Direct inverse nonequispaced fast Fourier transforms

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joint work with Daniel Potts

Workshop and Summer School on Applied Analysis 2024
Chemnitz, September 16—20, 2024

Motivation – discrete problem

phantom

$\rightsquigarrow$ given measurements $\rightsquigarrow$

reconstruction

$\hat{f}_{\boldsymbol{k}}$

$$f(\boldsymbol{x}_j) = \sum_{\boldsymbol{k} \in \mathcal{I}_M} \hat{f}_{\boldsymbol{k}} \, \mathrm{e}^{2\pi \mathrm{i} \boldsymbol{k} \boldsymbol{x}_j}$$

$\tilde{h}_{\boldsymbol{k}} \approx \hat{f}_{\boldsymbol{k}}$

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$\tilde{h}_{\boldsymbol{k}} \approx \hat{f}_{\boldsymbol{k}}$

$\boldsymbol{x}_j$ equispaced $\Longrightarrow$ FFT (Fast Fourier Transform)

$\boldsymbol{x}_j$ nonequispaced $\Longrightarrow$ inverse NFFT (Nonequispaced Fast Fourier Transform) **?**

Motivation

	ground truth	given measurements	aim
discrete problem	$\hat{f}_{\boldsymbol{k}}$	$f(\boldsymbol{x}_j) = \sum_{\boldsymbol{k} \in \mathcal{I}_{\boldsymbol{M}}} \hat{f}_{\boldsymbol{k}} \, \mathrm{e}^{2\pi\mathrm{i}\boldsymbol{k}\boldsymbol{x}_j}$	$\tilde{h}_{\boldsymbol{k}} \approx \hat{f}_{\boldsymbol{k}}$

$\boldsymbol{v} \in \mathbb{R}^d, \; \boldsymbol{k} \in \mathcal{I}_{\boldsymbol{M}}$

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continuous problem	$\hat{f}(\boldsymbol{v})$	$f(\boldsymbol{x}_j) = \int_{[-\frac{M}{2}, \frac{M}{2})^d} \hat{f}(\boldsymbol{v}) \, \mathrm{e}^{2\pi\mathrm{i}\boldsymbol{v}\boldsymbol{x}_j} \, \mathrm{d}\boldsymbol{v}$	$\tilde{h}(\boldsymbol{k}) \approx \hat{f}(\boldsymbol{k})$

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iterated methods
(multiple applications of the NFFT needed)

vs.

direct methods
(realized with a single NFFT)

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iterated methods (multiple applications of the NFFT needed) **vs.** direct methods (realized with a single NFFT)

special setting: evaluation points $\boldsymbol{x}_j, j = 1, \ldots, N$, fixed $\Longrightarrow$ highly profit from direct method

1. precomputation: only once for fixed $\boldsymbol{x}_j$
2. reconstruction: for each measurement $\rightsquigarrow$ very efficient

Overview

1. Introduction
2. Discrete problem
3. Continuous problem

NFFT (Nonequispaced Fast Fourier Transform)

Fast algorithm to evaluate a trigonometric polynomial $\mathcal{O}(|\mathcal{I}_{\boldsymbol{M}}|\log(|\mathcal{I}_{\boldsymbol{M}}|) + N)$

$$f(\boldsymbol{x}) = \sum_{\boldsymbol{k}\in\mathcal{I}_{\boldsymbol{M}}} \hat{f}_{\boldsymbol{k}}\, \mathrm{e}^{2\pi\mathrm{i}\boldsymbol{k}\boldsymbol{x}}$$

[Dutt, Rokhlin 93], [Beylkin 95], [Potts, Steidl, Tasche 01]

- index set $\mathcal{I}_{\boldsymbol{M}} := \mathbb{Z}^d \cap \left[-\frac{M}{2}, \frac{M}{2}\right)^d$ with cardinality $|\mathcal{I}_{\boldsymbol{M}}| = M^d$, $M \in 2\mathbb{N}$,
- Fourier coefficients $\hat{f}_{\boldsymbol{k}} \in \mathbb{C}$, $\boldsymbol{k} \in \mathcal{I}_{\boldsymbol{M}}$,
- nonequispaced points $\boldsymbol{x}_j \in \mathbb{T}^d \cong \left[-\frac{1}{2}, \frac{1}{2}\right)^d$, $j = 1, \dots, N$, $N \in \mathbb{N}$

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Matrix notation:

$$\boldsymbol{f} = \boldsymbol{A}\hat{\boldsymbol{f}} \quad \text{with} \quad \boldsymbol{A} = \boldsymbol{A}_{|\mathcal{I}_{\boldsymbol{M}}|} := \left(\mathrm{e}^{2\pi\mathrm{i}\boldsymbol{k}\boldsymbol{x}_j}\right)_{j=1,\,\boldsymbol{k}\in\mathcal{I}_{\boldsymbol{M}}}^{N} \in \mathbb{C}^{N\times|\mathcal{I}_{\boldsymbol{M}}|}$$

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Factorizations: $\boldsymbol{A} \approx \boldsymbol{B}\boldsymbol{F}\boldsymbol{D}$ and $\boldsymbol{A}^* \approx \boldsymbol{D}^*\boldsymbol{F}^*\boldsymbol{B}^*$

↗ banded ↑ FFT ↖ diagonal

(in each column of $\boldsymbol{B}$ only $(2m+1)^d$ entries, $m \in \mathbb{N}$ given)

NFFT (Nonequispaced Fast Fourier Transform)

Fast algorithm to evaluate a trigonometric polynomial $\mathcal{O}(|\mathcal{I}_{\boldsymbol{M}}|\log(|\mathcal{I}_{\boldsymbol{M}}|)+N)$

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Factorizations: $\boldsymbol{A} \approx \boldsymbol{B}\boldsymbol{F}\boldsymbol{D}$ and $\boldsymbol{A}^* \approx \boldsymbol{D}^*\boldsymbol{F}^*\boldsymbol{B}^*$

$\boldsymbol{B}$: banded, $\boldsymbol{F}$: FFT, $\boldsymbol{D}$: diagonal

(in each column of $\boldsymbol{B}$ only $(2m+1)^d$ entries, $m \in \mathbb{N}$ given)

Inversion problem (iNFFT): **Given:** $\boldsymbol{f} := (f(\boldsymbol{x}_j))_{j=1}^N$ **Find:** $\hat{\boldsymbol{f}} := (\hat{f}_{\boldsymbol{k}})_{\boldsymbol{k}\in\mathcal{I}_{\boldsymbol{M}}}$ **Challenge:** in general $N \neq |\mathcal{I}_{\boldsymbol{M}}|$

Basic idea

Equispaced nodes: $\boldsymbol{A}^*\boldsymbol{A} = N\boldsymbol{I}_{|\mathcal{I}_M|}$

Nonequispaced nodes: $\boldsymbol{A}^*\boldsymbol{A} \neq N\boldsymbol{I}_{|\mathcal{I}_M|}$

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Nonequispaced nodes: $\boldsymbol{A}^*\boldsymbol{A} \neq N\boldsymbol{I}_{|\mathcal{I}_M|}$

$\Rightarrow$ Find suitable matrix $\boldsymbol{X}$ with

$$\boldsymbol{X}\boldsymbol{A} \approx \boldsymbol{I}_{|\mathcal{I}_M|},$$

since then

$$\hat{\boldsymbol{f}} \approx \boldsymbol{X}\boldsymbol{A}\hat{\boldsymbol{f}} = \boldsymbol{X}\boldsymbol{f}.$$

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Reminder – Equispaced nodes:

$$\boldsymbol{X} = \boldsymbol{A}^* \cdot \frac{1}{N}$$

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Simplest generalization:

$$\boldsymbol{X} = \boldsymbol{A}^*\boldsymbol{W} \approx \boldsymbol{D}^*\boldsymbol{F}^*\boldsymbol{B}^*\boldsymbol{W},$$

i. e., additional weighting $\boldsymbol{W} := \operatorname{diag}(w_j)_{j=1}^N$ due to nonequispaced sampling

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Nonequispaced nodes: $\boldsymbol{A}^*\boldsymbol{A} \neq N\boldsymbol{I}_{|\mathcal{I}_M|}$

Density compensation algorithm

0. Precompute weights $\boldsymbol{W}$ — ?
1. Compute scaled coefficients $\boldsymbol{W}\boldsymbol{f}$ — $\mathcal{O}(N)$
2. Adjoint NFFT — $\mathcal{O}(|\mathcal{I}_M|\log(|\mathcal{I}_M|) + N)$

Intuitive approach:
Voronoi weights based on geometry

Exact reconstruction for trigonometric polynomials

Theorem (K., Potts 23)

Let $|\mathcal{I}_{\mathbf{2M}}| \leq N$ *and* $\boldsymbol{x}_j \in \mathbb{T}^d$, $j = 1, \dots, N$, *be given.*

Then the density compensation factors $w_j \in \mathbb{C}$ *satisfying*

$$\sum_{j=1}^{N} w_j \, \mathrm{e}^{2\pi\mathrm{i}\boldsymbol{k}\boldsymbol{x}_j} = \delta_{\mathbf{0},\boldsymbol{k}}, \quad \boldsymbol{k} \in \mathcal{I}_{\mathbf{2M}}, \qquad \boldsymbol{A}^T_{|\mathcal{I}_{\mathbf{2M}}|} \boldsymbol{w} = \boldsymbol{e}_{\mathbf{0}}$$

are optimal,
i. e., for all trigonometric polynomials $f(\boldsymbol{x}) = \sum_{\boldsymbol{k} \in \mathcal{I}_{\boldsymbol{M}}} \hat{f}_{\boldsymbol{k}} \, \mathrm{e}^{2\pi\mathrm{i}\boldsymbol{k}\boldsymbol{x}}$ *an exact reconstruction of the Fourier coefficients* $\hat{f}_{\boldsymbol{k}} \in \mathbb{C}$ *is given by*

$$\hat{f}_{\boldsymbol{k}} = h^{\mathrm{w}}_{\boldsymbol{k}} := \sum_{j=1}^{N} w_j \, f(\boldsymbol{x}_j) \, \mathrm{e}^{-2\pi\mathrm{i}\boldsymbol{k}\boldsymbol{x}_j}, \quad \boldsymbol{k} \in \mathcal{I}_{\boldsymbol{M}}. \qquad \hat{\boldsymbol{f}} = \boldsymbol{A}^* \boldsymbol{W} \boldsymbol{f}$$

$\delta_{\mathbf{0},\boldsymbol{k}}$ ... Kronecker delta

Computation schemes

Aim: exact solution to

$$\boldsymbol{A}_{|\mathcal{I}_{\boldsymbol{2M}}|}^{T}\,\boldsymbol{w} = \boldsymbol{e_0} := (\delta_{\boldsymbol{0},\boldsymbol{k}})_{\boldsymbol{k}\in\mathcal{I}_{\boldsymbol{2M}}} \qquad (*)$$

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Friendly setting ($|\mathcal{I}_{\boldsymbol{2M}}| \leq N$): unique solution given by normal eqs. of 2nd kind [K., Potts 23]

$$\boldsymbol{A}_{|\mathcal{I}_{\boldsymbol{2M}}|}^{T} \overline{\boldsymbol{A}_{|\mathcal{I}_{\boldsymbol{2M}}|}} \boldsymbol{v} = \boldsymbol{e_0}, \quad \overline{\boldsymbol{A}_{|\mathcal{I}_{\boldsymbol{2M}}|}} \boldsymbol{v} = \boldsymbol{w}$$

$\rightsquigarrow$ efficient computation: CG algorithm combined with NFFT $\mathcal{O}(|\mathcal{I}_{\boldsymbol{2M}}| \log(|\mathcal{I}_{\boldsymbol{2M}}|) + N)$

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Unfriendly setting ($|\mathcal{I}_{\boldsymbol{2M}}| > N$): no theoretical guarantee!

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Unfriendly setting ($|\mathcal{I}_{\boldsymbol{2M}}| > N$): no theoretical guarantee!
least squares solution by normal eqs. of 1st kind

$$\overline{\boldsymbol{A}_{|\mathcal{I}_{\boldsymbol{2M}}|}} \boldsymbol{A}_{|\mathcal{I}_{\boldsymbol{2M}}|}^T \boldsymbol{w} = \overline{\boldsymbol{A}_{|\mathcal{I}_{\boldsymbol{2M}}|}} \boldsymbol{e_0}$$

[K., Potts 23]: not a good approximation...

Recapitulation

So far:

$$\hat{\boldsymbol{f}} \approx \boldsymbol{A}^*\boldsymbol{W}\boldsymbol{f} \approx \underset{\text{diagonal}}{\boldsymbol{D}^*}\,\underset{\text{FFT}}{\boldsymbol{F}^*}\,\underset{\text{banded}}{\boldsymbol{B}^*}\boldsymbol{W}\boldsymbol{f}$$

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Interpretation perspectives:

(i) Set $\boldsymbol{g} := \boldsymbol{W} \boldsymbol{f}$. $\Rightarrow$ $\hat{\boldsymbol{f}} \approx \boldsymbol{D}^* \boldsymbol{F}^* \boldsymbol{B}^* \boldsymbol{g}$ $\rightsquigarrow$ ordinary NFFT, modified coefficient vector

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(ii) Set $\tilde{\boldsymbol{B}} := \boldsymbol{W}\boldsymbol{B}$. $\Rightarrow$ $\hat{\boldsymbol{f}} \approx \boldsymbol{D}^* \boldsymbol{F}^* \tilde{\boldsymbol{B}}^* \boldsymbol{f}$ $\rightsquigarrow$ modified NFFT, ordinary coefficient vector

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$\Rightarrow$ density compensation $\widehat{=}$ optimization of the banded matrix $\boldsymbol{B}$ (only N degrees of freedom)

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Now: optimize each nonzero entry of the banded matrix $\boldsymbol{B}$ ($N(2m+1)$ degrees of freedom)

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Now: optimize each nonzero entry of the banded matrix $\boldsymbol{B}$ ($N(2m+1)$ degrees of freedom)

Reminder: seek to find a matrix $\boldsymbol{X}$ with

$$\boldsymbol{X}\boldsymbol{A} \approx \boldsymbol{I}_{|\mathcal{I}_M|}$$

such that

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Aim: $\boldsymbol{X} = \boldsymbol{D}^* \boldsymbol{F}^* \tilde{\boldsymbol{B}}^*$ [K., Potts 23]

- $\rightsquigarrow$ modification of matrix $\boldsymbol{B}$
- $\rightsquigarrow$ preserve band structure and arithmetic complexity

Precomputational step – Optimization procedure

[K., Potts 23]

Define $\tilde{\boldsymbol{h}} := \boldsymbol{D}^* \boldsymbol{F}^* \tilde{\boldsymbol{B}}^* \boldsymbol{f}$.

$$\Rightarrow \quad \|\tilde{\boldsymbol{h}} - \hat{\boldsymbol{f}}\|_2 = \|\boldsymbol{D}^* \boldsymbol{F}^* \tilde{\boldsymbol{B}}^* \boldsymbol{f} - \hat{\boldsymbol{f}}\|_2 = \|\boldsymbol{D}^* \boldsymbol{F}^* \tilde{\boldsymbol{B}}^* \boldsymbol{A} \hat{\boldsymbol{f}} - \hat{\boldsymbol{f}}\|_2$$

$$\leq \left\| \boldsymbol{D}^* \boldsymbol{F}^* \tilde{\boldsymbol{B}}^* \boldsymbol{A} - \boldsymbol{I}_{|\mathcal{I}_M|} \right\|_{\mathrm{F}} \|\hat{\boldsymbol{f}}\|_2$$

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Optimization problem:

$$\underset{\tilde{\boldsymbol{B}} \text{ banded}}{\text{Minimize}} \quad \left\| \boldsymbol{D}^* \boldsymbol{F}^* \tilde{\boldsymbol{B}}^* \boldsymbol{A} - \boldsymbol{I}_{|\mathcal{I}_M|} \right\|_{\mathrm{F}}^2 = \left\| \boldsymbol{A}^* \tilde{\boldsymbol{B}} \boldsymbol{F} \boldsymbol{D} - \boldsymbol{I}_{|\mathcal{I}_M|} \right\|_{\mathrm{F}}^2$$

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$$\underset{\tilde{\boldsymbol{B}} \text{ banded}}{\text{Minimize}} \quad \left\|\boldsymbol{D}^*\boldsymbol{F}^*\tilde{\boldsymbol{B}}^*\boldsymbol{A} - \boldsymbol{I}_{|\mathcal{I}_M|}\right\|_{\mathrm{F}}^2 = \left\|\boldsymbol{A}^*\tilde{\boldsymbol{B}}\boldsymbol{F}\boldsymbol{D} - \boldsymbol{I}_{|\mathcal{I}_M|}\right\|_{\mathrm{F}}^2$$

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Precomputational step – Optimization procedure

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$\rightsquigarrow \mathcal{O}(|\mathcal{I}_{\boldsymbol{M}}|)$

Discrete example – Shepp-Logan phantom

1. phantom data = Fourier coefficients $\hat{\boldsymbol{f}} := (\hat{f}_{\boldsymbol{k}})_{\boldsymbol{k}\in\mathcal{I}_M}$ of a trigonometric polynomial
2. compute the evaluations $f(\boldsymbol{x}_j) = \sum_{\boldsymbol{k}\in\mathcal{I}_M} \hat{f}_{\boldsymbol{k}}\, \mathrm{e}^{2\pi\mathrm{i}\boldsymbol{k}\boldsymbol{x}_j}$ by means of NFFT
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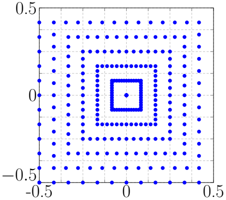
Friendly setting ($|\mathcal{I}_{\boldsymbol{2M}}| \leq N$):

- linogram grid of size $R = 2M, T = 2R$
- phantom size $M \times M$ with $M = 2^c, c = 3, \dots, 10$
- relative errors

$$e_2 := \frac{\|\tilde{\boldsymbol{h}} - \hat{\boldsymbol{f}}\|_2}{\|\hat{\boldsymbol{f}}\|_2}$$



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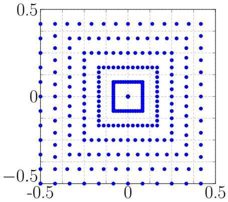
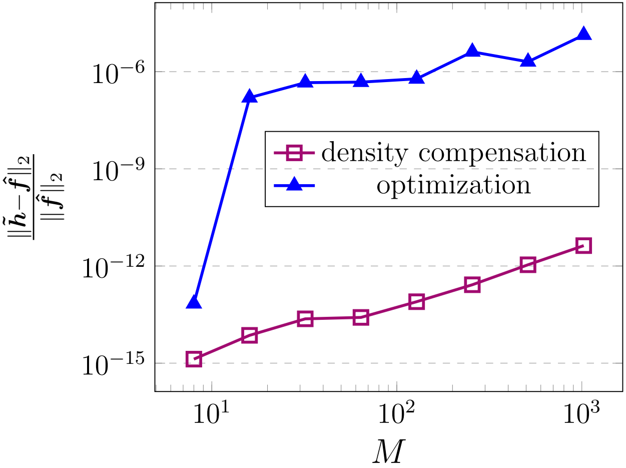
Unfriendly setting ($|\mathcal{I}_{2M}| > N$):

- linogram grid of size $R = M, T = 2R$
- phantom of size $M = 1024$

$\rightsquigarrow$ compare presented computation schemes
  - Voronoi weights
  - new density compensation factors
  - optimization approach



Friendly setting ($|\mathcal{I}_{\mathbf{2}M}| < N$)



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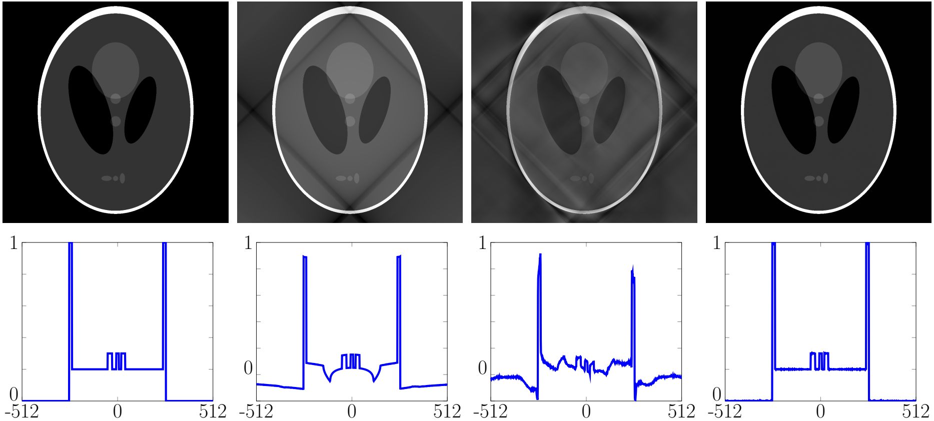

(a) Original phantom

(b) Voronoi weights $e_2 = 5.30e-01$

(c) Density compensation $e_2 = 5.06e-01$

(d) Optimization $e_2 = 2.27e-03$

Analogous problem for bandlimited functions

Application: MRI (Magnetic Resonance Imaging) [Rosenfeld 98], [Greengard, Lee, Inati 06], [Eggers, K., Potts 22]

no longer discrete (trigonometric polynomials) but continuous (bandlimited functions)

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(Continuous) Fourier transform:

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Reconstruct: evaluations $\hat{f}(\boldsymbol{k}) \in \mathbb{C}$, $\boldsymbol{k} \in \mathcal{I}_{\boldsymbol{M}}$

Given: measurements $f(\boldsymbol{x}_j)$, $j = 1, \dots, N$

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Now: extend previous methods

Reconsideration as trigonometric polynomials

Consider 1-periodization

$$\tilde{f}(\boldsymbol{x}) := \sum_{\boldsymbol{r}\in\mathbb{Z}^d} f(\boldsymbol{x}+\boldsymbol{r}) \in L_2(\mathbb{T}^d)$$

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$\Rightarrow$ uniquely representable by absolute convergent Fourier series [Plonka, Potts, Steidl, Tasche 18]

$$\tilde{f}(\boldsymbol{x}) := \sum_{\boldsymbol{k}\in\mathbb{Z}^d} c_{\boldsymbol{k}}(\tilde{f})\, \mathrm{e}^{2\pi\mathrm{i}\boldsymbol{k}\boldsymbol{x}},$$

with Fourier coefficients

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$\Rightarrow$ density compensation method:

$$\hat{f}(\boldsymbol{k}) = c_{\boldsymbol{k}}(\tilde{f}) = \sum_{j=1}^{N} w_j\, \tilde{f}(\boldsymbol{x}_j)\, \mathrm{e}^{-2\pi\mathrm{i}\boldsymbol{k}\boldsymbol{x}_j}, \quad \boldsymbol{k}\in\mathcal{I}_{\boldsymbol{M}} \qquad \hat{\boldsymbol{f}} = \boldsymbol{A}^*\boldsymbol{W}\tilde{\boldsymbol{f}}$$

Practical situation (e. g. MRI)

In practice: only hypothetical case !

periodization $\tilde{f}$ cannot be sampled $\iff$ f cannot be sampled on whole $\mathbb{R}^d$

Sampling: limited coverage of space

$\leadsto$ f only known on a bounded domain, w.l.o.g. for $\boldsymbol{x} \in \left[-\frac{1}{2}, \frac{1}{2}\right)^d$

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Continuous example – tensorized triangular pulse function

1. specify compactly supported $\hat{f}(\boldsymbol{v}) = g(v_1) \cdot g(v_2)$, with triangular pulse $g(v) := (1 - \left|\frac{v}{b}\right|) \cdot \chi_{[-b,b]}(v)$
2. compute inverse Fourier transform

$$f(\boldsymbol{x}) = \int_{\mathbb{R}^2} \hat{f}(\boldsymbol{v})\, \mathrm{e}^{2\pi \mathrm{i} \boldsymbol{v}\boldsymbol{x}}\, \mathrm{d}\boldsymbol{v} = b^2 \operatorname{sinc}^2(b\pi\boldsymbol{x}), \quad \boldsymbol{x} \in \mathbb{R}^2$$

$\rightsquigarrow$ bandlimited with bandwidth $\boldsymbol{M}$ for all $b \in \mathbb{N}$ with $b \leq \frac{M}{2}$

3. sample $f(\boldsymbol{x}_j)$ for given $\boldsymbol{x}_j \in \left[-\frac{1}{2}, \frac{1}{2}\right)^2, j = 1, \dots, N$
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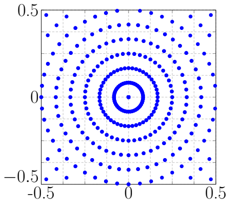
Setup:

- consider $|\mathcal{I}_{\boldsymbol{2M}}| \leq N$
- $M = 32$ and $b = 12$
- modified polar grid of size $R = 2M, T = 2R$
- pointwise errors $\left|\tilde{\boldsymbol{h}} - \hat{\boldsymbol{f}}\right|$



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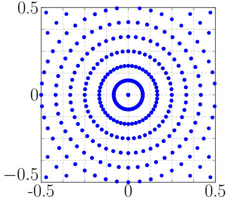
$$f(\boldsymbol{x}) = \int_{\mathbb{R}^2} \hat{f}(\boldsymbol{v})\, \mathrm{e}^{2\pi\mathrm{i}\boldsymbol{v}\boldsymbol{x}}\, \mathrm{d}\boldsymbol{v} = b^2 \operatorname{sinc}^2(b\pi\boldsymbol{x}), \quad \boldsymbol{x} \in \mathbb{R}^2$$

$\rightsquigarrow$ bandlimited with bandwidth $\boldsymbol{M}$ for all $b \in \mathbb{N}$ with $b \leq \frac{M}{2}$

3. sample $f(\boldsymbol{x}_j)$ for given $\boldsymbol{x}_j \in \left[-\frac{1}{2}, \frac{1}{2}\right)^2, j = 1, \ldots, N$
4. reconstruct $\tilde{h}(\boldsymbol{k}) \approx \hat{f}(\boldsymbol{k})$, $\boldsymbol{k} \in \mathcal{I}_{\boldsymbol{M}}$

Setup:

- consider $|\mathcal{I}_{\boldsymbol{2M}}| \leq N$
- $M = 32$ and $b = 12$
- modified polar grid of size $R = 2M, T = 2R$
- pointwise errors $\left|\tilde{\boldsymbol{h}} - \hat{\boldsymbol{f}}\right|$



Sampling data:

- real-world sampling $f(\boldsymbol{x}_j)$
- artificial sampling of the periodization

$$\tilde{f}(\boldsymbol{x}_j) = \sum_{\boldsymbol{k} \in \mathcal{I}_{\boldsymbol{M}}} \hat{f}(\boldsymbol{k})\, \mathrm{e}^{2\pi\mathrm{i}\boldsymbol{k}\boldsymbol{x}_j}$$

Results – pointwise errors $|\tilde{h} - \hat{f}|$

real-world sampling
$f(\boldsymbol{x}_j)$

artificial sampling
$$\tilde{f}(\boldsymbol{x}_j) = \sum_{\boldsymbol{k}\in\mathcal{I}_{\boldsymbol{M}}} \hat{f}(\boldsymbol{k})\, \mathrm{e}^{2\pi\mathrm{i}\boldsymbol{k}\boldsymbol{x}_j}$$
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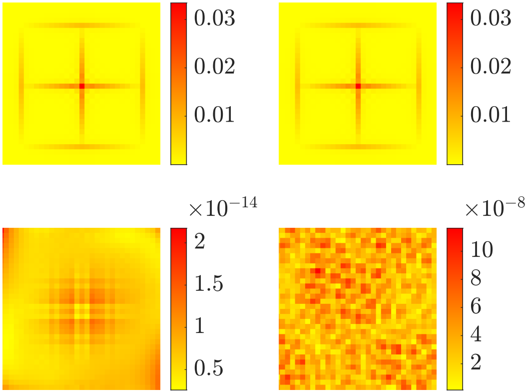

(b) Density compensation (c) Optimization

Results – pointwise errors $|\tilde{h} - \hat{f}|$

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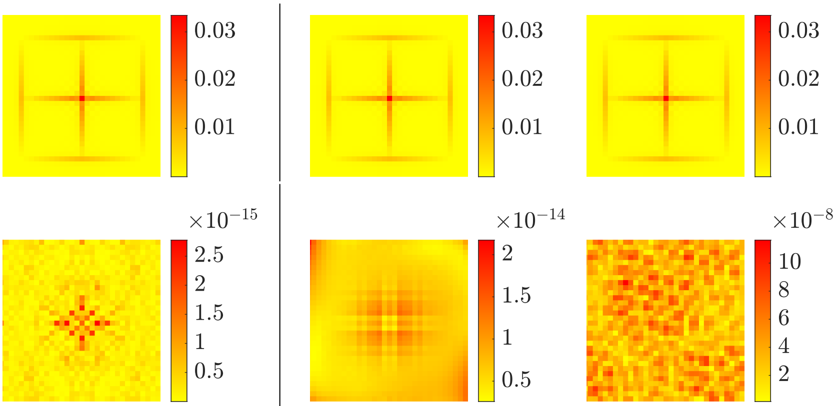

(a) Equispaced points $\boldsymbol{x}_j$ & FFT

(b) Density compensation

(c) Optimization

Summary

- new direct inversion methods for $d \geq 1$, introduced for discrete problem (trigonometric polynomials)
- sampling density compensation: exact reconstruction in case $|\mathcal{I}_{\boldsymbol{2M}}| \leq N$
- optimization: based on factorization $\boldsymbol{B}\,\boldsymbol{F}\boldsymbol{D}$ of NFFT, also works for $|\mathcal{I}_{\boldsymbol{M}}| < N$ ($\nwarrow$ optimized)
- fast algorithms of same complexity $\mathcal{O}(|\mathcal{I}_{\boldsymbol{M}}| \log(|\mathcal{I}_{\boldsymbol{M}}|) + N)$
- extendable to continuous problem (bandlimited functions)
- error solely occurs since f cannot be sampled on whole $\mathbb{R}^d$

- **K., Potts: Fast and direct inversion methods for the multivariate nonequispaced fast Fourier transform**. Front. Appl. Math. Stat. 9 (2023).
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Thank you for your attention!