol{x}) \\ \vdots \\ F_d(\boldsymbol{x}) \end{pmatrix} + \begin{pmatrix} G_{11}(\boldsymbol{x}) & \dots & G_{1m}(\boldsymbol{x}) \\ \vdots & & \vdots \\ G_{d1}(\boldsymbol{x}) & \dots & G_{dm}(\boldsymbol{x}) \end{pmatrix} \boldsymbol{e}_\ell \\ &= F(\boldsymbol{x}) + G_\ell(\boldsymbol{x}), \quad \ell = 1, \dots, m, \end{aligned}$$

i. e.,

$$\begin{aligned} G_\ell(\boldsymbol{x}) &= H(\boldsymbol{x}, \boldsymbol{e}_\ell) - F(\boldsymbol{x}) \\ &= H(\boldsymbol{x}, \boldsymbol{e}_\ell) - H(\boldsymbol{x}, \mathbf{0}), \quad \ell = 1, \dots, m \end{aligned}$$

$\Rightarrow$ may produce samples of F and G artificially

Remark: ANOVA only for scalar-valued functions

$\rightsquigarrow$ proceed rowwise:

$$H_j(\boldsymbol{x}, \boldsymbol{u}) = F_j(\boldsymbol{x}) + \sum_{\ell=1}^{m} G_{j\ell}(\boldsymbol{x})\, u_\ell, \quad j = 1, \dots, d$$

- samples $\left(H_j(\boldsymbol{x}^i, \mathbf{0})\right)_{i=1}^M$ for $\tilde{F}_j$

Unit vectors as control

Simplest approach: $\mathbb{U} = \{\mathbf{0}, \boldsymbol{e}_1, \dots, \boldsymbol{e}_m\}$ with $\{\boldsymbol{e}_\ell\}_{\ell=1}^m$ unit vectors of $\mathbb{R}^m$

evaluation of $H(\boldsymbol{x}, \boldsymbol{u}) = F(\boldsymbol{x}) + G(\boldsymbol{x})\boldsymbol{u}$ yields

$$H(\boldsymbol{x}, \mathbf{0}) = F(\boldsymbol{x}),$$

$$\begin{aligned} H(\boldsymbol{x}, \boldsymbol{e}_\ell) &= \begin{pmatrix} F_1(\boldsymbol{x}) \\ \vdots \\ F_d(\boldsymbol{x}) \end{pmatrix} + \begin{pmatrix} G_{11}(\boldsymbol{x}) & \dots & G_{1m}(\boldsymbol{x}) \\ \vdots & & \vdots \\ G_{d1}(\boldsymbol{x}) & \dots & G_{dm}(\boldsymbol{x}) \end{pmatrix} \boldsymbol{e}_\ell \\ &= F(\boldsymbol{x}) + G_\ell(\boldsymbol{x}), \quad \ell = 1, \dots, m, \end{aligned}$$

i. e.,

$$\begin{aligned} G_\ell(\boldsymbol{x}) &= H(\boldsymbol{x}, \boldsymbol{e}_\ell) - F(\boldsymbol{x}) \\ &= H(\boldsymbol{x}, \boldsymbol{e}_\ell) - H(\boldsymbol{x}, \mathbf{0}), \quad \ell = 1, \dots, m \end{aligned}$$

$\Rightarrow$ may produce samples of F and G artificially

Remark: ANOVA only for scalar-valued functions

$\rightsquigarrow$ proceed rowwise:

$$H_j(\boldsymbol{x}, \boldsymbol{u}) = F_j(\boldsymbol{x}) + \sum_{\ell=1}^m G_{j\ell}(\boldsymbol{x})\, u_\ell, \quad j = 1, \dots, d$$

- samples $\left(H_j(\boldsymbol{x}^i, \mathbf{0})\right)_{i=1}^M$ for $\tilde{F}_j$
- samples $\left(H_j(\boldsymbol{x}^i, \boldsymbol{e}_\ell) - H_j(\boldsymbol{x}^i, \mathbf{0})\right)_{i=1}^M$ for $\tilde{G}_{j\ell}$

Unit vectors as control

Simplest approach: $\mathbb{U} = \{\mathbf{0}, \boldsymbol{e}_1, \dots, \boldsymbol{e}_m\}$ with $\{\boldsymbol{e}_\ell\}_{\ell=1}^m$ unit vectors of $\mathbb{R}^m$

evaluation of $H(\boldsymbol{x}, \boldsymbol{u}) = F(\boldsymbol{x}) + G(\boldsymbol{x})\boldsymbol{u}$ yields

$$H(\boldsymbol{x}, \mathbf{0}) = F(\boldsymbol{x}),$$

$$\begin{aligned} H(\boldsymbol{x}, \boldsymbol{e}_\ell) &= \begin{pmatrix} F_1(\boldsymbol{x}) \\ \vdots \\ F_d(\boldsymbol{x}) \end{pmatrix} + \begin{pmatrix} G_{11}(\boldsymbol{x}) & \dots & G_{1m}(\boldsymbol{x}) \\ \vdots & & \vdots \\ G_{d1}(\boldsymbol{x}) & \dots & G_{dm}(\boldsymbol{x}) \end{pmatrix} \boldsymbol{e}_\ell \\ &= F(\boldsymbol{x}) + G_\ell(\boldsymbol{x}), \quad \ell = 1, \dots, m, \end{aligned}$$

i. e.,

$$\begin{aligned} G_\ell(\boldsymbol{x}) &= H(\boldsymbol{x}, \boldsymbol{e}_\ell) - F(\boldsymbol{x}) \\ &= H(\boldsymbol{x}, \boldsymbol{e}_\ell) - H(\boldsymbol{x}, \mathbf{0}), \quad \ell = 1, \dots, m \end{aligned}$$

$\Rightarrow$ may produce samples of F and G artificially

Remark: ANOVA only for scalar-valued functions

$\rightsquigarrow$ proceed rowwise:

$$H_j(\boldsymbol{x}, \boldsymbol{u}) = F_j(\boldsymbol{x}) + \sum_{\ell=1}^{m} G_{j\ell}(\boldsymbol{x})\, u_\ell, \quad j = 1, \dots, d$$

- samples $\left(H_j(\boldsymbol{x}^i, \mathbf{0})\right)_{i=1}^M$ for $\tilde{F}_j$
- samples $\left(H_j(\boldsymbol{x}^i, \boldsymbol{e}_\ell) - H_j(\boldsymbol{x}^i, \mathbf{0})\right)_{i=1}^M$ for $\tilde{G}_{j\ell}$

✓ preserves structure (linearity in $\boldsymbol{u}$)

Unit vectors as control

Simplest approach: $\mathbb{U} = \{\mathbf{0}, \boldsymbol{e}_1, \ldots, \boldsymbol{e}_m\}$ with $\{\boldsymbol{e}_\ell\}_{\ell=1}^m$ unit vectors of $\mathbb{R}^m$

evaluation of $H(\boldsymbol{x}, \boldsymbol{u}) = F(\boldsymbol{x}) + G(\boldsymbol{x})\boldsymbol{u}$ yields

$$H(\boldsymbol{x}, \mathbf{0}) = F(\boldsymbol{x}),$$

$$H(\boldsymbol{x}, \boldsymbol{e}_\ell) = \begin{pmatrix} F_1(\boldsymbol{x}) \\ \vdots \\ F_d(\boldsymbol{x}) \end{pmatrix} + \begin{pmatrix} G_{11}(\boldsymbol{x}) & \ldots & G_{1m}(\boldsymbol{x}) \\ \vdots & & \vdots \\ G_{d1}(\boldsymbol{x}) & \ldots & G_{dm}(\boldsymbol{x}) \end{pmatrix} \boldsymbol{e}_\ell$$
$$= F(\boldsymbol{x}) + G_\ell(\boldsymbol{x}), \quad \ell = 1, \ldots, m,$$

i. e.,

$$G_\ell(\boldsymbol{x}) = H(\boldsymbol{x}, \boldsymbol{e}_\ell) - F(\boldsymbol{x})$$
$$= H(\boldsymbol{x}, \boldsymbol{e}_\ell) - H(\boldsymbol{x}, \mathbf{0}), \quad \ell = 1, \ldots, m$$

$\Rightarrow$ may produce samples of F and G artificially

Remark: ANOVA only for scalar-valued functions

$\rightsquigarrow$ proceed rowwise:

$$H_j(\boldsymbol{x}, \boldsymbol{u}) = F_j(\boldsymbol{x}) + \sum_{\ell=1}^m G_{j\ell}(\boldsymbol{x})\, u_\ell, \quad j = 1, \ldots, d$$

- samples $\left(H_j(\boldsymbol{x}^i, \mathbf{0})\right)_{i=1}^M$ for $\tilde{F}_j$
- samples $\left(H_j(\boldsymbol{x}^i, \boldsymbol{e}_\ell) - H_j(\boldsymbol{x}^i, \mathbf{0})\right)_{i=1}^M$ for $\tilde{G}_{j\ell}$

✓ preserves structure (linearity in $\boldsymbol{u}$)

✗ $M(m+1)$ function evaluations necessary
  - at each unit vector all x^i
  - non-flexible

Matrix approach

Aim: preserve structure + use only one set of samples of H

Matrix approach

Aim: preserve structure + use only one set of samples of H

Observation: ANOVA approximation separately for F and G would mean

$$H_j(\boldsymbol{x},\boldsymbol{u}) = F_j(\boldsymbol{x}) + \sum_{\ell=1}^{m} G_{j\ell}(\boldsymbol{x})\,u_\ell \qquad , \quad j = 1,\dots,d$$

Matrix approach

Aim: preserve structure + use only one set of samples of H

Observation: ANOVA approximation separately for F and G would mean

$$H_j(\boldsymbol{x},\boldsymbol{u}) = F_j(\boldsymbol{x}) + \sum_{\ell=1}^{m} G_{j\ell}(\boldsymbol{x})\, u_\ell \approx \sum_{\boldsymbol{k}\in\mathcal{I}(V_1)} c^1_{\boldsymbol{k}}\varphi^1_{\boldsymbol{k}}(\boldsymbol{x}) + \qquad\qquad , \quad j = 1,\dots,d$$

Matrix approach

Aim: preserve structure + use only one set of samples of H

Observation: ANOVA approximation separately for F and G would mean

$$H_j(\boldsymbol{x},\boldsymbol{u}) = F_j(\boldsymbol{x}) + \sum_{\ell=1}^{m} G_{j\ell}(\boldsymbol{x})\, u_\ell \approx \sum_{\boldsymbol{k}\in\mathcal{I}(V_1)} c_{\boldsymbol{k}}^1 \varphi_{\boldsymbol{k}}^1(\boldsymbol{x}) + \sum_{\ell=1}^{m}\left(\sum_{\boldsymbol{s}\in\mathcal{I}(V_{\ell+1})} c_{\boldsymbol{s}}^{\ell+1}\varphi_{\boldsymbol{s}}^{\ell+1}(\boldsymbol{x})\right) u_\ell, \quad j=1,\dots,d$$

Matrix approach

Aim: preserve structure + use only one set of samples of H

Observation: ANOVA approximation separately for F and G would mean

$$H_j(\boldsymbol{x},\boldsymbol{u}) = F_j(\boldsymbol{x}) + \sum_{\ell=1}^{m} G_{j\ell}(\boldsymbol{x})\, u_\ell \approx \sum_{\boldsymbol{k}\in\mathcal{I}(V_1)} c_{\boldsymbol{k}}^1 \varphi_{\boldsymbol{k}}^1(\boldsymbol{x}) + \sum_{\ell=1}^{m}\left(\sum_{\boldsymbol{s}\in\mathcal{I}(V_{\ell+1})} c_{\boldsymbol{s}}^{\ell+1}\varphi_{\boldsymbol{s}}^{\ell+1}(\boldsymbol{x})\right)u_\ell, \quad j=1,\dots,d$$

$\Rightarrow$ for $(\boldsymbol{x}^i,\boldsymbol{u}^i)$ i.i.d. random:

$$\boldsymbol{c}^\ell := \left(c_{\boldsymbol{k}}^\ell\right)_{\boldsymbol{k}\in\mathcal{I}(V_\ell)}, \quad \boldsymbol{A}_\ell := \left(\varphi_{\boldsymbol{k}}^\ell(\boldsymbol{x}^i)\right)_{i=1,\,\boldsymbol{k}\in\mathcal{I}(V_\ell)}^{M}, \quad \boldsymbol{U}_\ell := \operatorname{diag}(u_\ell^1,\dots,u_\ell^M)$$

Matrix approach

Aim: preserve structure + use only one set of samples of H

Observation: ANOVA approximation separately for F and G would mean

$$H_j(\boldsymbol{x},\boldsymbol{u}) = F_j(\boldsymbol{x}) + \sum_{\ell=1}^{m} G_{j\ell}(\boldsymbol{x})\, u_\ell \approx \sum_{\boldsymbol{k}\in\mathcal{I}(V_1)} c_{\boldsymbol{k}}^1 \varphi_{\boldsymbol{k}}^1(\boldsymbol{x}) + \sum_{\ell=1}^{m} \left(\sum_{\boldsymbol{s}\in\mathcal{I}(V_{\ell+1})} c_{\boldsymbol{s}}^{\ell+1} \varphi_{\boldsymbol{s}}^{\ell+1}(\boldsymbol{x}) \right) u_\ell, \quad j = 1,\dots,d$$

$\Rightarrow$ for $(\boldsymbol{x}^i, \boldsymbol{u}^i)$ i.i.d. random:

$$\boldsymbol{c}^\ell := \left(c_{\boldsymbol{k}}^\ell\right)_{\boldsymbol{k}\in\mathcal{I}(V_\ell)}, \quad \boldsymbol{A}_\ell := \left(\varphi_{\boldsymbol{k}}^\ell(\boldsymbol{x}^i)\right)_{i=1,\,\boldsymbol{k}\in\mathcal{I}(V_\ell)}^{M}, \quad \boldsymbol{U}_\ell := \operatorname{diag}(u_\ell^1,\dots,u_\ell^M)$$

yields

$$\left(H_j(\boldsymbol{x}^i, \boldsymbol{u}^i)\right)_{i=1}^{M} \approx \begin{pmatrix} \boldsymbol{A}_1 & \boldsymbol{U}_1\boldsymbol{A}_2 & \dots & \boldsymbol{U}_m\boldsymbol{A}_{m+1} \end{pmatrix} \begin{pmatrix} \boldsymbol{c}^1 \\ \vdots \\ \boldsymbol{c}^{m+1} \end{pmatrix}$$

Matrix approach

Aim: preserve structure + use only one set of samples of H

Observation: ANOVA approximation separately for F and G would mean

$$H_j(\boldsymbol{x},\boldsymbol{u}) = F_j(\boldsymbol{x}) + \sum_{\ell=1}^{m} G_{j\ell}(\boldsymbol{x})\, u_\ell \approx \sum_{\boldsymbol{k}\in\mathcal{I}(V_1)} c_{\boldsymbol{k}}^1 \varphi_{\boldsymbol{k}}^1(\boldsymbol{x}) + \sum_{\ell=1}^{m} \left(\sum_{\boldsymbol{s}\in\mathcal{I}(V_{\ell+1})} c_{\boldsymbol{s}}^{\ell+1} \varphi_{\boldsymbol{s}}^{\ell+1}(\boldsymbol{x}) \right) u_\ell, \quad j = 1,\dots,d$$

$\Rightarrow$ for $(\boldsymbol{x}^i, \boldsymbol{u}^i)$ i.i.d. random:

$$\boldsymbol{c}^\ell := \left(c_{\boldsymbol{k}}^\ell\right)_{\boldsymbol{k}\in\mathcal{I}(V_\ell)}, \quad \boldsymbol{A}_\ell := \left(\varphi_{\boldsymbol{k}}^\ell(\boldsymbol{x}^i)\right)_{i=1,\,\boldsymbol{k}\in\mathcal{I}(V_\ell)}^{M}, \quad \boldsymbol{U}_\ell := \operatorname{diag}(u_\ell^1,\dots,u_\ell^M)$$

yields

$$\left(H_j(\boldsymbol{x}^i,\boldsymbol{u}^i)\right)_{i=1}^{M} \approx \begin{pmatrix} \boldsymbol{A}_1 & \boldsymbol{U}_1\boldsymbol{A}_2 & \dots & \boldsymbol{U}_m\boldsymbol{A}_{m+1} \end{pmatrix} \begin{pmatrix} \boldsymbol{c}^1 \\ \vdots \\ \boldsymbol{c}^{m+1} \end{pmatrix}$$

$\Rightarrow$ find least squares solution

Efficient implementation

[https://github.com/NFFT/ANOVAapprox.jl]

$$H_j(\boldsymbol{x},\boldsymbol{u}) = F_j(\boldsymbol{x}) + \sum_{\ell=1}^{m} G_{j\ell}(\boldsymbol{x})\, u_\ell, \quad j = 1,\dots,d$$

Recap – ANOVA approximation:

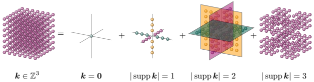
Truncation

$$\begin{aligned} f(\boldsymbol{x}) = {} & f_\varnothing \\ & + \cancel{f_1(x_1)} + f_2(x_2) + \dots + f_d(x_d) \\ & + f_{1,2}(x_1,x_2) + \cancel{f_{1,3}(x_1,x_3) + \dots + f_{d-1,d}(x_{d-1},x_d)} \\ & \cancel{+ f_{1,2,3}(x_1,x_2,x_3) + \dots + f_{d-2,d-1,d}(x_{d-2},x_{d-1},x_d)} \\ & \quad \vdots \\ & \cancel{+ f_{[d]}(\boldsymbol{x})} \\ \approx {} & \sum_{\substack{\boldsymbol{v}\subseteq[d] \\ \boldsymbol{v}\in V}} f_{\boldsymbol{v}}(\boldsymbol{x}_{\boldsymbol{v}}) = \mathrm{T}_V f \end{aligned}$$

reduce number of ANOVA terms

Projection

choose finite number of basis functions $\{\varphi_{\boldsymbol{k}}\}_{\boldsymbol{k}\in\mathbb{Z}^d}$



Regression

$$\underbrace{\begin{pmatrix} \varphi_{\boldsymbol{k}_1}(\boldsymbol{x}^1) & \cdots & \varphi_{\boldsymbol{k}_N}(\boldsymbol{x}^1) \\ \vdots & & \vdots \\ \varphi_{\boldsymbol{k}_1}(\boldsymbol{x}^M) & \cdots & \varphi_{\boldsymbol{k}_N}(\boldsymbol{x}^M) \end{pmatrix}}_{\boldsymbol{A}\in\mathbb{R}^{M\times|\mathcal{I}_{\boldsymbol{N}}|}} \underbrace{\begin{pmatrix} c_{\boldsymbol{k}_1} \\ \vdots \\ c_{\boldsymbol{k}_N} \end{pmatrix}}_{\boldsymbol{c}} \approx \underbrace{\begin{pmatrix} f(\boldsymbol{x}^1) \\ \vdots \\ f(\boldsymbol{x}^M) \end{pmatrix}}_{\boldsymbol{f}}$$

$\rightsquigarrow$ minimize $\|\boldsymbol{A}\boldsymbol{c}-\boldsymbol{f}\|_2^2$

Efficient implementation

[https://github.com/NFFT/ANOVAapprox.jl]

$$H_j(\boldsymbol{x},\boldsymbol{u}) = F_j(\boldsymbol{x}) + \sum_{\ell=1}^{m} G_{j\ell}(\boldsymbol{x})\, u_\ell, \quad j = 1,\dots,d$$

Recap – ANOVA approximation:

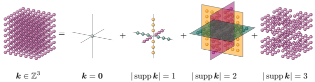
Truncation

$$\begin{aligned} f(\boldsymbol{x}) = {} & f_\varnothing \\ & + \cancel{f_1(x_1)} + f_2(x_2) + \ldots + f_d(x_d) \\ & + f_{1,2}(x_1,x_2) + \cancel{f_{1,3}(x_1,x_3)} + \ldots + \cancel{f_{d-1,d}(x_{d-1},x_d)} \\ & + f_{1,2,3}(x_1,x_2,x_3) + \ldots + \cancel{f_{d-2,d-1,d}(x_{d-2},x_{d-1},x_d)} \\ & \quad \vdots \\ & + \cancel{f_{[d]}(\boldsymbol{x})} \\ \approx {} & \sum_{\substack{\boldsymbol{v}\subseteq[d] \\ \boldsymbol{v}\in V}} f_{\boldsymbol{v}}(\boldsymbol{x}_{\boldsymbol{v}}) = \mathcal{T}_V f \end{aligned}$$

$\rightsquigarrow$ set V without interactions between the u_ℓ

Projection

choose finite number of basis functions $\{\varphi_{\boldsymbol{k}}\}_{\boldsymbol{k}\in\mathbb{Z}^d}$



Regression

$$\underbrace{\begin{pmatrix} \varphi_{\boldsymbol{k}_1}(\boldsymbol{x}^1) & \cdots & \varphi_{\boldsymbol{k}_N}(\boldsymbol{x}^1) \\ \vdots & & \vdots \\ \varphi_{\boldsymbol{k}_1}(\boldsymbol{x}^M) & \cdots & \varphi_{\boldsymbol{k}_N}(\boldsymbol{x}^M) \end{pmatrix}}_{\boldsymbol{A}\in\mathbb{R}^{M\times|\mathcal{I}_{\boldsymbol{N}}|}} \underbrace{\begin{pmatrix} c_{\boldsymbol{k}_1} \\ \vdots \\ c_{\boldsymbol{k}_N} \end{pmatrix}}_{\boldsymbol{c}} \approx \underbrace{\begin{pmatrix} f(\boldsymbol{x}^1) \\ \vdots \\ f(\boldsymbol{x}^M) \end{pmatrix}}_{\boldsymbol{f}}$$

$\rightsquigarrow$ minimize $\|\boldsymbol{A}\boldsymbol{c}-\boldsymbol{f}\|_2^2$

Efficient implementation

[https://github.com/NFFT/ANOVAapprox.jl]

$$H_j(\boldsymbol{x}, \boldsymbol{u}) = F_j(\boldsymbol{x}) + \sum_{\ell=1}^{m} G_{j\ell}(\boldsymbol{x})\, u_\ell, \quad j = 1, \dots, d$$

Recap – ANOVA approximation:

Truncation

$$\begin{aligned} f(\boldsymbol{x}) = & f_\varnothing \\ & + \cancel{f_1(x_1)} + f_2(x_2) + \ldots + f_d(x_d) \\ & + f_{1,2}(x_1, x_2) + \cancel{f_{1,3}(x_1, x_3)} + \ldots + \cancel{f_{d-1,d}(x_{d-1}, x_d)} \\ & \cancel{+ f_{1,2,3}(x_1, x_2, x_3) + \ldots + f_{d-2,d-1,d}(x_{d-2}, x_{d-1}, x_d)} \\ & \quad \vdots \\ & \cancel{+ f_{[d]}(x)} \\ \approx & \sum_{\substack{\boldsymbol{v} \subseteq [d] \\ \boldsymbol{v} \in V}} f_{\boldsymbol{v}}(\boldsymbol{x}_{\boldsymbol{v}}) = \mathcal{T}_V f \end{aligned}$$

$\rightsquigarrow$ set V without interactions between the u_ℓ

Projection

use the Chebyshev basis

$\{\varphi_k^{\mathrm{cheb}}\}_{k \in \mathbb{N}_0} = \{1, \sqrt{2}x, \dots\}$

+

choose only one basis function for all u_ℓ

$\Rightarrow$ exact reconstruction of all u_ℓ

Regression

$$\underbrace{\begin{pmatrix} \varphi_{\boldsymbol{k}_1}(\boldsymbol{x}^1) & \cdots & \varphi_{\boldsymbol{k}_N}(\boldsymbol{x}^1) \\ \vdots & & \vdots \\ \varphi_{\boldsymbol{k}_1}(\boldsymbol{x}^M) & \cdots & \varphi_{\boldsymbol{k}_N}(\boldsymbol{x}^M) \end{pmatrix}}_{\boldsymbol{A} \in \mathbb{R}^{M \times |\mathcal{I}_{\boldsymbol{N}}|}} \underbrace{\begin{pmatrix} c_{\boldsymbol{k}_1} \\ \vdots \\ c_{\boldsymbol{k}_N} \end{pmatrix}}_{\boldsymbol{c}} \approx \underbrace{\begin{pmatrix} f(\boldsymbol{x}^1) \\ \vdots \\ f(\boldsymbol{x}^M) \end{pmatrix}}_{\boldsymbol{f}}$$

$\rightsquigarrow$ minimize $\|\boldsymbol{A}\boldsymbol{c} - \boldsymbol{f}\|_2^2$

Efficient implementation

[https://github.com/NFFT/ANOVAapprox.jl]

$$H_j(\boldsymbol{x}, \boldsymbol{u}) = F_j(\boldsymbol{x}) + \sum_{\ell=1}^{m} G_{j\ell}(\boldsymbol{x})\, u_\ell, \quad j = 1, \dots, d$$

Recap – ANOVA approximation:

Truncation

$$\begin{aligned} f(\boldsymbol{x}) = f_\varnothing & + \cancel{f_1(x_1)} + f_2(x_2) + \ldots + f_d(x_d) \\ & + f_{1,2}(x_1, x_2) + \cancel{f_{1,3}(x_1, x_3)} + \ldots + \cancel{f_{d-1,d}(x_{d-1}, x_d)} \\ & + f_{1,2,3}(x_1, x_2, x_3) + \cancel{\ldots + f_{d-2,d-1,d}(x_{d-2}, x_{d-1}, x_d)} \\ & \quad \vdots \\ & + \cancel{f_{[d]}(\boldsymbol{x})} \\ \approx & \sum_{\substack{\boldsymbol{u} \subseteq [d] \\ \boldsymbol{u} \in V}} f_{\boldsymbol{u}}(\boldsymbol{x}_{\boldsymbol{u}}) = \mathcal{T}_V f \end{aligned}$$

$\rightsquigarrow$ set V without interactions between the u_ℓ

Projection

use the Chebyshev basis

$\{\varphi_k^{\text{cheb}}\}_{k \in \mathbb{N}_0} = \{1, \sqrt{2}x, \dots\}$

+

choose only one basis function for all u_ℓ

$\Rightarrow$ exact reconstruction of all u_ℓ

Regression

$$\underbrace{\begin{pmatrix} \varphi_{\boldsymbol{k}_1}(\boldsymbol{x}^1) & \cdots & \varphi_{\boldsymbol{k}_N}(\boldsymbol{x}^1) \\ \vdots & & \vdots \\ \varphi_{\boldsymbol{k}_1}(\boldsymbol{x}^M) & \cdots & \varphi_{\boldsymbol{k}_N}(\boldsymbol{x}^M) \end{pmatrix}}_{\boldsymbol{A} \in \mathbb{R}^{M \times |\mathcal{I}_{\boldsymbol{N}}|}} \underbrace{\begin{pmatrix} c_{\boldsymbol{k}_1} \\ \vdots \\ c_{\boldsymbol{k}_N} \end{pmatrix}}_{\boldsymbol{c}} \approx \underbrace{\begin{pmatrix} f(\boldsymbol{x}^1) \\ \vdots \\ f(\boldsymbol{x}^M) \end{pmatrix}}_{\boldsymbol{f}}$$

$\rightsquigarrow$ minimize $\|\boldsymbol{A}\boldsymbol{c} - \boldsymbol{f}\|_2^2$

$\Rightarrow$ only need suitable preprocessing step (adjust the index sets V and $\mathcal{I}_{\boldsymbol{N}^{\boldsymbol{v}}}$ appropriately)

Numerical Example

Toy example

- $d = 8$ and $m = 1$
- consider only one component

$$H_j(\boldsymbol{x}, u) = \underbrace{(x_4 - x_1)\, x_2}_{F(\boldsymbol{x})} - \underbrace{x_3}_{G(\boldsymbol{x})} u$$

- maximum two-dimensional terms
  $\rightsquigarrow q = 2$

Toy example

- $d = 8$ and $m = 1$
- consider only one component

$$H_j(\boldsymbol{x}, u) = \underbrace{(x_4 - x_1)\, x_2}_{F(\boldsymbol{x})} - \underbrace{x_3}_{G(\boldsymbol{x})}\, u$$

- maximum two-dimensional terms
  $\rightsquigarrow q = 2$
- Chebyshev basis for both approaches
  - ▷ approach 1 – unit vectors
  - ▷ approach 2 – matrix approach

Toy example

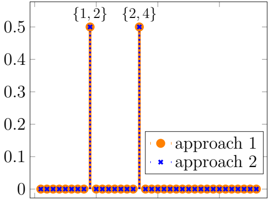
- $d = 8$ and $m = 1$
- consider only one component

$$H_j(\boldsymbol{x}, u) = \underbrace{(x_4 - x_1)\, x_2}_{F(\boldsymbol{x})} - \underbrace{x_3}_{G(\boldsymbol{x})} u$$

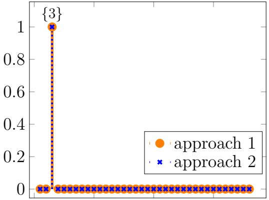
- maximum two-dimensional terms
  $\rightsquigarrow q = 2$
- Chebyshev basis for both approaches
  - approach 1 – unit vectors
  - approach 2 – matrix approach

1. relevant ANOVA terms

of F



of G



$\Rightarrow$ correctly identified

Toy example

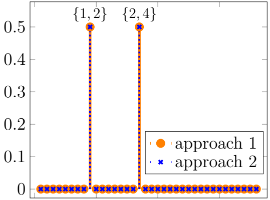
- $d = 8$ and $m = 1$
- consider only one component

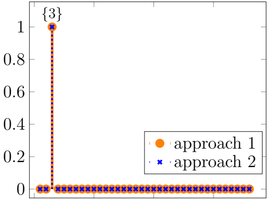
$$H_j(\boldsymbol{x}, u) = \underbrace{(x_4 - x_1)\, x_2}_{F(\boldsymbol{x})} - \underbrace{x_3}_{G(\boldsymbol{x})}\, u$$

- maximum two-dimensional terms
  $\rightsquigarrow q = 2$
- Chebyshev basis for both approaches
  - approach 1 – unit vectors
  - approach 2 – matrix approach

1 relevant ANOVA terms

of F / of G





$\Rightarrow$ correctly identified

2 error results $\rightsquigarrow$ same order of magnitude 10^{-10}

Summary

- investigated ANOVA in the context of control problems $H(\boldsymbol{x}, \boldsymbol{u}) = F(\boldsymbol{x}) + G(\boldsymbol{x})\boldsymbol{u}$

Summary

- investigated ANOVA in the context of control problems $H(\boldsymbol{x}, \boldsymbol{u}) = F(\boldsymbol{x}) + G(\boldsymbol{x})\boldsymbol{u}$
- introduced 2 methods to exploit low-dimensional structures of F and G, especially linearity in $\boldsymbol{u}$
  - ✓ easy to use due to free software package [https://github.com/NFFT/ANOVAapprox.jl]
  - ✓ efficient computation

Summary

- investigated ANOVA in the context of control problems $H(\boldsymbol{x}, \boldsymbol{u}) = F(\boldsymbol{x}) + G(\boldsymbol{x})\boldsymbol{u}$
- introduced 2 methods to exploit low-dimensional structures of F and G, especially linearity in $\boldsymbol{u}$
  - ✓ easy to use due to free software package [`https://github.com/NFFT/ANOVAapprox.jl`]
  - ✓ efficient computation

1. simple approach using unit vectors as control
   - ✓ allows to approximate F and G separately due to artificial samples
   - ✗ many samples needed $\rightsquigarrow$ possibly unfeasible in practice

Summary

- investigated ANOVA in the context of control problems $H(\boldsymbol{x}, \boldsymbol{u}) = F(\boldsymbol{x}) + G(\boldsymbol{x})\boldsymbol{u}$
- introduced 2 methods to exploit low-dimensional structures of F and G, especially linearity in $\boldsymbol{u}$
  - ✓ easy to use due to free software package [https://github.com/NFFT/ANOVAapprox.jl]
  - ✓ efficient computation

1. simple approach using unit vectors as control
   - ✓ allows to approximate F and G separately due to artificial samples
   - ✗ many samples needed $\rightsquigarrow$ possibly unfeasible in practice
2. additional approach to only need one set of samples $\rightsquigarrow$ determined suitable preprocessing
   - ✓ preserve linearity – exact reconstruction of components of $\boldsymbol{u}$
   - ✓ only one set of samples
   - ✓ no additional ANOVA terms introduced

Summary

- investigated ANOVA in the context of control problems $H(\boldsymbol{x}, \boldsymbol{u}) = F(\boldsymbol{x}) + G(\boldsymbol{x})\boldsymbol{u}$
- introduced 2 methods to exploit low-dimensional structures of F and G, especially linearity in $\boldsymbol{u}$
  - ✓ easy to use due to free software package [https://github.com/NFFT/ANOVAapprox.jl]
  - ✓ efficient computation

1. simple approach using unit vectors as control
   - ✓ allows to approximate F and G separately due to artificial samples
   - ✗ many samples needed $\rightsquigarrow$ possibly unfeasible in practice
2. additional approach to only need one set of samples $\rightsquigarrow$ determined suitable preprocessing
   - ✓ preserve linearity – exact reconstruction of components of $\boldsymbol{u}$
   - ✓ only one set of samples
   - ✓ no additional ANOVA terms introduced

- **K., Potts, Schaller, Worthmann: ANOVA approximation for control systems**. In preparation.

Summary

- investigated ANOVA in the context of control problems $H(\boldsymbol{x}, \boldsymbol{u}) = F(\boldsymbol{x}) + G(\boldsymbol{x})\boldsymbol{u}$
- introduced 2 methods to exploit low-dimensional structures of F and G, especially linearity in $\boldsymbol{u}$
  - ✓ easy to use due to free software package [https://github.com/NFFT/ANOVAapprox.jl]
  - ✓ efficient computation

1. simple approach using unit vectors as control
   - ✓ allows to approximate F and G separately due to artificial samples
   - ✗ many samples needed $\rightsquigarrow$ possibly unfeasible in practice
2. additional approach to only need one set of samples $\rightsquigarrow$ determined suitable preprocessing
   - ✓ preserve linearity – exact reconstruction of components of $\boldsymbol{u}$
   - ✓ only one set of samples
   - ✓ no additional ANOVA terms introduced

- **K., Potts, Schaller, Worthmann: ANOVA approximation for control systems**. In preparation.

Thank you for your attention!