Efficient numerical realizations of Shannon's sampling theorem

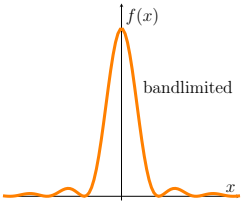
Melanie Kircheis

Chemnitz University of Technology
Faculty of Mathematics

joint work with
Daniel Potts and Manfred Tasche

Workshop on Approximation Theory and Fast Algorithms
Hasenwinkel, April 7–11, 2025

Motivation – Bandlimited functions



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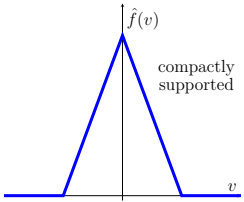
$$\hat{f}(v) := \int_{\mathbb{R}} f(x)\, \mathrm{e}^{-2\pi \mathrm{i} x v}\, \mathrm{d}x,$$



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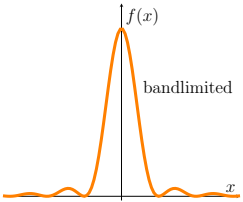

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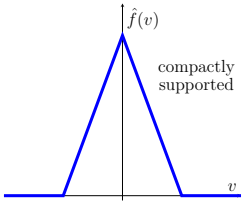

$\rightsquigarrow$ space of all bandlimited functions with **bandwidth** $M \in \mathbb{N}$ (**Paley–Wiener space**)

$$\mathcal{B}_{M/2}(\mathbb{R}) := \left\{ f \in L_2(\mathbb{R}) \colon \operatorname{supp}(\hat{f}) \subseteq \left[-\tfrac{M}{2},\, \tfrac{M}{2}\right] \right\} \subseteq L_2(\mathbb{R}) \cap C_0(\mathbb{R}) \cap C^\infty(\mathbb{R})$$

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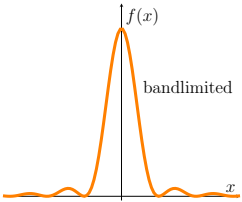

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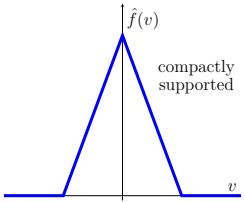

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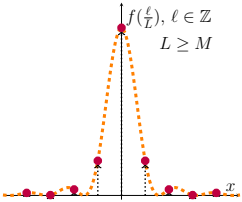
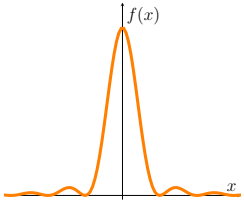


reconstruction of f
$\rightsquigarrow$ exact



Sampling Theorem of Shannon–Whittaker–Kotelnikov

[Whittaker 1915], [Kotelnikov 1933], [Shannon 1949]

Sampling Theorem

Let $f \in \mathcal{B}_{M/2}(\mathbb{R})$ and $L = M(1+\lambda) \in \mathbb{N}$, $\lambda \geq 0$. Then f is completely determined by its samples $f\left(\frac{\ell}{L}\right)$, $\ell \in \mathbb{Z}$, and we have

$$f(x) = \sum_{\ell \in \mathbb{Z}} f\left(\tfrac{\ell}{L}\right) \operatorname{sinc}\left(L\pi\left(x - \tfrac{\ell}{L}\right)\right), \quad x \in \mathbb{R},$$

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$$(S_T f)(x) := \sum_{\ell=-T}^{T} f\big(\tfrac{\ell}{L}\big)\, \operatorname{sinc}\big(L\pi\big(x - \tfrac{\ell}{L}\big)\big)$$

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Numerical Shortcomings:

- slow convergence: [Jagerman 66], [K. 24]

$$\max_{|x| \leq 1} \big|f(x) - (S_T f)(x)\big| \leq \frac{\sqrt{2L}}{\pi}\, (T-L)^{-1/2}\, \|f\|_{L_2(\mathbb{R})}$$

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- non-robustness w.r.t. erroneous samples

$$\tilde{f}_\ell := \begin{cases} f\big(\frac{\ell}{L}\big) + \varepsilon_\ell & : |\ell| \leq T \\ f\big(\frac{\ell}{L}\big) & : |\ell| > T \end{cases}$$

[Feichtinger 92], [Daubechies, DeVore 03], [K., Potts, Tasche 24a], [K. 24]

Numerical realization

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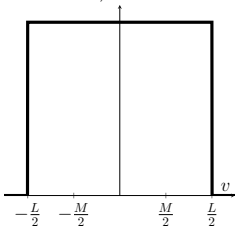

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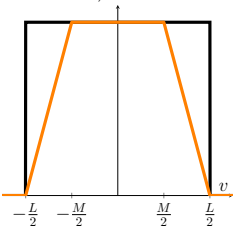

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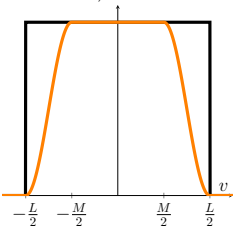

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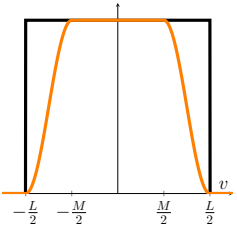

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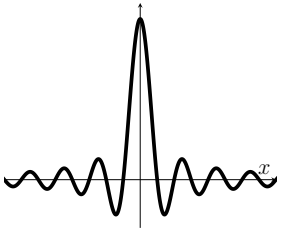

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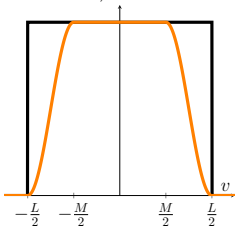

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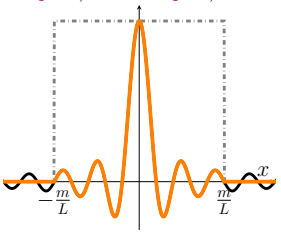

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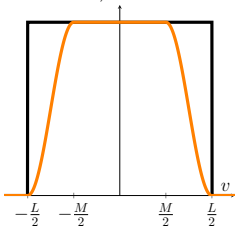

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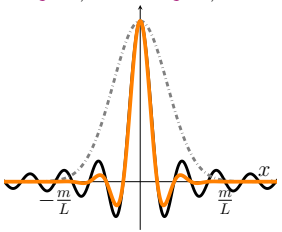

Overview

1. Regularization in frequency domain
2. Regularization in spatial domain
3. Numerical Example

Regularization in frequency domain

Frequency window functions

For $L := M(1+\lambda)$, $\lambda > 0$, we consider window functions of the form

$$\hat{\psi}(v) := \begin{cases} 1 & : |v| \leq \frac{M}{2}, \\ \xi(|v|) & : \frac{M}{2} < |v| < \frac{L}{2}, \\ 0 & : |v| \geq \frac{L}{2}, \end{cases}$$

with

1. $\xi|_{\left[\frac{M}{2}, \frac{L}{2}\right]}$ continuous and monotonously decreasing
2. $\xi\left(\frac{M}{2}\right) = 1,\ \xi\left(\frac{L}{2}\right) = 0$

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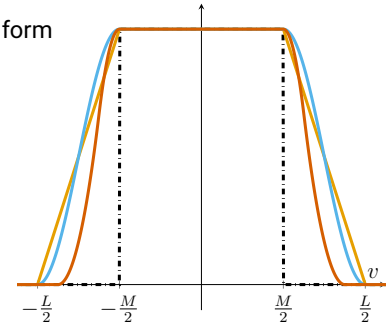

Examples:

$$\xi_{\mathrm{lin}}(v) := 1 - \tfrac{2v-M}{L-M}$$

$$\xi_{\mathrm{cub}}(v) := \tfrac{16}{(L-M)^3}\left(v - \tfrac{L}{2}\right)^2\left(v - \tfrac{3M-L}{4}\right)$$

$$\hat{\psi}_{\mathrm{conv,n}} := \chi_{\left[-\frac{L+M}{4}, \frac{L+M}{4}\right]} * \rho_n$$

with $\rho_n(v) = \frac{2n}{L-M}\, B_n\left(\frac{2n}{L-M} v\right)$, $n \in \mathbb{N}$,

$n = 1$, $n = 2$

Definition & properties

Condition: $\displaystyle\sum_{\ell\in\mathbb{Z}} \left|f\left(\tfrac{\ell}{L}\right)\right| < \infty \qquad (*)$

$$\implies \quad f(x) = \sum_{\ell\in\mathbb{Z}} f\left(\tfrac{\ell}{L}\right) \frac{1}{L}\, \psi\big(x - \tfrac{\ell}{L}\big), \quad x \in \mathbb{R}$$

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No interpolation: $f\left(\frac{k}{L}\right) \neq (P_{\psi,T} f)\left(\frac{k}{L}\right), k \in \mathbb{Z}$

since $\frac{1}{L}\, \psi\big(x - \frac{\ell}{L}\big)\big|_{x=\frac{k}{L}} \neq \delta_{k,\ell}, k, \ell \in \mathbb{Z}$

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Theorem (Approximation error estimate) [K. 24]

Let $f \in \mathcal{B}_{M/2}(\mathbb{R})$ with $M \in \mathbb{N}$, $L = M\,(1+\lambda) \in \mathbb{N}$ with $\lambda > 0$. If additionally to $(*)$ the decay condition

$$\big|\psi(x)\big| \leq c\,|x|^{-n}, \quad x \in \mathbb{R} \setminus \{0\},$$

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Condition: $\sum_{\ell\in\mathbb{Z}} \left|f\left(\frac{\ell}{L}\right)\right| < \infty \qquad (*)$

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$\rightsquigarrow$ only algebraic decay rates possible

Regularization in spatial domain

Spatial window functions

For $L \coloneqq M(1+\lambda)$, $\lambda \geq 0$, and $m \in \mathbb{N} \setminus \{1\}$ with $2m \ll L$ we introduce the set Φ of window functions

1. $\varphi : \mathbb{R} \to [0,\, 1]$ is positive and even
2. $\varphi \in L_1(\mathbb{R}) \cap C_0(\mathbb{R})$
3. $\varphi|_{[0,\,\infty)}$ is monotonously non-increasing with $\varphi(0) = 1$

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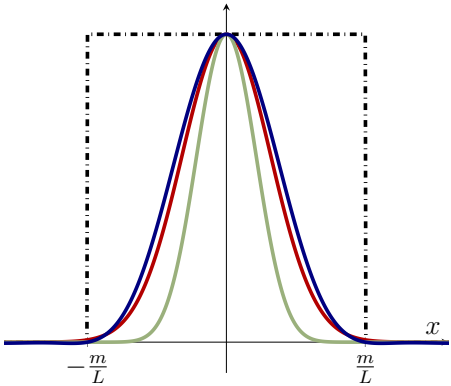
Examples:

$$\varphi_{\mathrm{Gauss}}(t) := \mathrm{e}^{-t^2/(2\alpha^2)}\,, \alpha > 0$$

$$\varphi_{\sinh}(t) := \frac{\sinh\left(\beta\sqrt{1-(Lt/m)^2}\right)}{\sinh\beta}\,\chi_{[-m/L,\,m/L]}(t)\,, \beta > 0$$

$$\varphi_{\mathrm{cKB}}(t) := \frac{I_0\left(\beta\sqrt{1-(Lt/m)^2}\right)-1}{I_0(\beta)-1}\,\chi_{[-m/L,\,m/L]}(t)\,, \beta > 0$$



Definition & properties

Truncation:

$$(R_{\varphi,m} f)(x) = \sum_{\ell \in \mathbb{Z}} f\left(\tfrac{\ell}{L}\right) \operatorname{sinc}\left(L\pi \left(x - \tfrac{\ell}{L}\right)\right) \varphi_m\left(x - \tfrac{\ell}{L}\right), \quad x \in \mathbb{R},$$

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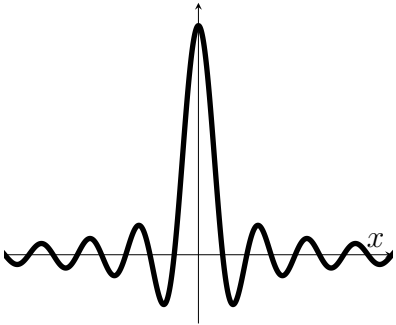

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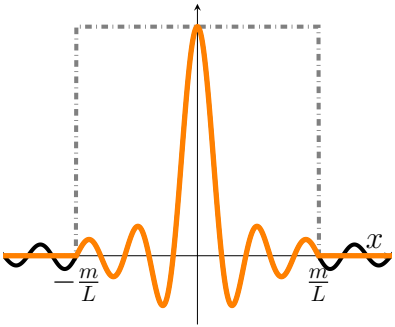

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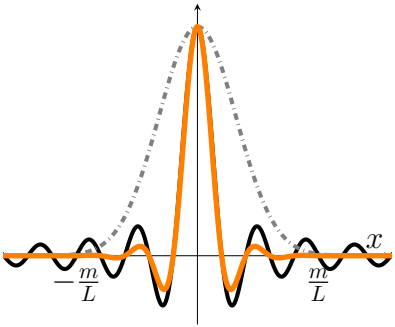

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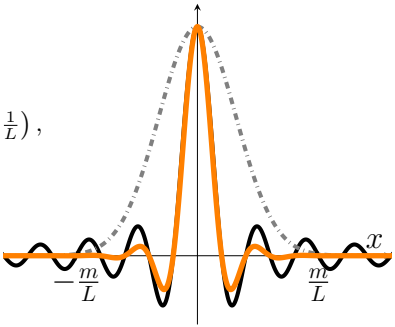

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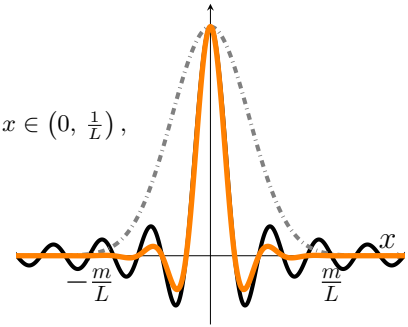

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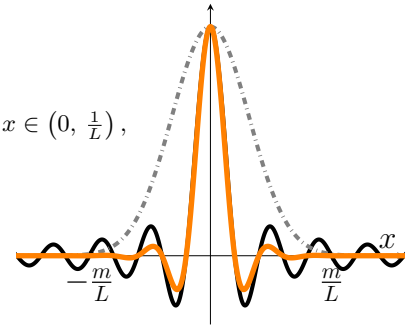
since $\operatorname{sinc}(\pi(k-\ell)) = \delta_{k,\ell}$ and $\varphi(0) = 1$



General error estimates

Theorem (Uniform approximation error estimate) [K., Potts, Tasche 22]

Let $f \in \mathcal{B}_{M/2}(\mathbb{R})$ with $M \in \mathbb{N}$, $L = M\,(1+\lambda) \in \mathbb{N}$ with $\lambda \geq 0$, and $m \in \mathbb{N} \setminus \{1\}$ be given. Further let $\varphi \in \Phi$ with $\varphi_m(x) := \varphi(x)\,\chi_{\left[-\frac{m}{L},\,\frac{m}{L}\right]}(x)$. Then the regularized Shannon sampling formula satisfies

$$\|f - R_{\varphi,m} f\|_{C_0(\mathbb{R})} \leq \big(E_1(\varphi, M, L) + E_2(\varphi, M, L)\big)\,\|f\|_{L_2(\mathbb{R})},$$

with the error constants

$$E_1(\varphi, M, L) := \sqrt{M} \max_{v \in \left[-\frac{M}{2},\,\frac{M}{2}\right]} \left| 1 - \int_{v-\frac{L}{2}}^{v+\frac{L}{2}} \hat{\varphi}(u)\,\mathrm{d}u \right|, \qquad E_2(\varphi, M, L) := \frac{\sqrt{2L}}{\pi} \sqrt{\frac{\varphi^2\left(\frac{m}{L}\right)}{m^2} + \int_{\frac{m}{L}}^{\infty} \frac{\varphi^2(t)}{Lt^2}\,\mathrm{d}t}.$$

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Theorem (Uniform perturbation error estimate) [K., Potts, Tasche 22]

Let additionally the noisy samples $\tilde{f}_\ell := f\left(\frac{\ell}{L}\right) + \varepsilon_\ell$ with $|\varepsilon_\ell| \leq \varepsilon$, $\ell \in \mathbb{Z}$, and $\varepsilon > 0$ be given. Then we have

$$\|R_{\varphi,m} \tilde{f} - R_{\varphi,m} f\|_{C_0(\mathbb{R})} \leq \varepsilon\,\big(2 + L\,\hat{\varphi}(0)\big).$$

Simplified result

This theorem can be simplified, if the window function $\varphi \in \Phi$ vanishes on $\mathbb{R} \setminus \left[-\frac{m}{L}, \frac{m}{L} \right]$.

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Examples:

✓ sinh-type

✓ continuous Kaiser–Bessel

✗ Gaussian
→ does not vanish at all

Results for special window functions (approximation error) & parameter choice

Only have to estimate the error constants $E_j(\varphi, M, L), j = 1, 2.$

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$$\|R_{\sinh,m}\tilde{f} - R_{\sinh,m}f\|_{C_0(\mathbb{R})} \leq \varepsilon \left(2 + \sqrt{\frac{2+2\lambda}{\lambda}}\,\frac{1}{1-\mathrm{e}^{-2\beta}}\,\sqrt{m}\right)$$

- continuous Kaiser–Bessel window function [K., Potts, Tasche 24a]

$$\|R_{\mathrm{cKB},m}\tilde{f} - R_{\mathrm{cKB},m}f\|_{C_0(\mathbb{R})} \leq \varepsilon \left(2 + \sqrt{\frac{2+2\lambda}{\lambda}}\,\sqrt{m}\right)$$

Numerical Example

Numerical example – Comparison of regularized Shannon sampling formulas

[K., Potts, Tasche 24a]

1. choose bandlimited function $f(x) = \sqrt{\frac{4M}{5}} \left[\operatorname{sinc}(M\pi x) + \frac{1}{2}\operatorname{sinc}(M\pi(x-1))\right]$ with $M = 256$
2. compute the approximation $\sum_{\ell\in\mathbb{Z}} f\left(\frac{\ell}{L}\right) \psi\left(x - \frac{\ell}{L}\right)$ using
   - $L = 3M$,
   - regularizations ψ in frequency and spatial domain,
   - same number of samples $\rightsquigarrow$ set $T = L + m$
3. compute the error

$$\max_{s=1,\dots,10^5} \left| f(x_s) - \sum_{\ell\in\mathbb{Z}} f\left(\frac{\ell}{L}\right) \psi\left(x_s - \frac{\ell}{L}\right) \right|$$

on a fine equispaced grid $x_s \in [-1,\, 1]$, $s = 1, \dots, 10^5$

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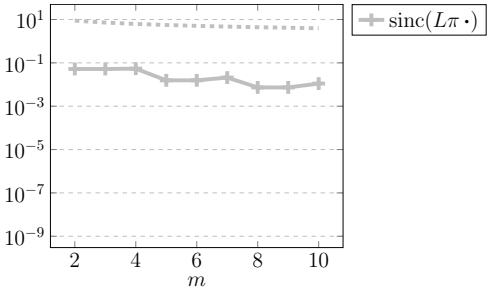

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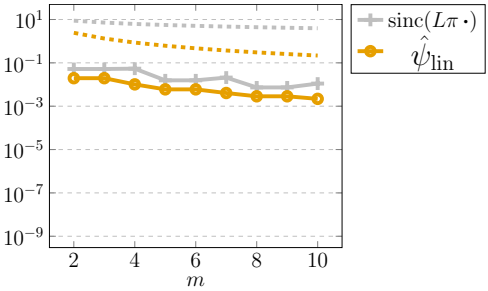

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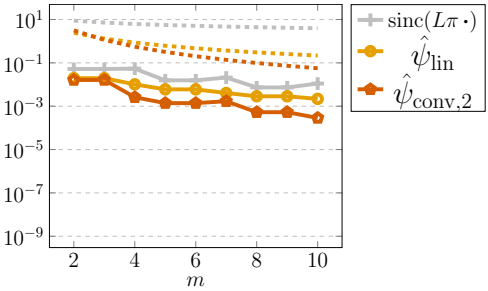

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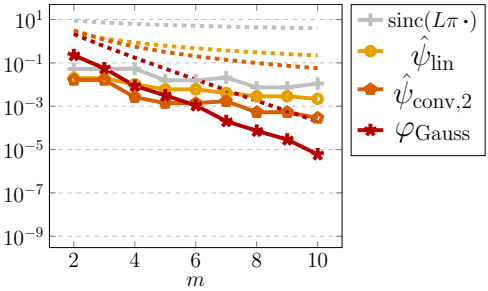

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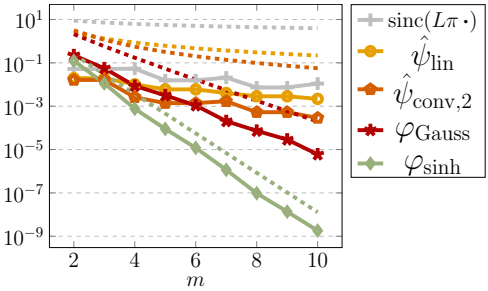


$\rightsquigarrow$ superiority of the new windows evident

Summary – Regularized Shannon sampling formulas

Regularization in frequency domain

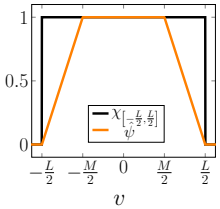
Regularization in spatial domain

Summary – Regularized Shannon sampling formulas

Regularization in frequency domain

bandlimited
(compactly supported in frequency domain)

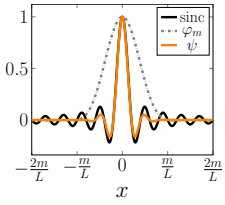
choose $\hat{\psi}$



Regularization in spatial domain

timelimited
(compactly supported in spatial domain)

multiply sinc with convenient window function φ_m



Summary – Regularized Shannon sampling formulas

Regularization in frequency domain

bandlimited
(compactly supported in frequency domain)

choose $\hat{\psi}$ and compute inverse Fourier transform

Regularization in spatial domain

timelimited
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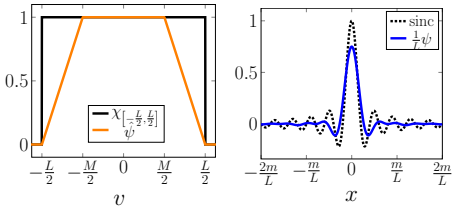
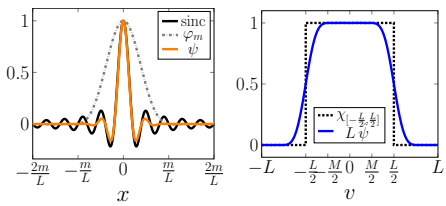
multiply sinc with convenient window function φ_m





Summary – Regularized Shannon sampling formulas

Regularization in frequency domain

bandlimited
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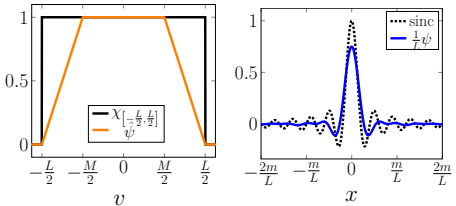
choose $\hat{\psi}$ and compute inverse Fourier transform

no localization
$\rightsquigarrow$ still truncation parameter T needed

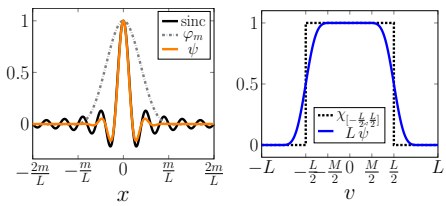


Regularization in spatial domain

timelimited
(compactly supported in spatial domain)

multiply sinc with convenient window function φ_m

localization
$\rightsquigarrow$ only $2m+1$ samples needed



Summary – Regularized Shannon sampling formulas

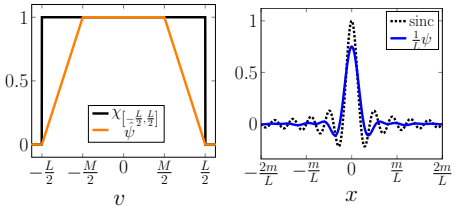
Regularization in frequency domain

bandlimited
(compactly supported in frequency domain)

choose $\hat{\psi}$ and compute inverse Fourier transform

no localization
$\rightsquigarrow$ still truncation parameter T needed

$\rightsquigarrow$ only algebraic decay rates
for all known window functions



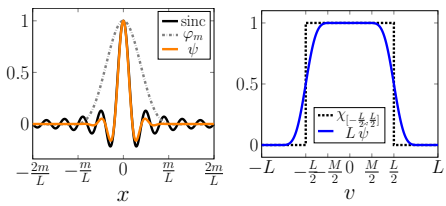
Regularization in spatial domain

timelimited
(compactly supported in spatial domain)

multiply sinc with convenient window function φ_m

localization
$\rightsquigarrow$ only $2m+1$ samples needed

$\rightsquigarrow$ exponential decay rates
for new window functions



Publications

- **K., Potts, Tasche: On regularized Shannon sampling formulas with localized sampling**. Sampl. Theory Signal Process. Data Anal. 20 (20), 2022.
- **K., Potts, Tasche: On numerical realizations of Shannon's sampling theorem**. Sampl. Theory Signal Process. Data Anal. 22 (13), 2024.
- **K., Potts, Tasche: Optimal parameter choice for regularized Shannon sampling formulas**. arXiv:2407.16401, 2024.
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Thank you for your attention!