# ANOVA approximation for control systems

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joint work with
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# Motivation

**Unknown function (control system):**

$$H : \mathbb{R}^d \times \mathbb{R}^m \to \mathbb{R}^d,$$
$$H(\boldsymbol{x}, \boldsymbol{u}) := F(\boldsymbol{x}) + G(\boldsymbol{x})\boldsymbol{u},$$

with

- $F : \mathbb{R}^d \to \mathbb{R}^d, G : \mathbb{R}^d \to \mathbb{R}^{d \times m},$
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**Example:** Euler approximation of a controlled Duffing oscillator

$$\begin{aligned} H(\boldsymbol{x},\boldsymbol{u}) &= \boldsymbol{x} + \Delta t \begin{pmatrix} x_2 \\ x_1 - 3x_1^3 u \end{pmatrix} \\ &= \begin{pmatrix} x_1 + \Delta t x_2 \\ x_2 + \Delta t x_1 \end{pmatrix} + \begin{pmatrix} 0 \\ -3\Delta t x_1^3 \end{pmatrix} u \\ &= F(\boldsymbol{x}) + G(\boldsymbol{x}) u \end{aligned}$$

i. e.,

- $d = 2$,
- $m = 1$,
- only 1D terms $\rightsquigarrow$ low-dimensional

# Overview

# ANOVA approximation

# ANalysis Of VAriance (ANOVA) decomposition

[Caflish, Morokoff, Owen 97], [Rabitz, Alis 99], [Liu, Owen 06], [Kuo, Sloan, Wasilkowski, Wozniakowski 10], ...

Decompose a $d$-dimensional function $f$ into

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→ in general multiple representations possible

→ conditions for uniqueness?

## Definition

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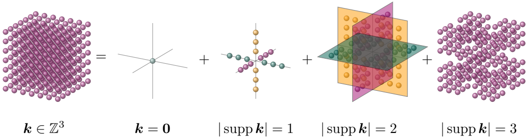


[image credits: Laura Weidensager]

$\Rightarrow$ decomposition in the space of basis coefficients

# Numerical realization – ANOVA approximation

[Potts, Schmischke 21]

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**Truncation**

reduce number of ANOVA terms

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**Projection**
choose finite number of basis functions $\{\varphi_{\boldsymbol{k}}\}_{\boldsymbol{k} \in \mathbb{Z}^d}$

$$P_{\boldsymbol{N}} f = \sum_{\substack{\boldsymbol{v} \subseteq [d] \\ \boldsymbol{v} \in V}} \tilde{f}_{\boldsymbol{v}}$$

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**Regression**

compute coefficients $c_{\boldsymbol{k}}^{\star}$ from samples

$$f^{\star} = \sum_{\substack{\boldsymbol{v} \subseteq [d] \\ \boldsymbol{v} \in V}} f_{\boldsymbol{v}}^{\star}$$

$$f_{\boldsymbol{v}}^{\star} = \sum_{\boldsymbol{k} \in \mathcal{I}_{\boldsymbol{N} \boldsymbol{v}}} c_{\boldsymbol{k}}^{\star}\, \varphi_{\boldsymbol{k}}$$

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$$f(\boldsymbol{x}) = f_\varnothing$$ constant

$$+ f_1(x_1) + f_2(x_2) + \ldots + f_d(x_d)$$ one-dimensional terms

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$$\vdots$$

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$$\approx \sum_{\substack{\boldsymbol{v} \subseteq [d] \\ |\boldsymbol{v}| \leq q}} f_{\boldsymbol{v}}(\boldsymbol{x}_{\boldsymbol{v}})$$

**Problem:** $2^d$ many terms (curse of dimensionality) $\Rightarrow$ introduce $q \in \mathbb{N}, q < d$ (superposition dimension)

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set $V$ according to sparsity

# Projection

**Recap:** reduced the number of ANOVA terms

$$f_{\boldsymbol{v}}(\boldsymbol{x}_{\boldsymbol{v}}) = \sum_{\substack{\boldsymbol{k}\in\mathbb{Z}^d \\ \operatorname{supp}\boldsymbol{k}=\boldsymbol{v}}} c_{\boldsymbol{k}}(f)\,\varphi_{\boldsymbol{k}}(\boldsymbol{x}_{\boldsymbol{v}})$$

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$\rightsquigarrow$ introduce finite index sets $\mathcal{I}_{\boldsymbol{N}^{\boldsymbol{v}}}$ and approximate by

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# Regression

**Goal:** approximate coefficients $c_{\boldsymbol{k}}(f)$ in $f(\boldsymbol{x}) \approx \sum_{\boldsymbol{k}\in\mathcal{I}_{\boldsymbol{N}}} c_{\boldsymbol{k}}(f)\,\varphi_{\boldsymbol{k}}(\boldsymbol{x})$

- from samples of the function $f$
- at points $\{\boldsymbol{x}^1, \dots, \boldsymbol{x}^M\}$ i.i.d. random according to the density $\omega$

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**Goal:** approximate coefficients $c_{\boldsymbol{k}}(f)$ in $f(\boldsymbol{x}) \approx \sum_{\boldsymbol{k}\in\mathcal{I}_{\boldsymbol{N}}} c_{\boldsymbol{k}}(f)\,\varphi_{\boldsymbol{k}}(\boldsymbol{x})$

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- least squares solution $\boldsymbol{c}^\star = (\boldsymbol{A}^\top \boldsymbol{A})^{-1}\boldsymbol{A}^\top \boldsymbol{f}$
  - [Kämmerer, Ullrich, Volkmer 21]: good condition number with high probability, if $|\mathcal{I}_{\boldsymbol{N}}| < \frac{M}{\log M}$
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ANOVA approximation for control systems

# ANOVA approximation for control systems

ANOVA approximation for control systems

# Setting

**Unknown function:**

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**Important:** avoid ANOVA approximation of $H$ in $\boldsymbol{z} = (\boldsymbol{x}, \boldsymbol{u})$
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ANOVA approximation for control systems

# Unit vectors as control

**Simplest approach:** $\mathbb{U} = \{\mathbf{0}, \boldsymbol{e}_1, \ldots, \boldsymbol{e}_m\}$ with $\{\boldsymbol{e}_\ell\}_{\ell=1}^m$ unit vectors of $\mathbb{R}^m$

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$\rightsquigarrow$ proceed rowwise:

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- ✗ $M(m+1)$ function evaluations necessary
  - at each unit vector all $x^i$
  - non-flexible

ANOVA approximation for control systems

# Matrix approach

**Aim:** preserve structure **+** use only one set of samples of $H$

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ANOVA approximation for control systems

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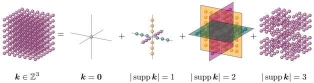
$$\begin{aligned} f(\boldsymbol{x}) = {} & f_\varnothing \\ & + f_1(x_1) + f_2(x_2) + \dots + f_d(x_d) \\ & + f_{1,2}(x_1,x_2) + f_{1,3}(x_1,x_3) + \dots + f_{d-1,d}(x_{d-1},x_d) \\ & + f_{1,2,3}(x_1,x_2,x_3) + \dots + f_{d-2,d-1,d}(x_{d-2},x_{d-1},x_d) \\ & \vdots \\ & + f_{[d]}(\boldsymbol{x}) \\ \approx {} & \sum_{\substack{\boldsymbol{v}\subseteq[d] \\ \boldsymbol{v}\in V}} f_{\boldsymbol{v}}(\boldsymbol{x}_{\boldsymbol{v}}) = T_V f \end{aligned}$$

reduce number of ANOVA terms

### Projection

choose finite number of basis functions $\{\varphi_{\boldsymbol{k}}\}_{\boldsymbol{k}\in\mathbb{Z}^d}$

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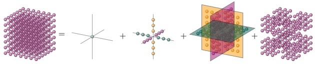


$\boldsymbol{k} \in \mathbb{Z}^3$ | $\boldsymbol{k} = \boldsymbol{0}$ | $|\operatorname{supp} \boldsymbol{k}| = 1$ | $|\operatorname{supp} \boldsymbol{k}| = 2$ | $|\operatorname{supp} \boldsymbol{k}| = 3$

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$\Rightarrow$ only need suitable preprocessing step (adjust the index sets $V$ and $\mathcal{I}_{\boldsymbol{N}^{\boldsymbol{v}}}$ appropriately)

# Numerical Example

# Toy example

- $d = 8$ and $m = 1$
- consider only one component

$$H_j(\boldsymbol{x}, u) = \underbrace{(x_4 - x_1)\, x_2}_{F(\boldsymbol{x})} - \underbrace{x_3}_{G(\boldsymbol{x})} u$$

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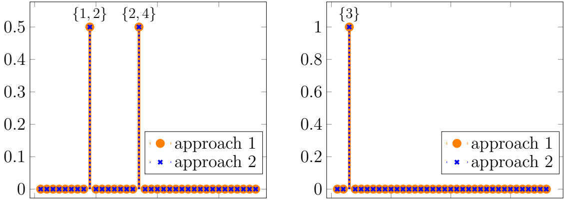


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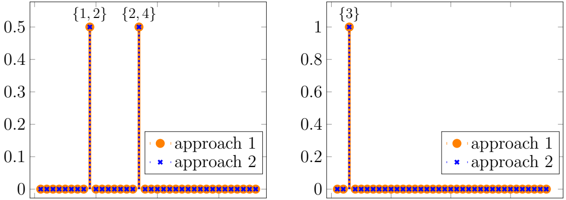


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(2) error results $\rightsquigarrow$ same order of magnitude $10^{-10}$

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