Fast Fourier methods for trigonometric polynomials and bandlimited functions

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Motivation

phantom

$\rightsquigarrow$ given measurements $\rightsquigarrow$

reconstruction

$\hat{f}_{\boldsymbol{k}}$ $\qquad$ $f(\boldsymbol{x}_j) = \sum_{\boldsymbol{k} \in \mathcal{I}_M} \hat{f}_{\boldsymbol{k}} \, \mathrm{e}^{2\pi \mathrm{i} \boldsymbol{k} \boldsymbol{x}_j}, \; j = 1, \ldots, N$ $\qquad$ $\tilde{h}_{\boldsymbol{k}} \approx \hat{f}_{\boldsymbol{k}}$

Discrete Fourier Transform (DFT)

e. g. $\mathcal{I}_M = \{-512, \ldots, 512\}^2$

$$f(\boldsymbol{x}_j) = \sum_{\boldsymbol{k} \in \mathcal{I}_{\boldsymbol{M}}} \hat{f}_{\boldsymbol{k}} \, \mathrm{e}^{2\pi \mathrm{i} \boldsymbol{k} \boldsymbol{x}_j}, \quad j = 1, \ldots, N \qquad \mathcal{O}(|\mathcal{I}_{\boldsymbol{M}}| \cdot N)$$

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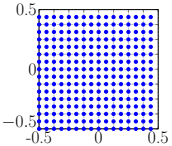
$\boldsymbol{x}_j$ equispaced and $|\mathcal{I}_{\boldsymbol{M}}| = N$



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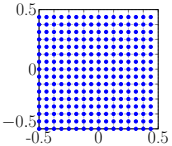

Evaluation: Fast Fourier Transform (FFT)

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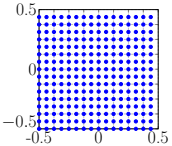

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Inversion: iFFT

(basically the same algorithm)

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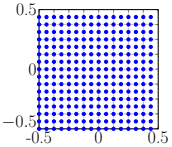
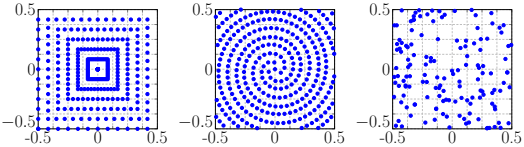


$\boldsymbol{x}_j$ nonequispaced



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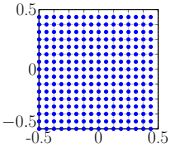

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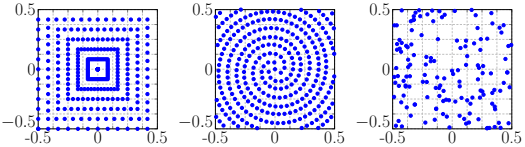

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Nonequispaced Fast Fourier Transform (NFFT)

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[Dutt, Rokhlin 93], [Beylkin 95], [Potts, Steidl, Tasche 01]

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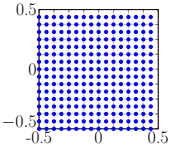

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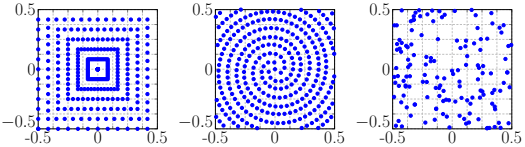

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$\boldsymbol{v} \in \mathbb{R}^2$, $\boldsymbol{k} \in \mathcal{I}_{\boldsymbol{M}}$

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iterative methods
(multiple applications of the NFFT needed)

vs.

direct methods
(realized with a single NFFT)

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special setting: evaluation points $\boldsymbol{x}_j$, $j = 1, \ldots, N$, fixed $\implies$ highly profit from direct method

1. precomputation: only once for fixed $\boldsymbol{x}_j$
2. reconstruction: for each measurement $\rightsquigarrow$ very efficient

Overview

1. Discrete inversion problem
2. Continuous evaluation problem
3. Continuous inversion problem

NFFT (Nonequispaced Fast Fourier Transform) / NUFFT

Fast algorithm to evaluate a trigonometric polynomial

$$f(\boldsymbol{x}) = \sum_{\boldsymbol{k} \in \mathcal{I}_{\boldsymbol{M}}} \hat{f}_{\boldsymbol{k}} \, \mathrm{e}^{2\pi \mathrm{i} \boldsymbol{k} \boldsymbol{x}}$$

$\mathcal{O}(|\mathcal{I}_{\boldsymbol{M}}| \log(|\mathcal{I}_{\boldsymbol{M}}|) + N)$

[Dutt, Rokhlin 93], [Beylkin 95], [Potts, Steidl, Tasche 01]

- index set $\mathcal{I}_{\boldsymbol{M}} := \mathbb{Z}^d \cap \left[-\frac{M}{2}, \frac{M}{2}\right)^d$ with cardinality $|\mathcal{I}_{\boldsymbol{M}}| = M^d$, $M \in 2\mathbb{N}$,
- Fourier coefficients $\hat{f}_{\boldsymbol{k}} \in \mathbb{C}$, $\boldsymbol{k} \in \mathcal{I}_{\boldsymbol{M}}$,
- nonequispaced points $\boldsymbol{x}_j \in \mathbb{T}^d \cong \left[-\frac{1}{2}, \frac{1}{2}\right)^d$, $j = 1, \ldots, N$, $N \in \mathbb{N}$

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Matrix notation:

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Factorizations: $\boldsymbol{A} \approx \boldsymbol{B}\boldsymbol{F}\boldsymbol{D}$ and $\boldsymbol{A}^* \approx \boldsymbol{D}^*\boldsymbol{F}^*\boldsymbol{B}^*$

$(2m+1)^d$–sparse ↗ FFT ↑ diagonal ↖

(in each column of $\boldsymbol{B}$ only $(2m+1)^d$ entries, $m \in \mathbb{N}$ given)

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Inversion problem (iNFFT): **Given:** $\boldsymbol{f} := (f(\boldsymbol{x}_j))_{j=1}^N$ **Find:** $\hat{\boldsymbol{f}} := (\hat{f}_{\boldsymbol{k}})_{\boldsymbol{k}\in\mathcal{I}_{\boldsymbol{M}}}$ **Challenge:** in general $N \neq |\mathcal{I}_{\boldsymbol{M}}|$

Previous approaches for $d = 1$:

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  [Ruiz-Antolin, Townsend 18]: CG algorithm & low-rank approximation, $N = |\mathcal{I}_M|$

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  - general case: [K., Potts 19]: optimization of the sparse matrix $\boldsymbol{B}^*$ by minimizing a certain Frobenius norm
    $\rightsquigarrow$ connection to frame approach
    [Wilber, Epperly, Barnett 24]: low-rank approximation of structured matrices
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Now: generalization of our direct method to $d > 1$

Basic idea

Equispaced nodes: $\boldsymbol{A}^* \boldsymbol{A} = N \boldsymbol{I}_{|\mathcal{I}_M|}$

Nonequispaced nodes: $\boldsymbol{A}^* \boldsymbol{A} \neq N \boldsymbol{I}_{|\mathcal{I}_M|}$

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$\Rightarrow$ Find suitable matrix $\boldsymbol{X}$ with

$$\boldsymbol{X} \boldsymbol{A} \approx \boldsymbol{I}_{|\mathcal{I}_M|},$$

since then

$$\hat{\boldsymbol{f}} \approx \boldsymbol{X} \boldsymbol{A} \hat{\boldsymbol{f}} = \boldsymbol{X} \boldsymbol{f}.$$

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Reminder – Equispaced nodes:

$$\boldsymbol{X} = \boldsymbol{A}^* \cdot \frac{1}{N}$$

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Simplest generalization:

$$\boldsymbol{X} = \boldsymbol{A}^*\boldsymbol{W} \approx \boldsymbol{D}^*\boldsymbol{F}^*\boldsymbol{B}^*\boldsymbol{W},$$

i. e., additional weighting $\boldsymbol{W} := \operatorname{diag}(w_j)_{j=1}^N$ due to nonequispaced sampling

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Density compensation algorithm

0. Precompute weights $\boldsymbol{W}$?
1. Compute scaled coefficients $\boldsymbol{W}\boldsymbol{f}$ $\mathcal{O}(N)$
2. Adjoint NFFT $\mathcal{O}(|\mathcal{I}_M|\log(|\mathcal{I}_M|) + N)$

Intuitive approach:
Voronoi weights based on geometry

i. e., additional weighting $\boldsymbol{W} := \operatorname{diag}(w_j)_{j=1}^N$ due to nonequispaced sampling

Exact reconstruction for trigonometric polynomials

Theorem (Exactness condition) [K., Potts 23a]

Let $|\mathcal{I}_{\boldsymbol{2M}}| \leq N$ and $\boldsymbol{x}_j \in \mathbb{T}^d$, $j = 1, \ldots, N$, be given.

If the density compensation factors $w_j \in \mathbb{C}$ satisfy

$$\sum_{j=1}^{N} w_j \, \mathrm{e}^{2\pi \mathrm{i} \boldsymbol{k} \boldsymbol{x}_j} = \delta_{\boldsymbol{0},\boldsymbol{k}}, \quad \boldsymbol{k} \in \mathcal{I}_{\boldsymbol{2M}}, \qquad \boldsymbol{A}^{T}_{|\mathcal{I}_{\boldsymbol{2M}}|} \, \boldsymbol{w} = \boldsymbol{e_0}$$

then for all trigonometric polynomials $f(\boldsymbol{x}) = \sum_{\boldsymbol{k} \in \mathcal{I}_{\boldsymbol{M}}} \hat{f}_{\boldsymbol{k}} \, \mathrm{e}^{2\pi \mathrm{i} \boldsymbol{k} \boldsymbol{x}}$ an exact reconstruction of the Fourier coefficients $\hat{f}_{\boldsymbol{k}} \in \mathbb{C}$ is given by

$$\hat{f}_{\boldsymbol{k}} = h^{\mathrm{w}}_{\boldsymbol{k}} := \sum_{j=1}^{N} w_j \, f(\boldsymbol{x}_j) \, \mathrm{e}^{-2\pi \mathrm{i} \boldsymbol{k} \boldsymbol{x}_j}, \quad \boldsymbol{k} \in \mathcal{I}_{\boldsymbol{M}}. \qquad \hat{\boldsymbol{f}} = \boldsymbol{A}^{*} \boldsymbol{W} \boldsymbol{f}$$

$\delta_{\boldsymbol{0},\boldsymbol{k}}$ ... Kronecker delta

Computation schemes

Aim: exact solution to

$$\boldsymbol{A}^{T}_{|\mathcal{I}_{\boldsymbol{2M}}|}\,\boldsymbol{w} = \boldsymbol{e_0} \coloneqq (\delta_{\boldsymbol{0},\boldsymbol{k}})_{\boldsymbol{k}\in\mathcal{I}_{\boldsymbol{2M}}} \qquad (*)$$

Computation schemes

Aim: exact solution to

$$\boldsymbol{A}_{|\mathcal{I}_{\boldsymbol{2M}}|}^{T}\,\boldsymbol{w} = \boldsymbol{e_0} \coloneqq (\delta_{\boldsymbol{0},\boldsymbol{k}})_{\boldsymbol{k}\in\mathcal{I}_{\boldsymbol{2M}}} \tag{$*$}$$

Friendly setting ($|\mathcal{I}_{\boldsymbol{2M}}| \leq N$): unique minimum norm solution given by normal eqs. of 2nd kind [Huybrechs 09]

$$\boldsymbol{A}_{|\mathcal{I}_{\boldsymbol{2M}}|}^{T}\,\overline{\boldsymbol{A}_{|\mathcal{I}_{\boldsymbol{2M}}|}}\,\boldsymbol{v} = \boldsymbol{e_0}, \quad \overline{\boldsymbol{A}_{|\mathcal{I}_{\boldsymbol{2M}}|}}\,\boldsymbol{v} = \boldsymbol{w}$$

$\rightsquigarrow$ efficient computation: CG algorithm combined with NFFT $\mathcal{O}(|\mathcal{I}_{\boldsymbol{2M}}|\log(|\mathcal{I}_{\boldsymbol{2M}}|) + N)$

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Unfriendly setting ($|\mathcal{I}_{2M}| > N$): no theoretical guarantee!

Computation schemes

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Unfriendly setting ($|\mathcal{I}_{\boldsymbol{2M}}| > N$): no theoretical guarantee!
least squares solution by normal eqs. of 1st kind

$$\overline{\boldsymbol{A}_{|\mathcal{I}_{\boldsymbol{2M}}|}}\, \boldsymbol{A}_{|\mathcal{I}_{\boldsymbol{2M}}|}^{T}\, \boldsymbol{w} = \overline{\boldsymbol{A}_{|\mathcal{I}_{\boldsymbol{2M}}|}}\, \boldsymbol{e_0}$$

[K., Potts 23a]: not a good approximation...

Recapitulation

So far:

$$\hat{\boldsymbol{f}} \approx \boldsymbol{A}^* \boldsymbol{W} \boldsymbol{f} \approx \boldsymbol{D}^* \boldsymbol{F}^* \boldsymbol{B}^* \boldsymbol{W} \boldsymbol{f}$$

$\boldsymbol{D}^*$: diagonal; $\boldsymbol{F}^*$: FFT; $\boldsymbol{B}^*$: $(2m+1)^d$-sparse

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So far:

$$\hat{f} \approx A^* W f \approx D^* F^* B^* W f$$

diagonal — FFT — $(2m+1)^d$–sparse

Interpretation perspectives:

(i) Set $g := Wf$. $\Rightarrow$ $\hat{f} \approx D^* F^* B^* g$ $\rightsquigarrow$ ordinary NFFT, modified coefficient vector

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$\Rightarrow$ density compensation $\widehat{=}$ optimization of the sparse matrix B (only N degrees of freedom)

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Now: optimize each nonzero entry of the sparse matrix $\boldsymbol{B}$ ($N(2m+1)^d$ degrees of freedom)

Aim: [K., Potts 21/23a]

- $\rightsquigarrow$ modification of matrix $\boldsymbol{B}$
- $\rightsquigarrow$ preserve sparse structure and thereby arithmetic complexity

Optimization approach

0. Precompute $(2m+1)^d$–sparse matrix $\boldsymbol{B}$?

1. (Modified) adjoint NFFT $\mathcal{O}(|\mathcal{I}_{\boldsymbol{M}}| \log(|\mathcal{I}_{\boldsymbol{M}}|) + N)$

Precomputational step – Optimization procedure

[K., Potts 21/23a]

Define $\tilde{\boldsymbol{h}} := \boldsymbol{D}^* \boldsymbol{F}^* \tilde{\boldsymbol{B}}^* \boldsymbol{f}$.

$$\begin{aligned} \Rightarrow \quad \|\tilde{\boldsymbol{h}} - \hat{\boldsymbol{f}}\|_2 &= \|\boldsymbol{D}^* \boldsymbol{F}^* \tilde{\boldsymbol{B}}^* \boldsymbol{f} - \hat{\boldsymbol{f}}\|_2 = \|\boldsymbol{D}^* \boldsymbol{F}^* \tilde{\boldsymbol{B}}^* \boldsymbol{A} \hat{\boldsymbol{f}} - \hat{\boldsymbol{f}}\|_2 \\ &\leq \left\| \boldsymbol{D}^* \boldsymbol{F}^* \tilde{\boldsymbol{B}}^* \boldsymbol{A} - \boldsymbol{I}_{|\mathcal{I}_M|} \right\|_{\mathrm{F}} \|\hat{\boldsymbol{f}}\|_2 \end{aligned}$$

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Optimization problem:

$$\underset{\tilde{\boldsymbol{B}}\ (2m+1)^d\text{-sparse}}{\text{Minimize}} \left\| \boldsymbol{D}^* \boldsymbol{F}^* \tilde{\boldsymbol{B}}^* \boldsymbol{A} - \boldsymbol{I}_{|\mathcal{I}_M|} \right\|_{\mathrm{F}}^2 = \left\| \boldsymbol{A}^* \tilde{\boldsymbol{B}} \boldsymbol{F} \boldsymbol{D} - \boldsymbol{I}_{|\mathcal{I}_M|} \right\|_{\mathrm{F}}^2$$

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Frobenius norm submultiplicative

$$\left\| \boldsymbol{A}^* \tilde{\boldsymbol{B}} \boldsymbol{F} \boldsymbol{D} - \boldsymbol{I}_{|\mathcal{I}_M|} \right\|_{\mathrm{F}}^2 = \left\| \left(\boldsymbol{A}^* \tilde{\boldsymbol{B}} - \tfrac{1}{|\mathcal{I}_{M_\sigma}|} \boldsymbol{D}^{-1} \boldsymbol{F}^* \right) \boldsymbol{F} \boldsymbol{D} \right\|_{\mathrm{F}}^2 \leq \left\| \boldsymbol{A}^* \tilde{\boldsymbol{B}} - \tfrac{1}{|\mathcal{I}_{M_\sigma}|} \boldsymbol{D}^{-1} \boldsymbol{F}^* \right\|_{\mathrm{F}}^2 \|\boldsymbol{F} \boldsymbol{D}\|_{\mathrm{F}}^2$$

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Frobenius norm submultiplicative

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$$\Longrightarrow \quad \underset{\tilde{\boldsymbol{B}}\ (2m+1)^d\text{-sparse}}{\text{Minimize}} \left\| \boldsymbol{A}^* \tilde{\boldsymbol{B}} - \tfrac{1}{|\mathcal{I}_{M_\sigma}|} \boldsymbol{D}^{-1} \boldsymbol{F}^* \right\|_{\mathrm{F}}^2 = \sum_{\boldsymbol{\ell} \in \mathcal{I}_{M_\sigma}} \left\| \boldsymbol{A}_{\boldsymbol{\ell}}^* \tilde{\boldsymbol{b}}_{\boldsymbol{\ell}} - \tfrac{1}{|\mathcal{I}_{M_\sigma}|} \boldsymbol{D}^{-1} \boldsymbol{f}_{\boldsymbol{\ell}} \right\|_2^2$$

$\rightsquigarrow \mathcal{O}(|\mathcal{I}_M|)$

Numerical example – Shepp-Logan phantom

[K., Potts 23a]

1. phantom data $=$ Fourier coefficients $\hat{\boldsymbol{f}} := (\hat{f}_{\boldsymbol{k}})_{\boldsymbol{k}\in\mathcal{I}_{\boldsymbol{M}}}$ of a trigonometric polynomial
2. compute the evaluations $f(\boldsymbol{x}_j) = \sum_{\boldsymbol{k}\in\mathcal{I}_{\boldsymbol{M}}} \hat{f}_{\boldsymbol{k}}\, \mathrm{e}^{2\pi \mathrm{i} \boldsymbol{k}\boldsymbol{x}_j}$ by means of NFFT
3. reconstruct $\tilde{h}_{\boldsymbol{k}} \approx \hat{f}_{\boldsymbol{k}}$, $\boldsymbol{k} \in \mathcal{I}_{\boldsymbol{M}}$

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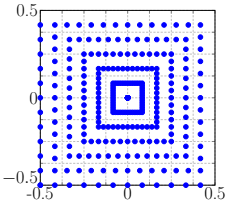
Friendly setting ($|\mathcal{I}_{2M}| \leq N$):

- linogram grid of size $R = 2M$, $T = 2R$
- phantom size $M \times M$ with $M = 2^c$, $c = 3, \ldots, 10$
- relative errors

$$e_2 := \frac{\|\tilde{\boldsymbol{h}} - \hat{\boldsymbol{f}}\|_2}{\|\hat{\boldsymbol{f}}\|_2}$$



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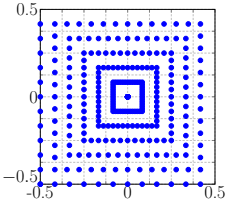
Unfriendly setting ($|\mathcal{I}_{\boldsymbol{2M}}| > N$):

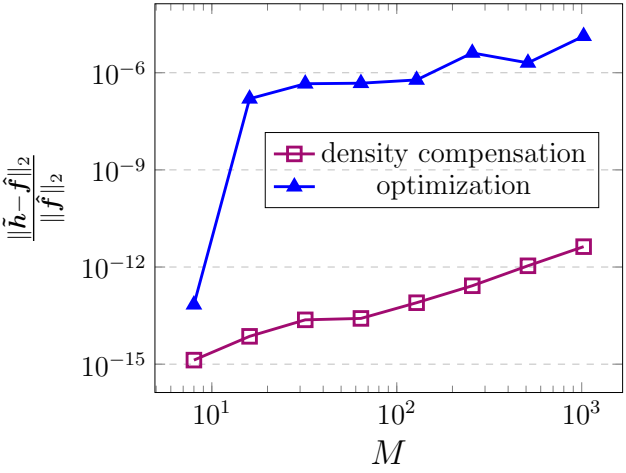
- linogram grid of size $R = M, T = 2R$
- phantom of size $M = 1024$

⇝ compare presented computation schemes

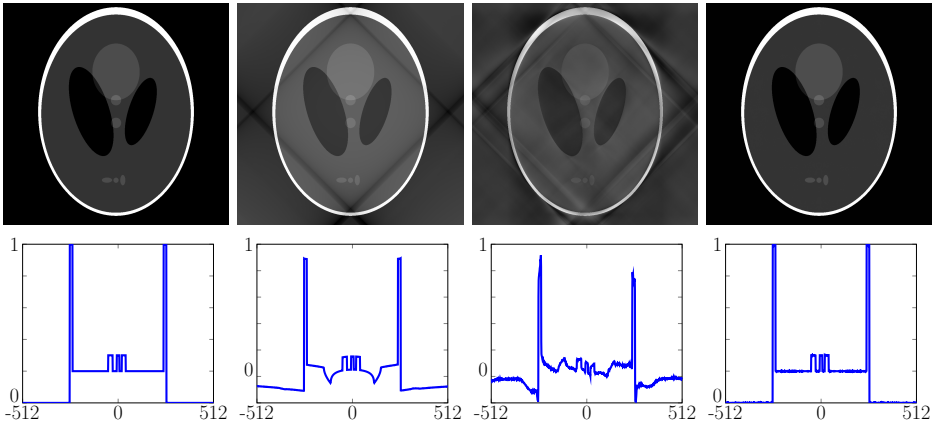
- Voronoi weights
- new density compensation factors
- optimization approach



Results – Friendly setting ($|\mathcal{I}_{2M}| < N$) covered by theory



Results – Unfriendly setting ($|\mathcal{I}_{2M}| > N$) testing approaches



(a) Original phantom

(b) Voronoi weights
$e_2 = 5.30e-01$

(c) Density compensation
$e_2 = 5.06e-01$

(d) Optimization
$e_2 = 2.27e-03$

Continuous evaluation problem

Bandlimited functions

Application: MRI (Magnetic Resonance Imaging) [Rosenfeld 98], [Greengard, Lee, Inati 06], [Eggers, K., Potts 22]

no longer discrete (trigonometric polynomials) but continuous (bandlimited functions)

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(Continuous) Fourier transform:

$$\hat{f}(\boldsymbol{v}) := \int_{\mathbb{R}^d} f(\boldsymbol{x})\, \mathrm{e}^{-2\pi \mathrm{i} \boldsymbol{v}\boldsymbol{x}}\, \mathrm{d}\boldsymbol{x}, \quad \boldsymbol{v} \in \mathbb{R}^d$$

$\rightsquigarrow$ space of all bandlimited functions with **bandwidth** $M \in \mathbb{N}$ (**Paley–Wiener space**)

$$\mathcal{B}_{M/2}(\mathbb{R}^d) := \left\{ f \in L_2(\mathbb{R}^d) \colon \operatorname{supp}(\hat{f}) \subseteq \left[-\tfrac{M}{2}, \tfrac{M}{2}\right]^d \right\} \subseteq L_2(\mathbb{R}^d) \cap C_0(\mathbb{R}^d) \cap C^\infty(\mathbb{R}^d)$$

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In this talk: for simplicity only $d = 1$

Sampling Theorem of Shannon–Whittaker–Kotelnikov

[Whittaker 1915], [Kotelnikov 1933], [Shannon 1949]

Sampling Theorem

Let $f \in \mathcal{B}_{M/2}(\mathbb{R})$ and $L = M(1+\lambda) \in \mathbb{N}$, $\lambda \geq 0$. Then f is completely determined by its samples $f\left(\frac{\ell}{L}\right)$, $\ell \in \mathbb{Z}$, and we have

$$f(x) = \sum_{\ell \in \mathbb{Z}} f\left(\tfrac{\ell}{L}\right) \operatorname{sinc}\left(L\pi\left(x - \tfrac{\ell}{L}\right)\right), \quad x \in \mathbb{R},$$

with the cardinal sine function

$$\operatorname{sinc}(x) := \begin{cases} \frac{\sin x}{x} & : x \in \mathbb{R} \setminus \{0\}, \\ 1 & : x = 0. \end{cases}$$

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Truncation:

$$(S_T f)(x) := \sum_{\ell=-T}^{T} f\big(\tfrac{\ell}{L}\big)\, \operatorname{sinc}\big(L\pi\big(x - \tfrac{\ell}{L}\big)\big)$$

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Numerical Shortcomings:

- slow convergence: [Jagerman 66], [K. 24]

$$\max_{|x| \leq 1} \big|f(x) - (S_T f)(x)\big| \leq \frac{\sqrt{2L}}{\pi}\, (T-L)^{-1/2}\, \|f\|_{L_2(\mathbb{R})}$$

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Sampling Theorem

Let $f \in \mathcal{B}_{M/2}(\mathbb{R})$ and $L = M(1+\lambda) \in \mathbb{N}$, $\lambda \geq 0$. Then f is completely determined by its samples $f\big(\frac{\ell}{L}\big)$, $\ell \in \mathbb{Z}$, and we have

$$f(x) = \sum_{\ell \in \mathbb{Z}} f\big(\tfrac{\ell}{L}\big)\, \operatorname{sinc}\big(L\pi\big(x - \tfrac{\ell}{L}\big)\big), \quad x \in \mathbb{R},$$

with the cardinal sine function

$$\operatorname{sinc}(x) := \begin{cases} \frac{\sin x}{x} & : x \in \mathbb{R} \setminus \{0\}, \\ 1 & : x = 0. \end{cases}$$

Problem: infinitely many samples needed
⇝ impossible in practice

Truncation:

$$(S_T f)(x) := \sum_{\ell=-T}^{T} f\big(\tfrac{\ell}{L}\big)\, \operatorname{sinc}\big(L\pi\big(x - \tfrac{\ell}{L}\big)\big)$$

Numerical Shortcomings:

- slow convergence: [Jagerman 66], [K. 24]

$$\max_{|x| \leq 1} \big|f(x) - (S_T f)(x)\big| \leq \frac{\sqrt{2L}}{\pi}\,(T-L)^{-1/2}\,\|f\|_{L_2(\mathbb{R})}$$

- non-robustness w.r.t. erroneous samples

$$\tilde{f}_\ell := \begin{cases} f\big(\frac{\ell}{L}\big) + \varepsilon_\ell & : |\ell| \leq T \\ f\big(\frac{\ell}{L}\big) & : |\ell| > T \end{cases}$$

[Feichtinger 92], [Daubechies, DeVore 03], [K., Potts, Tasche 24], [K. 24]

Numerical realization

Substitute

$$\sum_{\ell\in\mathbb{Z}} f\left(\tfrac{\ell}{L}\right) \operatorname{sinc}\left(L\pi\left(x-\tfrac{\ell}{L}\right)\right) \quad \text{by} \quad \sum_{\ell\in\mathbb{Z}} f\left(\tfrac{\ell}{L}\right) \psi\left(x-\tfrac{\ell}{L}\right)$$

with suitable $\psi\colon \mathbb{R} \to [-1,\, 1]$

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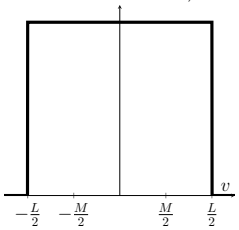
Regularization in frequency domain

[Natterer 86], [Daubechies 92], [Rappaport 96], [Partington 97], [Strohmer, Tanner 05], . . .



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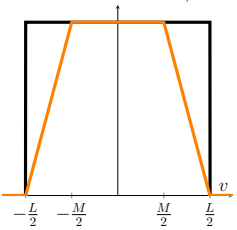

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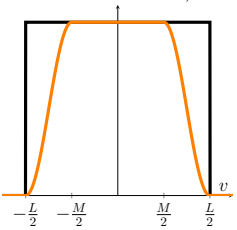

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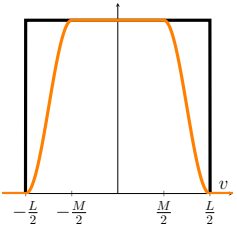
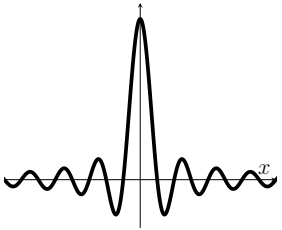


Regularization in spatial domain

[Qian 03-06], [Schmeisser, Stenger 07], [Micchelli, Xu, Zhang 09], [Lin, Zhang 17], [Chen, Zhang 19], . . .



Numerical realization

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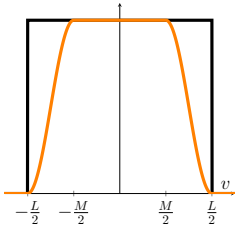

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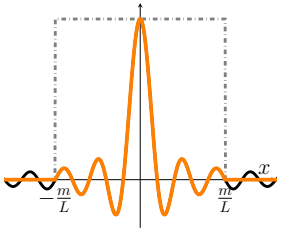

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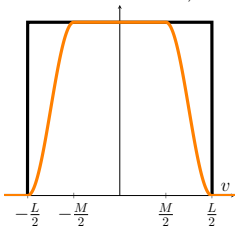

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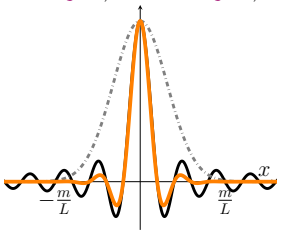

Regularized Shannon sampling formulas

[K., Potts, Tasche 24]

Regularization in frequency domain

Regularization in spatial domain

Regularized Shannon sampling formulas

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Regularization in frequency domain

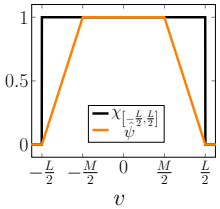
bandlimited
(compactly supported in frequency domain)

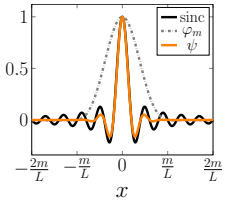
choose $\hat{\psi}$

Regularization in spatial domain

timelimited
(compactly supported in spatial domain)

multiply sinc with convenient window function φ_m





Regularized Shannon sampling formulas

[K., Potts, Tasche 24]

Regularization in frequency domain

bandlimited
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Regularization in spatial domain

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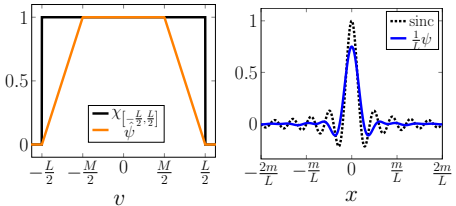
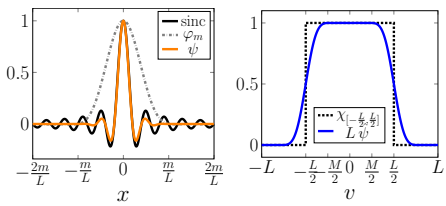




Regularized Shannon sampling formulas

[K., Potts, Tasche 24]

Regularization in frequency domain

bandlimited
(compactly supported in frequency domain)

choose $\hat{\psi}$ and compute inverse Fourier transform

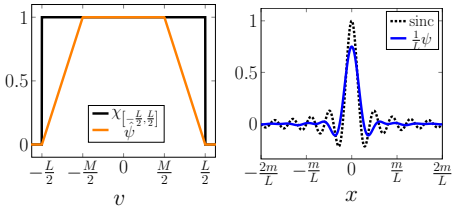
no localization
$\rightsquigarrow$ still truncation parameter T needed

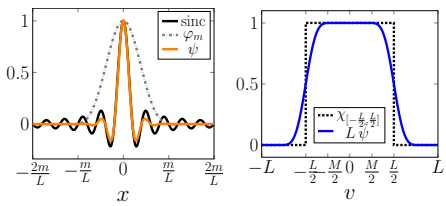
Regularization in spatial domain

timelimited
(compactly supported in spatial domain)

multiply sinc with convenient window function φ_m

localization
$\rightsquigarrow$ only $2m+1$ samples needed





Regularized Shannon sampling formulas

[K., Potts, Tasche 24]

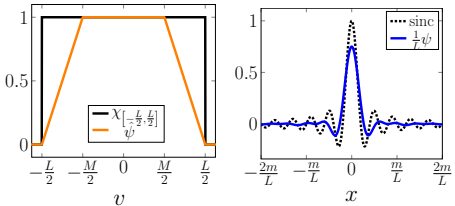
Regularization in frequency domain

bandlimited
(compactly supported in frequency domain)

choose $\hat{\psi}$ and compute inverse Fourier transform

no localization
$\rightsquigarrow$ still truncation parameter T needed

$\rightsquigarrow$ only algebraic decay rates
for all known window functions



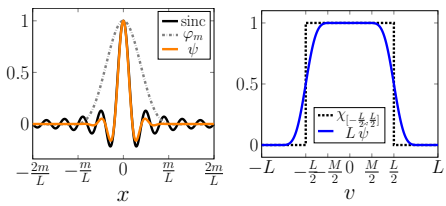
Regularization in spatial domain

timelimited
(compactly supported in spatial domain)

multiply sinc with convenient window function φ_m

localization
$\rightsquigarrow$ only $2m+1$ samples needed

$\rightsquigarrow$ exponential decay rates
for new window functions



General error estimates

Theorem (Uniform approximation error estimate) [K., Potts, Tasche 22]

Let $f \in \mathcal{B}_{M/2}(\mathbb{R})$ with $M \in \mathbb{N}$, $L = M\,(1+\lambda) \in \mathbb{N}$ with $\lambda \geq 0$, and $m \in \mathbb{N} \setminus \{1\}$ be given. Further let $\varphi \in \Phi$ with $\varphi_m(x) \coloneqq \varphi(x)\,\chi_{\left[-\frac{m}{L},\,\frac{m}{L}\right]}(x)$. Then the regularized Shannon sampling formula satisfies

$$\|f - R_{\varphi,m} f\|_{C_0(\mathbb{R})} \leq \big(E_1(\varphi, M, L) + E_2(\varphi, M, L)\big)\,\|f\|_{L_2(\mathbb{R})},$$

with the error constants

$$E_1(\varphi, M, L) \coloneqq \sqrt{M} \max_{v \in \left[-\frac{M}{2},\,\frac{M}{2}\right]} \left| 1 - \int_{v-\frac{L}{2}}^{v+\frac{L}{2}} \hat{\varphi}(u)\,\mathrm{d}u \right|, \quad E_2(\varphi, M, L) \coloneqq \frac{\sqrt{2L}}{\pi} \sqrt{\frac{1}{m^2}\varphi^2\big(\tfrac{m}{L}\big) + \frac{1}{L}\int_{\frac{m}{L}}^{\infty} \frac{\varphi^2(t)}{t^2}\,\mathrm{d}t}.$$

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Theorem (Uniform perturbation error estimate) [K., Potts, Tasche 22]

Let additionally the noisy samples $\tilde{f}_\ell := f\left(\frac{\ell}{L}\right) + \varepsilon_\ell$ with $|\varepsilon_\ell| \leq \varepsilon$, $\ell \in \mathbb{Z}$, and $\varepsilon > 0$ be given. Then we have

$$\|R_{\varphi,m}\tilde{f} - R_{\varphi,m} f\|_{C_0(\mathbb{R})} \leq \varepsilon\,\big(2 + L\,\hat{\varphi}(0)\big).$$

Results for the Gaussian window function

Theorem (Uniform aproximation error estimate) [K., Potts, Tasche 22]

Let $f \in \mathcal{B}_{M/2}(\mathbb{R})$ with $M \in \mathbb{N}$, $L = M\,(1+\lambda) \in \mathbb{N}$ with $\lambda > 0$, and $m \in \mathbb{N} \setminus \{1\}$ be given. Then the regularized Shannon sampling formula using the Gaussian window function satisfies the error estimate

$$\|f - R_{\mathrm{Gauss},m} f\|_{C_0(\mathbb{R})} \leq \frac{\sqrt{2Lm} + \sqrt{2L\lambda(1+m)}}{\pi m \sqrt{\lambda}}\, \mathrm{e}^{-m\pi\lambda/(2+2\lambda)}\, \|f\|_{L_2(\mathbb{R})}.$$

Theorem (Uniform perturbation error estimate) [Qian, Craemer 06], [K., Potts, Tasche 22]

Let additionally the noisy samples $\tilde{f}_\ell = f\left(\frac{\ell}{L}\right) + \varepsilon_\ell$ with $|\varepsilon_\ell| \leq \varepsilon$, $\ell \in \mathbb{Z}$, and $0 < \varepsilon \ll 1$ be given. Then the regularized Shannon sampling formula using the Gaussian window function satisfies

$$\|R_{\mathrm{Gauss},m} \tilde{f} - R_{\mathrm{Gauss},m} f\|_{C_0(\mathbb{R})} \leq \varepsilon \left(2 + \sqrt{\frac{2+2\lambda}{\lambda}}\, \sqrt{m}\right).$$

$\rightsquigarrow$ improves decay rate in [Qian 03] and [Lin, Zhang 17] from $(m-1)$ to m

Results for the sinh-type window function

Theorem (Uniform aproximation error estimate) [K., Potts, Tasche 22]

Let $f \in \mathcal{B}_{M/2}(\mathbb{R})$ with $M \in \mathbb{N}$, $L = M\,(1+\lambda) \in \mathbb{N}$ with $\lambda > 0$, and $m \in \mathbb{N} \setminus \{1\}$ be given. Then the regularized Shannon sampling formula using the $\sinh$-type window function satisfies the error estimate

$$\|f - R_{\sinh,m} f\|_{C_0(\mathbb{R})} \leq \sqrt{M}\, \mathrm{e}^{-m\pi\lambda/(1+\lambda)}\, \|f\|_{L_2(\mathbb{R})}.$$

Theorem (Uniform perturbation error estimate) [K., Potts, Tasche 22]

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$\rightsquigarrow$ similar results also for the continuous Kaiser–Bessel window function in [K., Potts, Tasche 24]

Numerical example – Comparison of regularized Shannon sampling formulas

[K., Potts, Tasche 24]

1. choose bandlimited function $f(x) = \sqrt{\frac{4M}{5}} \left[\operatorname{sinc}(M\pi x) + \frac{1}{2}\operatorname{sinc}(M\pi(x-1))\right]$ with $M = 256$
2. compute the approximation $\sum_{\ell\in\mathbb{Z}} f\left(\frac{\ell}{L}\right) \psi\left(x - \frac{\ell}{L}\right)$ using
   - $L = 3M$,
   - regularizations ψ in frequency and spatial domain,
   - fine equispaced grid $x_s \in [-1, 1]$, $s = 1, \dots, 10^5$,
   - same number of samples $\rightsquigarrow$ set $T = L + m$
3. compute the error

$$\max_{s=1,\dots,10^5} \left| f(x_s) - \sum_{\ell\in\mathbb{Z}} f\left(\frac{\ell}{L}\right) \psi\left(x_s - \frac{\ell}{L}\right) \right|$$

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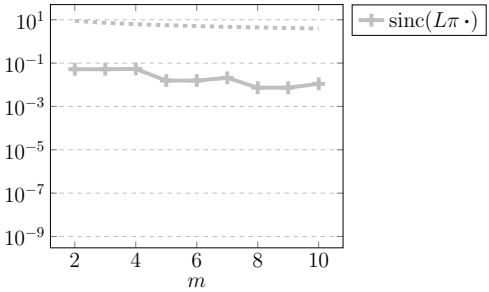

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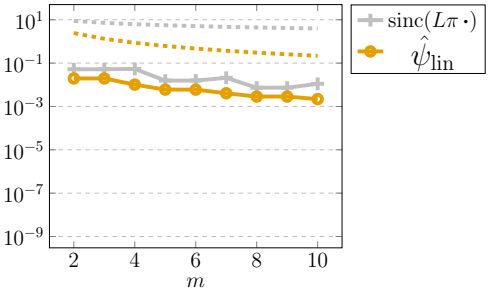

Numerical example – Comparison of regularized Shannon sampling formulas

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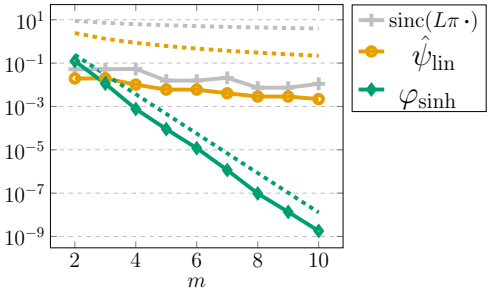

$\rightsquigarrow$ superiority of the new windows evident

Continuous inversion problem

Inversion problem for bandlimited functions

Fourier transform:

$$\hat{f}(\boldsymbol{v}) \coloneqq \int_{\mathbb{R}^d} f(\boldsymbol{x})\, \mathrm{e}^{-2\pi \mathrm{i} \boldsymbol{v}\boldsymbol{x}}\, \mathrm{d}\boldsymbol{x}, \quad \boldsymbol{v} \in \mathbb{R}^d$$

Inversion problem for bandlimited functions

Fourier transform:

$$\hat{f}(\boldsymbol{v}) := \int_{\mathbb{R}^d} f(\boldsymbol{x})\, \mathrm{e}^{-2\pi \mathrm{i} \boldsymbol{v}\boldsymbol{x}}\, \mathrm{d}\boldsymbol{x}, \quad \boldsymbol{v} \in \mathbb{R}^d$$

$\Longrightarrow$ **Inverse Fourier transform:** [Fourier inversion theorem]

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$\rightsquigarrow$ bandlimited functions with maximum bandwidth $\boldsymbol{M}$ $\Longleftrightarrow$ $\operatorname{supp}(\hat{f}\,) = \left[-\frac{M}{2}, \frac{M}{2}\right)^d$

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Application: MRI (Magnetic Resonance Imaging) [Rosenfeld 98], [Greengard, Lee, Inati 06], [Eggers, K., Potts 22]

Given: measurements $f(\boldsymbol{x}_j)$, $j = 1, \ldots, N$ **Reconstruct:** evaluations $\hat{f}(\boldsymbol{k}) \in \mathbb{C}$, $\boldsymbol{k} \in \mathcal{I}_{\boldsymbol{M}}$

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Now: study usability of previous direct inversion methods

Reconsideration as trigonometric polynomials

For $f \in L_1(\mathbb{R}^d)$ consider the 1-periodization

$$\tilde{f}(\boldsymbol{x}) \coloneqq \sum_{\boldsymbol{r} \in \mathbb{Z}^d} f(\boldsymbol{x} + \boldsymbol{r}) \in L_1(\mathbb{T}^d)$$

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$\Rightarrow$ density compensation method: [K., Potts 23b]

$$\hat{f}(\boldsymbol{k}) = c_{\boldsymbol{k}}(\tilde{f}) = \sum_{j=1}^{N} w_j\, \tilde{f}(\boldsymbol{x}_j)\, \mathrm{e}^{-2\pi \mathrm{i} \boldsymbol{k}\boldsymbol{x}_j}, \quad \boldsymbol{k} \in \mathcal{I}_{\boldsymbol{M}} \qquad \hat{\boldsymbol{f}} = \boldsymbol{A}^* \boldsymbol{W} \tilde{\boldsymbol{f}}$$

Practical situation (e. g. MRI)

In practice: only hypothetical case !

periodization $\tilde{f}$ cannot be sampled $\iff$ f cannot be sampled on whole $\mathbb{R}^d$

Sampling: limited coverage of space

$\rightsquigarrow$ f only known on a bounded domain, w.l.o.g. for $\boldsymbol{x} \in \left[-\frac{1}{2}, \frac{1}{2}\right)^d$

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$\rightsquigarrow$ have to deal with the approximation

$$\hat{f}(\boldsymbol{k}) \approx \sum_{j=1}^{N} w_j \, f(\boldsymbol{x}_j) \, \mathrm{e}^{-2\pi \mathrm{i} \boldsymbol{k} \boldsymbol{x}_j}, \quad \boldsymbol{k} \in \mathcal{I}_{\boldsymbol{M}}$$

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Numerical example – Tensorized triangular pulse function

[K., Potts 23a/b]

1. specify compactly supported $\hat{f}(\boldsymbol{v}) = g(v_1) \cdot g(v_2)$, with triangular pulse $g(v) \coloneqq (1 - \left|\frac{v}{b}\right|) \cdot \chi_{[-b,b]}(v)$
2. compute inverse Fourier transform

$$f(\boldsymbol{x}) = \int_{\mathbb{R}^2} \hat{f}(\boldsymbol{v})\, \mathrm{e}^{2\pi \mathrm{i} \boldsymbol{v}\boldsymbol{x}}\, \mathrm{d}\boldsymbol{v} = b^2\, \mathrm{sinc}^2(b\pi\boldsymbol{x}), \quad \boldsymbol{x} \in \mathbb{R}^2$$

$\rightsquigarrow$ bandlimited with bandwidth $\boldsymbol{M}$ for all $b \in \mathbb{N}$ with $b \leq \frac{M}{2}$

3. sample $f(\boldsymbol{x}_j)$ for given $\boldsymbol{x}_j \in \left[-\frac{1}{2}, \frac{1}{2}\right)^2$, $j = 1, \ldots, N$
4. reconstruct $\tilde{h}(\boldsymbol{k}) \approx \hat{f}(\boldsymbol{k})$, $\boldsymbol{k} \in \mathcal{I}_{\boldsymbol{M}}$, and compute pointwise errors $\left|\tilde{h}(\boldsymbol{k}) - \hat{f}(\boldsymbol{k})\right|$

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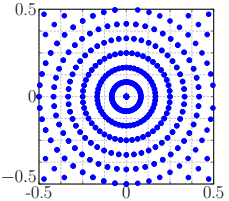
Setup:

- friendly setting ($|\mathcal{I}_{2\boldsymbol{M}}| \leq N$) and unfriendly setting ($|\mathcal{I}_{2\boldsymbol{M}}| > N$)
- $M = 32$ and $b = 12$
- modified polar grid of size $R \in \{2M, M\}$, $T = 2R$



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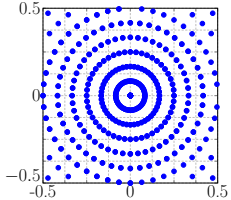

Sampling data:

- real-world sampling $f(\boldsymbol{x}_j)$
- artificial sampling of the periodization

$$\tilde{f}(\boldsymbol{x}_j) = \sum_{\boldsymbol{k} \in \mathcal{I}_{\boldsymbol{M}}} \hat{f}(\boldsymbol{k})\, \mathrm{e}^{2\pi \mathrm{i} \boldsymbol{k}\boldsymbol{x}_j}$$

Results – Friendly setting ($|\mathcal{I}_{\boldsymbol{2M}}| < N$) covered by theory

real-world sampling
$f(\boldsymbol{x}_j)$

artificial sampling
$$\tilde{f}(\boldsymbol{x}_j) = \sum_{\boldsymbol{k} \in \mathcal{I}_{\boldsymbol{M}}} \hat{f}(\boldsymbol{k})\, \mathrm{e}^{2\pi \mathrm{i} \boldsymbol{k} \boldsymbol{x}_j}$$
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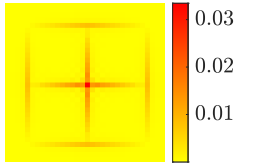
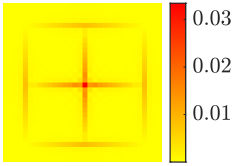


(b) Density compensation



(c) Optimization

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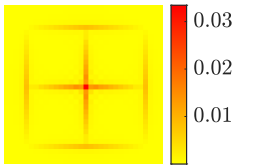
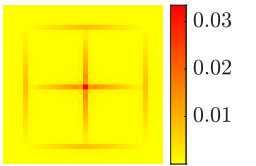


(a) Equispaced points $\boldsymbol{x}_j$ & FFT



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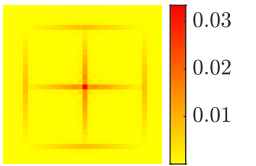

(c) Optimization

Results – Unfriendly setting ($|\mathcal{I}_{\boldsymbol{2M}}| > N$) testing approaches

real-world sampling
$f(\boldsymbol{x}_j)$

artificial sampling
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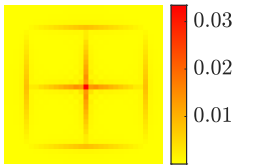

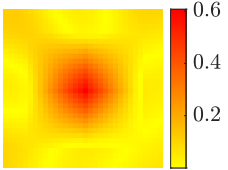

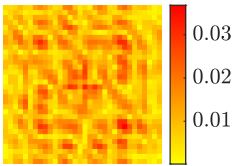

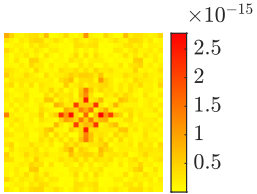

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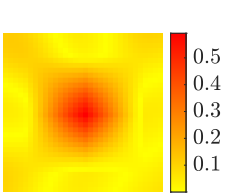

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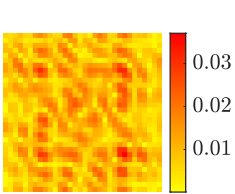

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Summary

- new direct inversion methods for $d \geq 1$, introduced for discrete problem (trigonometric polynomials)
- sampling density compensation: exact reconstruction in case $|\mathcal{I}_{\boldsymbol{2M}}| \leq N$
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- seen superiority of new window functions with compact support (NFFT windows)
- general setting: unified approach to error estimates and numerical robustness
- special windows: uniform approximation error $\sim \mathcal{O}(\mathrm{e}^{-m})$
  uniform perturbation error $\sim \mathcal{O}(\sqrt{m})$

Publications

- **K., Potts: Direct inversion of the nonequispaced fast Fourier transform**. Linear Algebra Appl. 575 (2019).
- **K., Potts: Efficient multivariate inversion of the nonequispaced fast Fourier transform**. PAMM 20, 1 (2021).
- **Eggers, K., Potts: Non-Cartesian MRI Reconstruction**. In: Doneva, Akcakaya, Prieto (eds.). Magnetic Resonance Image Reconstruction, Academic Press (2022).
- **K., Potts, Tasche: On regularized Shannon sampling formulas with localized sampling**. Sampl. Theory Signal Process. Data Anal. 20, 20 (2022).
- **K., Potts, Tasche: Nonuniform fast Fourier transforms with nonequispaced spatial and frequency data and fast sinc transforms**. Numer. Algor. 92 (2023).
- **K., Potts: Fast and direct inversion methods for the multivariate nonequispaced fast Fourier transform**. Front. Appl. Math. Stat. 9 (2023).
- **K., Potts: Optimal density compensation factors for the reconstruction of the Fourier transform of bandlimited functions**. 14th International Conference on Sampling Theory and Applications (2023).
- **K., Potts, Tasche: On numerical realizations of Shannon's sampling theorem**. Sampl. Theory Signal Process. Data Anal. 22, 13 (2024).
- **K., Potts, Tasche: Optimal parameter choice for regularized Shannon sampling formulas**. arXiv:2407.16401 (07/2024).
- **K., Potts: Nonequispaced fast Fourier transforms for bandlimited functions**. arXiv:2502.07155 (02/2025).

Chemnitz Summer School 2025 (September 22–26)

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Thank you for your attention!