UNIVERSITY OF TECHNOLOGY
IN THE EUROPEAN CAPITAL OF CULTURE
CHEMNITZ

On numerical realizations of Shannon's sampling theorem

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Problem formulation

Bandlimited functions

space of all bandlimited functions with **bandwidth** $M \in \mathbb{N}$

$$\mathcal{B}_{M/2}(\mathbb{R}) := \left\{ f \in L_2(\mathbb{R}) \colon \operatorname{supp}(\hat f) \subseteq \left[-\tfrac{M}{2}, \tfrac{M}{2}\right] \right\}$$

with **Fourier transform**

$$\hat f(v) := \int_{\mathbb{R}} f(x)\, \mathrm{e}^{-2\pi i v x}\, \mathrm{d}x, \quad v \in \mathbb{R}$$

Embedding:

$$\mathcal{B}_{M/2}(\mathbb{R}) \subseteq L_2(\mathbb{R}) \cap C_0(\mathbb{R}) \cap C^\infty(\mathbb{R})$$

(**Paley–Wiener space**)

Sampling Theorem

Let $f \in \mathcal{B}_{M/2}(\mathbb{R})$ and $\mathbb{N} \ni L = M(1+\lambda)$, $\lambda \geq 0$. Then f is completely determined by its samples $f\left(\frac{\ell}{L}\right)$, $\ell \in \mathbb{Z}$, and

$$f(x) = \sum_{\ell \in \mathbb{Z}} f\left(\tfrac{\ell}{L}\right) \operatorname{sinc}\left(L\pi\left(x - \tfrac{\ell}{L}\right)\right), \quad x \in \mathbb{R},$$

with the cardinal sine function

$$\operatorname{sinc}(x) := \begin{cases} \frac{\sin x}{x} & : x \in \mathbb{R} \setminus \{0\}, \\ 1 & : x = 0, \end{cases}$$

where the series converges absolutely and uniformly on $\mathbb{R}$.

Numerical Shortcomings

Problem: infinitely many samples needed $\rightsquigarrow$ impossible in practice

T-th Shannon sampling sum:

$$(S_T f)(x) := \sum_{\ell=-T}^{T} f\left(\tfrac{\ell}{L}\right) \operatorname{sinc}\left(L\pi\left(x - \tfrac{\ell}{L}\right)\right)$$

$\Rightarrow$ slow convergence:

$$\max_{|x|\leq 1} \left| f(x) - (S_T f)(x) \right| \leq \frac{\sqrt{2L}}{\pi} (T-L)^{-1/2} \|f\|_2$$

$\Rightarrow$ non-robustness:

$$\varepsilon \left(\tfrac{2}{\pi} \ln T + \tfrac{2}{5}\right) \leq \|\tilde f - f\|_\infty < \varepsilon \left(\tfrac{2}{\pi} \ln T + \tfrac{5}{4} + \tfrac{1}{2T}\right)$$

Regularization techniques

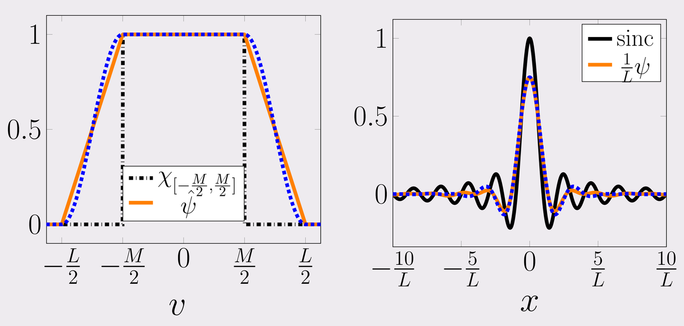
Regularization in frequency domain

Definition & properties

Frequency window functions:

$$\hat\psi(v) := \begin{cases} 1 & : |v| \leq \frac{M}{2}, \\ \xi(|v|) & : \frac{M}{2} < |v| < \frac{L}{2}, \\ 0 & : |v| \geq \frac{L}{2}, \end{cases}$$

with $\xi|_{[\frac{M}{2}, \frac{L}{2}]}$ continuous and decreasing, $\xi\left(\frac{M}{2}\right) = 1$, $\xi\left(\frac{L}{2}\right) = 0$.



Regularized sampling sum:

Assume $\sum_{\ell\in\mathbb{Z}} \left| f\left(\frac{\ell}{L}\right) \right| < \infty$.

$$\rightsquigarrow f(x) = \sum_{\ell \in \mathbb{Z}} f\left(\tfrac{\ell}{L}\right) \frac{1}{L} \psi\left(x - \tfrac{\ell}{L}\right)$$

with $\psi(x) = \int_{\mathbb{R}} \hat\psi(v)\, \mathrm{e}^{2\pi i v x}\, \mathrm{d}v$

$\Rightarrow$ not interpolating:

$$\tfrac{1}{L}\, \psi\left(x - \tfrac{\ell}{L}\right)\Big|_{x = \frac{k}{L}} \neq \delta_{k,\ell}, \quad k, \ell \in \mathbb{Z}$$

$\Rightarrow$ no localization:

$$\operatorname{supp}(\hat\psi) = \left[-\tfrac{L}{2}, \tfrac{L}{2}\right] \rightsquigarrow \operatorname{supp}(\psi) = \mathbb{R}$$

Truncation: $T > L$

$$(P_{\psi,T} f)(x) := \sum_{\ell=-T}^{T} f\left(\tfrac{\ell}{L}\right) \frac{1}{L} \psi\left(x - \tfrac{\ell}{L}\right)$$

Approximation error [4]

Assume that

$$|\psi(x)| \leq c\, |x|^{-r}, \quad x \in \mathbb{R} \setminus \{0\}.$$

Error bound:

$$\max_{|x|\leq 1} \left| f(x) - (P_{\psi,T} f)(x) \right| \leq c L^{r-1} \sqrt{\tfrac{2L}{2r-1}}\, (T-L)^{(-2r+1)/2} \|f\|_2$$

Sufficient condition:

$\hat\psi \in C^r(\mathbb{R})$ and $\xi \in C^{r+2}\left(\left[-\frac{M}{2}, \frac{L}{2}\right]\right)$

Examples [2, 4]

- linear $\hat\psi_{\mathrm{lin}}$ with $\xi_{\mathrm{lin}}(v) = 1 - \frac{2v - M}{L - M}$
- cubic $\hat\psi_{\mathrm{cub}}$ with
$$\xi_{\mathrm{cub}}(v) = \tfrac{16}{(L-M)^3} \left(v - \tfrac{L}{2}\right)^2 \left(v - \tfrac{3M-L}{4}\right)$$
- convolutional
$$\hat\psi_{\mathrm{conv}}(v) = \left(\chi_{[-\frac{L+M}{4}, \frac{L+M}{4}]} * \rho_n\right)(v)$$

with

$$(f * g)(x) := \int_{\mathbb{R}} f(x-t)\, g(t)\, \mathrm{d}t$$

and

$$\rho_n(v) = \tfrac{2n}{L-M} B_n\left(\tfrac{2n}{L-M} v\right), \quad n \in \mathbb{N}$$

centered cardinal B–spline

Regularization in spatial domain

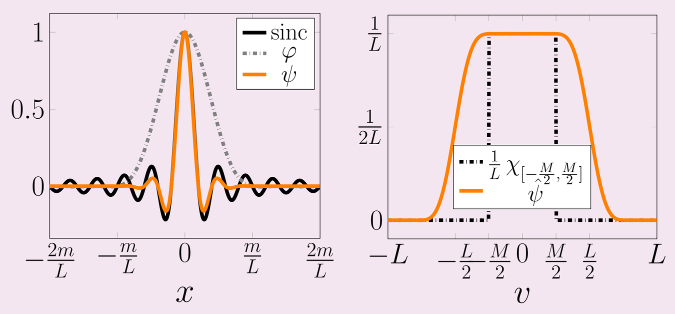
Definition & properties

Spatial window functions:

(i) $\varphi\colon \mathbb{R} \to [0, 1]$

(ii) $\varphi \in L_1(\mathbb{R}) \cap C_0(\mathbb{R})$ is even

(iii) $\varphi|_{[0,\infty)}$ is decreasing, $\varphi(0) = 1$



Truncation: $m \in \mathbb{N} \setminus \{1\}$

$\rightsquigarrow \varphi_m(t) := \varphi(t)\, \chi_{[-\frac{m}{L}, \frac{m}{L}]}(t)$

Regularized sampling sum:

$$(R_{\varphi,m} f)(x) := \sum_{\ell \in \mathbb{Z}} f\left(\tfrac{\ell}{L}\right) \psi_m\left(x - \tfrac{\ell}{L}\right)$$

with $\psi_m(t) := \operatorname{sinc}(L\pi t)\, \varphi_m(t)$

$\Rightarrow$ interpolating approximation:

$$f\left(\tfrac{k}{L}\right) = (R_{\varphi,m} f)\left(\tfrac{k}{L}\right), \quad k \in \mathbb{Z}$$

$\Rightarrow$ localized sampling: only $2m+1$ samples for fixed x

Robustness [1, 2, 4]

Let $\tilde f_\ell := f\left(\frac{\ell}{L}\right) + \varepsilon_\ell$ with $|\varepsilon_\ell| \leq \varepsilon$, $\ell \in \mathbb{Z}$, and $\varepsilon > 0$.

$$\|R_{\varphi,m} \tilde f - R_{\varphi,m} f\|_\infty \leq \varepsilon \left(2 + L\, \hat\varphi(0)\right)$$

Approximation error [4]

Error bound:

$$\|f - R_{\varphi,m} f\|_\infty \leq (E_1 + E_2)\, \|f\|_2$$

with regularization error constant

$$E_1 = \sqrt{M} \max_{|v| \leq \frac{M}{2}} \left| 1 - \int_{v-L/2}^{v+L/2} \hat\varphi(u)\, \mathrm{d}u \right|$$

and truncation error constant

$$E_2 = \frac{\sqrt{2L}}{\pi} \sqrt{\frac{\varphi^2\left(\frac{m}{L}\right)}{m^2} + \int_{m/L}^{\infty} \frac{\varphi^2(t)}{L t^2}\, \mathrm{d}t}$$

Examples [2, 3]

- Gaussian $\varphi_{\mathrm{Gauss}}(t) := \mathrm{e}^{-t^2/(2\alpha^2)}$

with $\alpha = \sqrt{\frac{m}{\pi L (L-M)}}$ optimal

- sinh-type

$$\varphi_{\sinh}(t) := \frac{\sinh\left(\beta \sqrt{1 - \left(\frac{Lt}{m}\right)^2}\right)}{\sinh \beta}\, \chi_{[-\frac{m}{L}, \frac{m}{L}]}(t)$$

with $\beta = \frac{\pi m \lambda}{1+\lambda}$ optimal

- continuous Kaiser–Bessel

$$\varphi_{\mathrm{cKB}}(t) := \frac{I_0\left(\beta \sqrt{1 - \left(\frac{Lt}{m}\right)^2}\right) - 1}{I_0(\beta) - 1}\, \chi_{[-\frac{m}{L}, \frac{m}{L}]}(t)$$

with $\beta = \frac{\pi m \lambda}{1+\lambda}$ optimal

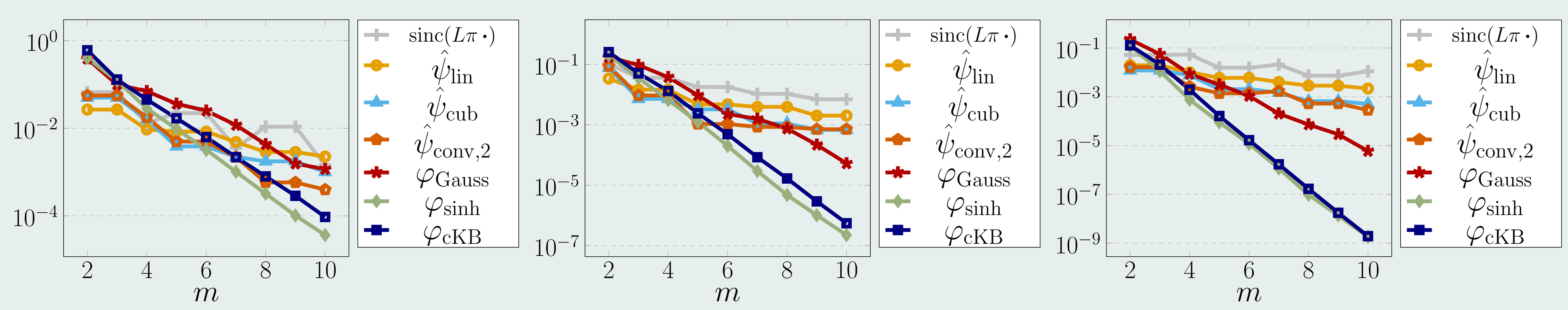
Comparison of the methods

Comparison of theoretical results

window function	error decay rate	see
$\operatorname{sinc}(L\pi\,\cdot)$	$(T-L)^{-1/2}$	
$\hat\psi_{\mathrm{lin}}$	$(T-L)^{-3/2}$	[2]
$\hat\psi_{\mathrm{cub}}$	$(T-L)^{-5/2}$	[2]
$\hat\psi_{\mathrm{conv},2}$	$(T-L)^{-5/2}$	[4]
φ_{const}	$\sqrt{\frac{1}{m} + \frac{1}{m^2}}$	[1,4]
φ_{Gauss}	$\mathrm{e}^{-m\pi\lambda/(2+2\lambda)}$	[1,4]
$\varphi_{\sinh}$	$\mathrm{e}^{-m\pi\lambda/(1+\lambda)}$	[1,4]
φ_{cKB}	$\mathrm{e}^{-m\pi\lambda/(1+\lambda)}$	[2]

Comparison of numerical results

- $\max_{|x|\leq 1} \left| f(x) - (P_{\psi,T} f)(x) \right|$ and $\max_{|x|\leq 1} \left| f(x) - (R_{\varphi,m} f)(x) \right|$
- fine grid $x_s \in [-1, 1]$, $s = 1, \ldots, 10^5$
- $f(x) = \sqrt{\frac{4M}{5}} \left[\operatorname{sinc}(M\pi x) + \frac{1}{2} \operatorname{sinc}(M\pi(x-1))\right]$ with $M = 256$
- $T = L + m \rightsquigarrow$ same number of samples



(a) $\lambda = 0.5$ (b) $\lambda = 1$ (c) $\lambda = 2$

[1] Kircheis, Potts, and Tasche. On regularized Shannon sampling formulas with localized sampling. *Sampl. Theory Signal Process. Data Anal.* 20(20), 2022
[2] Kircheis, Potts, and Tasche. On numerical realizations of Shannon's sampling theorem. *Sampl. Theory Signal Process. Data Anal.* 22(13), 2024.
[3] Kircheis, Potts, and Tasche. Optimal parameter choice for regularized Shannon sampling formulas. *upcoming, 2024.*
[4] Kircheis. PhD thesis. *upcoming, 2024.*

Related work

Sampling Teorem – [Whittaker 1915], [Kotelnikov 33], [Shannon 49]
Shortcomings – [Jagerman 66], [Feichtinger 90], [Daubechies, DeVore 03]
Frequency reg. – [Natterer 86], [Daubechies 92], [Partington 97]
Spatial reg. – [Qian 03/04], [Micchelli, Xu, Zhang 09], [Lin, Zhang 17]

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