Nonequispaced fast Fourier transforms for bandlimited functions

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joint work with Daniel Potts

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Nonequispaced Fast Fourier Transform (NFFT / NUFFT)

Fast algorithm to evaluate a trigonometric polynomial $\mathcal{O}(|\mathcal{I}_{\boldsymbol{M}}| \log(|\mathcal{I}_{\boldsymbol{M}}|) + N)$

$$f(\boldsymbol{x}) = \sum_{\boldsymbol{k} \in \mathcal{I}_{\boldsymbol{M}}} \hat{f}_{\boldsymbol{k}} \, \mathrm{e}^{2\pi \mathrm{i} \boldsymbol{k}\boldsymbol{x}}$$

[Dutt, Rokhlin 93], [Beylkin 95], [Potts, Steidl, Tasche 01]

- index set $\mathcal{I}_{\boldsymbol{M}} := \mathbb{Z}^d \cap \left[-\frac{M}{2}, \frac{M}{2}\right)^d$ with cardinality $|\mathcal{I}_{\boldsymbol{M}}| = M^d$, $M \in 2\mathbb{N}$,
- Fourier coefficients $\hat{f}_{\boldsymbol{k}} \in \mathbb{C}$, $\boldsymbol{k} \in \mathcal{I}_{\boldsymbol{M}}$,
- nonequispaced points $\boldsymbol{x}_j \in \mathbb{T}^d \cong \left[-\frac{1}{2}, \frac{1}{2}\right)^d$, $j = 1, \ldots, N$, $N \in \mathbb{N}$

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Matrix notation:

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Factorization:

$$\boldsymbol{A} \approx \underbrace{\boldsymbol{B}}_{(2m+1)^d\text{-sparse}}\ \underbrace{\boldsymbol{F}}_{\text{FFT}}\ \underbrace{\boldsymbol{D}}_{\text{diagonal}}$$

(in each column of $\boldsymbol{B}$ only $(2m+1)^d$ entries, $m \in \mathbb{N}$ given)

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no longer	discrete (trigonometric polynomials)	but	continuous (bandlimited functions)

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$\rightsquigarrow$ bandlimited functions with **bandwidth** $M \in \mathbb{N}$ (**Paley–Wiener space**)

$$\mathcal{B}_{M/2}(\mathbb{R}^d) := \left\{ f \in L_2(\mathbb{R}^d) \colon\ \mathrm{supp}(\hat{f}) \subseteq \left[-\tfrac{M}{2}, \tfrac{M}{2}\right]^d \right\} \quad \subseteq L_2(\mathbb{R}^d) \cap C_0(\mathbb{R}^d) \cap C^\infty(\mathbb{R}^d)$$

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In this talk: for simplicity only $d = 1$

Overview

1. Regularized Shannon sampling formulas
2. NFFT-like procedure for bandlimited functions
3. Numerical Example

Regularized Shannon sampling formulas

Sampling Theorem of Shannon–Whittaker–Kotelnikov

[Whittaker 1915], [Kotelnikov 1933], [Shannon 1949]

Sampling Theorem

Let $f \in \mathcal{B}_{M/2}(\mathbb{R})$ and $L = M(1+\lambda) \in \mathbb{N}, \lambda \geq 0$. Then f is completely determined by its samples $f\big(\frac{\ell}{L}\big)$, $\ell \in \mathbb{Z}$, and we have

$$f(x) = \sum_{\ell \in \mathbb{Z}} f\big(\tfrac{\ell}{L}\big) \operatorname{sinc}\big(L\pi\big(x - \tfrac{\ell}{L}\big)\big), \quad x \in \mathbb{R},$$

with the cardinal sine function

$$\operatorname{sinc}(x) := \begin{cases} \frac{\sin x}{x} & : x \in \mathbb{R} \setminus \{0\}, \\ 1 & : x = 0. \end{cases}$$

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Truncation:

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Numerical Shortcomings:

- slow convergence: [Jagerman 66], [K. 24]

$$\max_{|x| \leq 1} \left| f(x) - (S_T f)(x) \right| \leq \frac{\sqrt{2L}}{\pi} (T - L)^{-1/2} \, \|f\|_{L_2(\mathbb{R})}$$

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- non-robustness w.r.t. erroneous samples

$$\tilde{f}_\ell := \begin{cases} f\big(\frac{\ell}{L}\big) + \varepsilon_\ell & : |\ell| \leq T \\ f\big(\frac{\ell}{L}\big) & : |\ell| > T \end{cases}$$

[Feichtinger 92], [Daubechies, DeVore 03], [K., Potts, Tasche 24a], [K. 24]

Numerical realization

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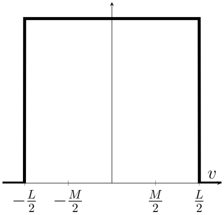
[Natterer 86], [Daubechies 92], [Rappaport 96], [Partington 97], [Schmeisser, Sickel 00], [Strohmer, Tanner 05], . . .



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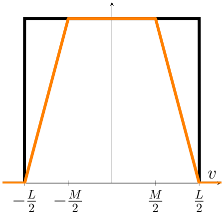

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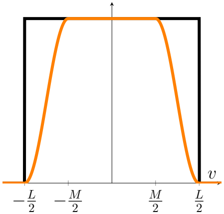

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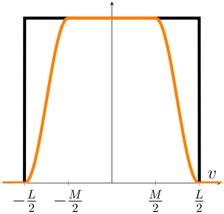

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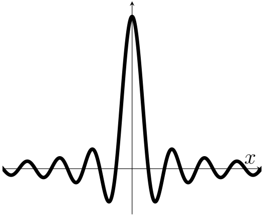

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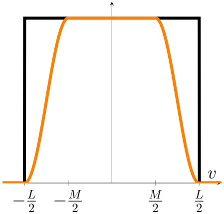

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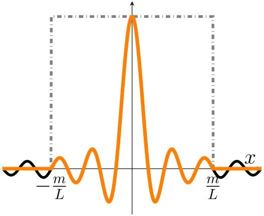

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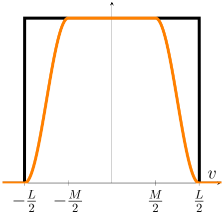

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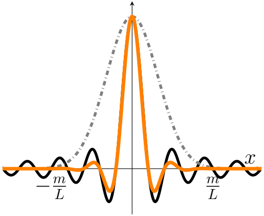

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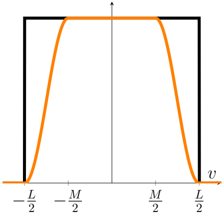

no localization

$\rightsquigarrow$ still truncation needed

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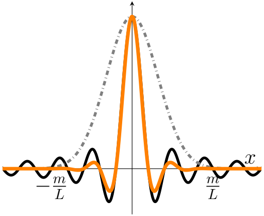

localization

$\rightsquigarrow$ only $2m$ samples needed

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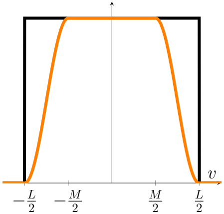

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$\rightsquigarrow$ only algebraic decay rates

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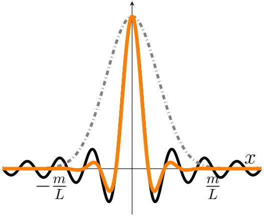

localization

$\rightsquigarrow$ only $2m$ samples needed

$\rightsquigarrow$ exponential decay rates

[K., Potts, Tasche 24a]

General error estimates for $(R_{\varphi,m}f)(x) = \sum_{\ell\in\mathbb{Z}} f\left(\frac{\ell}{L}\right)\psi\left(x-\frac{\ell}{L}\right)$

Theorem (Uniform approximation error estimate) [K., Potts, Tasche 22]

Let $f \in \mathcal{B}_{M/2}(\mathbb{R})$ with $M \in \mathbb{N}$, $L = M\,(1+\lambda) \in \mathbb{N}$ with $\lambda \geq 0$, and $\varphi \in \Phi_{m,L}$ with $m \in \mathbb{N}\setminus\{1\}$ be given. Then

$$\|f - R_{\varphi,m}f\|_{C_0(\mathbb{R})} \leq \big(E_1(\varphi,M,L) + E_2(\varphi,M,L)\big)\,\|f\|_{L_2(\mathbb{R})}$$

with the error constants

$$E_1(\varphi,M,L) := \sqrt{M}\max_{v\in\left[-\frac{M}{2},\frac{M}{2}\right]}\left|1-\int_{v-\frac{L}{2}}^{v+\frac{L}{2}}\hat{\varphi}(u)\,\mathrm{d}u\right|, \qquad E_2(\varphi,M,L) := \frac{\sqrt{2L}}{\pi}\sqrt{\frac{\varphi^2\left(\frac{m}{L}\right)}{m^2}+\int_{\frac{m}{L}}^{\infty}\frac{\varphi^2(t)}{Lt^2}\,\mathrm{d}t}.$$

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Theorem (Uniform perturbation error estimate) [K., Potts, Tasche 22]

Let additionally the noisy samples $\tilde{f}_\ell := f\left(\frac{\ell}{L}\right) + \varepsilon_\ell$ with $|\varepsilon_\ell| \leq \varepsilon$, $\ell\in\mathbb{Z}$, and $\varepsilon > 0$ be given. Then we have

$$\|R_{\varphi,m}\tilde{f} - R_{\varphi,m}f\|_{C_0(\mathbb{R})} \leq \varepsilon\,\big(2 + L\,\hat{\varphi}(0)\big).$$

NFFT-like procedure for bandlimited functions

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(well-defined for $f \in L_1(\mathbb{R})$, or more generally f with $\sum_{\ell\in\mathbb{Z}} \big|f\big(\frac{\ell}{L}\big)\big| < \infty$)

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$\Rightarrow$ need access to as many equispaced samples $f\big(\frac{\ell}{L}\big)$ as possible $\Rightarrow$ inversion formula for $\hat\nu(v)$

$$\hat{\nu}(v) = \sum_{\ell \in \mathbb{Z}} f\left(\tfrac{\ell}{L}\right) \mathrm{e}^{-2\pi \mathrm{i} v \ell / L}$$

$$\hat{\nu}(v) = \sum_{\ell\in\mathbb{Z}} f\left(\tfrac{\ell}{L}\right) \mathrm{e}^{-2\pi\mathrm{i}v\ell/L} = \sum_{\ell=-\frac{\Theta}{2}}^{\frac{\Theta}{2}-1} f\left(\tfrac{\ell}{L}\right) \mathrm{e}^{-2\pi\mathrm{i}v\ell/L} + \sum_{r\in\mathbb{Z}\setminus\{0\}} \sum_{\ell=-\frac{\Theta}{2}}^{\frac{\Theta}{2}-1} f\left(\tfrac{\ell+r\Theta}{L}\right) \mathrm{e}^{-2\pi\mathrm{i}v(\ell+r\Theta)/L}, \quad \Theta \in 2\mathbb{N}$$

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$\rightsquigarrow f \in \mathcal{B}_{M/2}(\mathbb{R}) \subseteq C_0(\mathbb{R})$

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Additionally: $\hat{f}(v) = 0,\, v\notin\big[-\tfrac{M}{2},\tfrac{M}{2}\big]$, and $\hat{\psi}(v)\neq 0,\, v\in\big[-\tfrac{L}{2},\tfrac{L}{2}\big]$

$$\Rightarrow \qquad \hat{\vartheta}(k) = \begin{cases} \frac{\hat{f}(k)}{\hat{\psi}(k)} & : \quad k = -\frac{M}{2},\dots,\frac{M}{2}-1, \\ 0 & : \quad k = -\frac{L}{2},\dots,-\frac{M}{2}-1,\, \frac{M}{2},\dots,\frac{L}{2}-1, \end{cases} \qquad \hat{\boldsymbol{\vartheta}} = \boldsymbol{D}_{\hat{\psi}}\hat{\boldsymbol{f}}$$

$$\hat{\nu}(v) = \sum_{\ell\in\mathbb{Z}} f\big(\tfrac{\ell}{L}\big)\,\mathrm{e}^{-2\pi\mathrm{i}v\ell/L} \approx \underbrace{\sum_{\ell=-\frac{L}{2}}^{\frac{L}{2}-1} f\big(\tfrac{\ell}{L}\big)\,\mathrm{e}^{-2\pi\mathrm{i}v\ell/L}}_{\hat{\vartheta}(v)}, \qquad \Rightarrow \hat{f}(v) \approx \hat{\vartheta}(v)\cdot\hat{\psi}(v),\ v\in\big[-\tfrac{L}{2},\tfrac{L}{2}\big]$$

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iFFT

$$f\big(\tfrac{\ell}{L}\big) \approx \vartheta_\ell := \frac{1}{L}\sum_{k=-\frac{L}{2}}^{\frac{L}{2}-1} \hat{\vartheta}(k)\,\mathrm{e}^{2\pi\mathrm{i}k\ell/L}, \quad \ell = -\tfrac{L}{2},\dots,\tfrac{L}{2}-1, \qquad \boldsymbol{\vartheta} = \boldsymbol{F}\boldsymbol{\hat{\vartheta}}$$

$$\hat{\nu}(v) = \sum_{\ell \in \mathbb{Z}} f\left(\tfrac{\ell}{L}\right) \mathrm{e}^{-2\pi\mathrm{i}v\ell/L} \approx \underbrace{\sum_{\ell=-\frac{L}{2}}^{\frac{L}{2}-1} f\left(\tfrac{\ell}{L}\right) \mathrm{e}^{-2\pi\mathrm{i}v\ell/L}}_{\hat{\vartheta}(v)}, \qquad \Rightarrow \hat{f}(v) \approx \hat{\vartheta}(v) \cdot \hat{\psi}(v),\, v \in \left[-\tfrac{L}{2}, \tfrac{L}{2}\right]$$

$\rightsquigarrow f \in \mathcal{B}_{M/2}(\mathbb{R}) \subseteq C_0(\mathbb{R}) \Rightarrow f\left(\frac{\ell}{L}\right) \approx 0,\, |\ell| \geq \frac{\Theta}{2}$ for suitable Θ $\rightsquigarrow$ to avoid aliasing assume $\Theta = L$ is sufficient

Additionally: $\hat{f}(v) = 0,\, v \notin \left[-\frac{M}{2}, \frac{M}{2}\right]$, and $\hat{\psi}(v) \neq 0,\, v \in \left[-\frac{L}{2}, \frac{L}{2}\right]$

$$\Rightarrow \qquad \hat{\vartheta}(k) = \begin{cases} \frac{\hat{f}(k)}{\hat{\psi}(k)} & : \quad k = -\frac{M}{2}, \dots, \frac{M}{2}-1, \\ 0 & : \quad k = -\frac{L}{2}, \dots, -\frac{M}{2}-1, \frac{M}{2}, \dots, \frac{L}{2}-1, \end{cases} \qquad \boldsymbol{\hat{\vartheta}} = \boldsymbol{D}_{\hat{\psi}} \boldsymbol{\hat{f}}$$

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short sums

$$(R_{\varphi,m} f)(x_j) \approx f_j \coloneqq \sum_{\ell=-\frac{L}{2}}^{\frac{L}{2}-1} \vartheta_\ell\, \psi\left(x_j - \tfrac{\ell}{L}\right), \quad j = 1, \dots, N, \qquad \boldsymbol{f}^\star = \boldsymbol{\Psi} \boldsymbol{\vartheta}$$

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Important: restriction to $\ell=-\frac{L}{2},\dots,\frac{L}{2}-1$, implies limitation to $x_j\in[-\frac{1}{2}+\frac{m}{L},\frac{1}{2}-\frac{m}{L})$

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Numerical Example

Comparison to classical NFFT

Also possible:

$$f(x) = \int_{\left[-\frac{M}{2}, \frac{M}{2}\right)} \hat{f}(v)\, \mathrm{e}^{2\pi \mathrm{i} v x}\, \mathrm{d}v$$

Comparison to classical NFFT

Also possible: equispaced quadrature rule

$$f(x) = \int_{[-\frac{M}{2},\frac{M}{2})} \hat{f}(v)\,\mathrm{e}^{2\pi\mathrm{i}vx}\,\mathrm{d}v \approx \sum_{k=-\frac{M}{2}}^{\frac{M}{2}-1} \hat{f}(k)\,\mathrm{e}^{2\pi\mathrm{i}kx} \qquad \Rightarrow \text{NFFT}$$

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Question: Which method is better suited?

classical NFFT – $\boldsymbol{BFD}$

new NFFT-like method – $\boldsymbol{\Psi F D}_{\hat{\psi}}$

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window function φ
in $\boldsymbol{B}$ and $\boldsymbol{D}$

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regularized sinc function $\psi = \operatorname{sinc}(L\pi\cdot)\varphi$
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$\rightsquigarrow$ only comparable for $\boldsymbol{x} \in [-\frac{1}{2} + \frac{m}{L}, \frac{1}{2} - \frac{m}{L})$

$\boldsymbol{B}$ non-periodic unlike usual	$\boldsymbol{\Psi}$ non-periodic inherently by definition

Numerical example – Reconstruction of function evaluations

- bandwidth parameters $M \in \{20, 40, \ldots, 1000\}$
- function $f(x) = \mathrm{sinc}^2\big(\frac{M}{2}\pi x\big)$ with the Fourier transform
$$\hat{f}(v) = \frac{2}{M} \cdot \begin{cases} 1 - \left|\frac{2v}{M}\right| & : |v| \le \frac{M}{2}, \\ 0 & : \text{otherwise} \end{cases}$$
- scaled Chebyshev nodes
$$x_j = \cos\left(\frac{(j-1)\pi}{N}\right) \cdot \big(\tfrac{1}{2} - \tfrac{m}{L}\big), \quad j = 1, \ldots, N,$$
with $N = \frac{M}{2}$, $m = 5$
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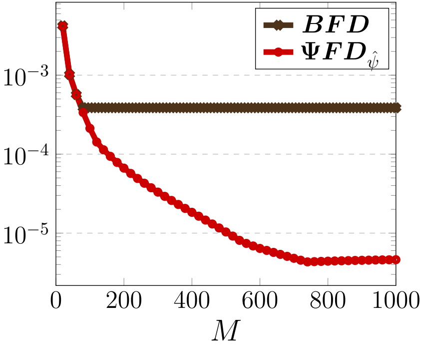

$\Rightarrow$ new method superior

Selected publications

- K., Potts, Tasche: **On regularized Shannon sampling formulas with localized sampling**. Sampl. Theory Signal Process. Data Anal. 20 (20), 2022.
- K., Potts, Tasche: **On numerical realizations of Shannon's sampling theorem**. Sampl. Theory Signal Process. Data Anal. 22 (13), 2024.
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- K.: **Fast Fourier Methods for Trigonometric Polynomials and Bandlimited Functions**. Dissertation. Shaker Verlag, Düren, 2024.
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Thank you for your attention!