Optimal density compensation factors for the reconstruction of the Fourier transform of bandlimited functions

Melanie Kircheis

Chemnitz University of Technology
Faculty of Mathematics

joint work with Daniel Potts

Sampling Theory and Applications
Yale, July 10–14, 2023

Motivation – discrete problem

phantom

$\rightsquigarrow$ given measurements $\rightsquigarrow$

reconstruction

$\hat{f}_{\boldsymbol{k}}$

$$f(\boldsymbol{x}_j) = \sum_{\boldsymbol{k} \in \mathcal{I}_{\boldsymbol{M}}} \hat{f}_{\boldsymbol{k}} \, \mathrm{e}^{2\pi \mathrm{i} \boldsymbol{k} \boldsymbol{x}_j}$$

$\tilde{h}_{\boldsymbol{k}} \approx \hat{f}_{\boldsymbol{k}}$

Motivation – discrete problem

phantom

$\rightsquigarrow$ given measurements $\rightsquigarrow$

reconstruction

$\hat{f}_{\boldsymbol{k}}$

$$f(\boldsymbol{x}_j) = \sum_{\boldsymbol{k} \in \mathcal{I}_M} \hat{f}_{\boldsymbol{k}} \, \mathrm{e}^{2\pi \mathrm{i} \boldsymbol{k} \boldsymbol{x}_j}$$

$\tilde{h}_{\boldsymbol{k}} \approx \hat{f}_{\boldsymbol{k}}$

$\boldsymbol{x}_j$ equispaced $\Longrightarrow$ FFT (Fast Fourier Transform)

$\boldsymbol{x}_j$ nonequispaced $\Longrightarrow$ inverse NFFT (Nonequispaced Fast Fourier Transform) **?**

Motivation

	ground truth	given measurements	aim
discrete problem	$\hat{f}_{\boldsymbol{k}}$	$f(\boldsymbol{x}_j) = \sum_{\boldsymbol{k}\in\mathcal{I}_M} \hat{f}_{\boldsymbol{k}}\, \mathrm{e}^{2\pi\mathrm{i}\boldsymbol{k}\boldsymbol{x}_j}$	$\tilde{h}_{\boldsymbol{k}} \approx \hat{f}_{\boldsymbol{k}}$

$$\boldsymbol{v} \in \mathbb{R}^d,\ \boldsymbol{k} \in \mathcal{I}_M$$

Motivation

	ground truth	given measurements	aim
discrete problem	$\hat{f}_{\boldsymbol{k}}$	$f(\boldsymbol{x}_j) = \sum_{\boldsymbol{k} \in \mathcal{I}_M} \hat{f}_{\boldsymbol{k}} \, \mathrm{e}^{2\pi \mathrm{i} \boldsymbol{k} \boldsymbol{x}_j}$	$\tilde{h}_{\boldsymbol{k}} \approx \hat{f}_{\boldsymbol{k}}$
continuous problem	$\hat{f}(\boldsymbol{v})$	$f(\boldsymbol{x}_j) = \int_{[-\frac{M}{2}, \frac{M}{2})^d} \hat{f}(\boldsymbol{v}) \, \mathrm{e}^{2\pi \mathrm{i} \boldsymbol{v} \boldsymbol{x}_j} \, \mathrm{d}\boldsymbol{v}$	$\tilde{h}(\boldsymbol{k}) \approx \hat{f}(\boldsymbol{k})$

$\boldsymbol{v} \in \mathbb{R}^d, \ \boldsymbol{k} \in \mathcal{I}_M$

Motivation

	ground truth	given measurements	aim
discrete problem	$\hat{f}_{\boldsymbol{k}}$	$f(\boldsymbol{x}_j) = \sum\limits_{\boldsymbol{k}\in\mathcal{I}_M} \hat{f}_{\boldsymbol{k}}\, \mathrm{e}^{2\pi\mathrm{i}\boldsymbol{k}\boldsymbol{x}_j}$	$\tilde{h}_{\boldsymbol{k}} \approx \hat{f}_{\boldsymbol{k}}$
continuous problem	$\hat{f}(\boldsymbol{v})$	$f(\boldsymbol{x}_j) = \int\limits_{[-\frac{M}{2},\frac{M}{2})^d} \hat{f}(\boldsymbol{v})\, \mathrm{e}^{2\pi\mathrm{i}\boldsymbol{v}\boldsymbol{x}_j}\, \mathrm{d}\boldsymbol{v}$	$\tilde{h}(\boldsymbol{k}) \approx \hat{f}(\boldsymbol{k})$

$\boldsymbol{v} \in \mathbb{R}^d,\ \boldsymbol{k} \in \mathcal{I}_M$

iterated methods
(multiple applications of the NFFT needed)

vs.

direct methods
(realized with a single NFFT)

Motivation

	ground truth	given measurements	aim
discrete problem	$\hat{f}_{\boldsymbol{k}}$	$f(\boldsymbol{x}_j) = \sum_{\boldsymbol{k}\in\mathcal{I}_{\boldsymbol{M}}} \hat{f}_{\boldsymbol{k}}\, \mathrm{e}^{2\pi\mathrm{i}\boldsymbol{k}\boldsymbol{x}_j}$	$\tilde{h}_{\boldsymbol{k}} \approx \hat{f}_{\boldsymbol{k}}$
continuous problem	$\hat{f}(\boldsymbol{v})$	$f(\boldsymbol{x}_j) = \int_{[-\frac{M}{2},\frac{M}{2})^d} \hat{f}(\boldsymbol{v})\, \mathrm{e}^{2\pi\mathrm{i}\boldsymbol{v}\boldsymbol{x}_j}\, \mathrm{d}\boldsymbol{v}$	$\tilde{h}(\boldsymbol{k}) \approx \hat{f}(\boldsymbol{k})$

$\boldsymbol{v} \in \mathbb{R}^d,\ \boldsymbol{k} \in \mathcal{I}_{\boldsymbol{M}}$

iterated methods (multiple applications of the NFFT needed) **vs.** direct methods (realized with a single NFFT)

special setting: evaluation points $\boldsymbol{x}_j$, $j = 1, \dots, N$, fixed $\Longrightarrow$ highly profit from direct method

1. precomputation: only once for fixed $\boldsymbol{x}_j$
2. reconstruction: for each measurement $\rightsquigarrow$ very efficient

Overview

NFFT (Nonequispaced Fast Fourier Transform)

Fast algorithm to evaluate a trigonometric polynomial $\mathcal{O}(|\mathcal{I}_{\boldsymbol{M}}|\log(|\mathcal{I}_{\boldsymbol{M}}|)+N)$

$$f(\boldsymbol{x}) = \sum_{\boldsymbol{k}\in\mathcal{I}_{\boldsymbol{M}}} \hat{f}_{\boldsymbol{k}}\, \mathrm{e}^{2\pi\mathrm{i}\boldsymbol{k}\boldsymbol{x}}$$

- index set $\mathcal{I}_{\boldsymbol{M}} := \mathbb{Z}^d \cap \left[-\frac{M}{2},\frac{M}{2}\right)^d$ with cardinality $|\mathcal{I}_{\boldsymbol{M}}| = M^d$, $M\in 2\mathbb{N}$,
- Fourier coefficients $\hat{f}_{\boldsymbol{k}}\in\mathbb{C}$, $\boldsymbol{k}\in\mathcal{I}_{\boldsymbol{M}}$,
- nonequispaced points $\boldsymbol{x}_j\in\mathbb{T}^d \cong \left[-\frac{1}{2},\frac{1}{2}\right)^d$, $j=1,\dots,N$, $N\in\mathbb{N}$

NFFT (Nonequispaced Fast Fourier Transform)

Fast algorithm to evaluate a trigonometric polynomial $\mathcal{O}(|\mathcal{I}_{\boldsymbol{M}}| \log(|\mathcal{I}_{\boldsymbol{M}}|) + N)$

$$f(\boldsymbol{x}) = \sum_{\boldsymbol{k} \in \mathcal{I}_{\boldsymbol{M}}} \hat{f}_{\boldsymbol{k}} \, \mathrm{e}^{2\pi \mathrm{i} \boldsymbol{k} \boldsymbol{x}}$$

- index set $\mathcal{I}_{\boldsymbol{M}} := \mathbb{Z}^d \cap \left[-\frac{M}{2}, \frac{M}{2}\right)^d$ with cardinality $|\mathcal{I}_{\boldsymbol{M}}| = M^d$, $M \in 2\mathbb{N}$,
- Fourier coefficients $\hat{f}_{\boldsymbol{k}} \in \mathbb{C}$, $\boldsymbol{k} \in \mathcal{I}_{\boldsymbol{M}}$,
- nonequispaced points $\boldsymbol{x}_j \in \mathbb{T}^d \cong \left[-\frac{1}{2}, \frac{1}{2}\right)^d$, $j = 1, \dots, N$, $N \in \mathbb{N}$

Inversion problem (iNFFT):

$$\boldsymbol{A}\hat{\boldsymbol{f}} = \boldsymbol{f} \qquad \text{with} \quad \boldsymbol{A} = \boldsymbol{A}_{|\mathcal{I}_{\boldsymbol{M}}|} := \left(\mathrm{e}^{2\pi \mathrm{i} \boldsymbol{k} \boldsymbol{x}_j}\right)_{j=1,\, \boldsymbol{k} \in \mathcal{I}_{\boldsymbol{M}}}^{N} \in \mathbb{C}^{N \times |\mathcal{I}_{\boldsymbol{M}}|}$$

Given: $\boldsymbol{f} := (f(\boldsymbol{x}_j))_{j=1}^N$

Find: $\hat{\boldsymbol{f}} := (\hat{f}_{\boldsymbol{k}})_{\boldsymbol{k} \in \mathcal{I}_{\boldsymbol{M}}}$

Challenge: in general $N \neq |\mathcal{I}_{\boldsymbol{M}}|$

Basic idea

Equispaced nodes: $\boldsymbol{A}^*\boldsymbol{A} = N\boldsymbol{I}_{|\mathcal{I}_{\boldsymbol{M}}|}$

Nonequispaced nodes: $\boldsymbol{A}^*\boldsymbol{A} \neq N\boldsymbol{I}_{|\mathcal{I}_{\boldsymbol{M}}|}$

Basic idea

Equispaced nodes: $\boldsymbol{A}^*\boldsymbol{A} = N\boldsymbol{I}_{|\mathcal{I}_M|}$

Nonequispaced nodes: $\boldsymbol{A}^*\boldsymbol{A} \neq N\boldsymbol{I}_{|\mathcal{I}_M|}$

$\Rightarrow$ Find suitable matrix $\boldsymbol{X}$ with

$$\boldsymbol{X}\boldsymbol{A} \approx \boldsymbol{I}_{|\mathcal{I}_M|},$$

since then

$$\hat{\boldsymbol{f}} \approx \boldsymbol{X}\boldsymbol{A}\hat{\boldsymbol{f}} = \boldsymbol{X}\boldsymbol{f}.$$

Basic idea

Equispaced nodes: $\boldsymbol{A}^*\boldsymbol{A} = N\boldsymbol{I}_{|\mathcal{I}_{\boldsymbol{M}}|}$

Nonequispaced nodes: $\boldsymbol{A}^*\boldsymbol{A} \neq N\boldsymbol{I}_{|\mathcal{I}_{\boldsymbol{M}}|}$

$\Rightarrow$ Find suitable matrix $\boldsymbol{X}$ with

$$\boldsymbol{X}\boldsymbol{A} \approx \boldsymbol{I}_{|\mathcal{I}_{\boldsymbol{M}}|},$$

since then

$$\hat{\boldsymbol{f}} \approx \boldsymbol{X}\boldsymbol{A}\hat{\boldsymbol{f}} = \boldsymbol{X}\boldsymbol{f}.$$

Reminder – Equispaced nodes:

$$\boldsymbol{X} = \boldsymbol{A}^* \cdot N^{-1}$$

Basic idea

Equispaced nodes: $\boldsymbol{A}^*\boldsymbol{A} = N\boldsymbol{I}_{|\mathcal{I}_{\boldsymbol{M}}|}$

$\Rightarrow$ Find suitable matrix $\boldsymbol{X}$ with

$$\boldsymbol{X}\boldsymbol{A} \approx \boldsymbol{I}_{|\mathcal{I}_{\boldsymbol{M}}|},$$

since then

$$\hat{\boldsymbol{f}} \approx \boldsymbol{X}\boldsymbol{A}\hat{\boldsymbol{f}} = \boldsymbol{X}\boldsymbol{f}.$$

Nonequispaced nodes: $\boldsymbol{A}^*\boldsymbol{A} \neq N\boldsymbol{I}_{|\mathcal{I}_{\boldsymbol{M}}|}$

Reminder – Equispaced nodes:

$$\boldsymbol{X} = \boldsymbol{A}^* \cdot N^{-1}$$

Simplest generalization:

$$\boldsymbol{X} = \boldsymbol{A}^*\boldsymbol{W},$$

i. e., additional weighting $\boldsymbol{W} := \operatorname{diag}(w_j)_{j=1}^N$ due to nonequispaced sampling

Basic idea

Equispaced nodes: $\boldsymbol{A}^*\boldsymbol{A} = N\boldsymbol{I}_{|\mathcal{I}_M|}$

$\Rightarrow$ Find suitable matrix $\boldsymbol{X}$ with

$$\boldsymbol{X}\boldsymbol{A} \approx \boldsymbol{I}_{|\mathcal{I}_M|},$$

since then

$$\hat{\boldsymbol{f}} \approx \boldsymbol{X}\boldsymbol{A}\hat{\boldsymbol{f}} = \boldsymbol{X}\boldsymbol{f}.$$

Reminder – Equispaced nodes:

$$\boldsymbol{X} = \boldsymbol{A}^* \cdot N^{-1}$$

Simplest generalization:

$$\boldsymbol{X} = \boldsymbol{A}^*\boldsymbol{W},$$

i. e., additional weighting $\boldsymbol{W} := \mathrm{diag}(w_j)_{j=1}^N$ due to nonequispaced sampling

Nonequispaced nodes: $\boldsymbol{A}^*\boldsymbol{A} \neq N\boldsymbol{I}_{|\mathcal{I}_M|}$

Algorithm

0. Precompute weights $\boldsymbol{W}$ — ?
1. Compute scaled coefficients $\boldsymbol{W}\boldsymbol{f}$ — $\mathcal{O}(N)$
2. Adjoint NFFT — $\mathcal{O}(|\mathcal{I}_M| \log(|\mathcal{I}_M|) + N)$

Exact reconstruction for trigonometric polynomials

Theorem (K., Potts 23)

Let $|\mathcal{I}_{\mathbf{2M}}| \leq N$ *and* $\boldsymbol{x}_j \in \mathbb{T}^d$, $j = 1, \dots, N$, *be given.*

Then the density compensation factors $w_j \in \mathbb{C}$ *satisfying*

$$\sum_{j=1}^{N} w_j \,\mathrm{e}^{2\pi\mathrm{i}\boldsymbol{k}\boldsymbol{x}_j} = \delta_{\mathbf{0},\boldsymbol{k}}, \quad \boldsymbol{k} \in \mathcal{I}_{\mathbf{2M}}, \qquad \boldsymbol{A}^T_{|\mathcal{I}_{\mathbf{2M}}|}\,\boldsymbol{w} = \boldsymbol{e}_{\mathbf{0}}$$

are optimal,
i. e., for all trigonometric polynomials $f(\boldsymbol{x}) = \sum_{\boldsymbol{k}\in\mathcal{I}_{\boldsymbol{M}}} \hat{f}_{\boldsymbol{k}}\,\mathrm{e}^{2\pi\mathrm{i}\boldsymbol{k}\boldsymbol{x}}$, *an exact reconstruction of the Fourier coefficients* $\hat{f}_{\boldsymbol{k}} \in \mathbb{C}$ *is given by*

$$\hat{f}_{\boldsymbol{k}} = h^{\mathrm{w}}_{\boldsymbol{k}} := \sum_{j=1}^{N} w_j\, f(\boldsymbol{x}_j)\,\mathrm{e}^{-2\pi\mathrm{i}\boldsymbol{k}\boldsymbol{x}_j}, \quad \boldsymbol{k} \in \mathcal{I}_{\boldsymbol{M}}. \qquad \hat{\boldsymbol{f}} = \boldsymbol{A}^*\boldsymbol{W}\boldsymbol{f}$$

$\delta_{\mathbf{0},\boldsymbol{k}}$ ... Kronecker delta

Proof

- $\{\mathrm{e}^{2\pi\mathrm{i}\boldsymbol{\ell}\boldsymbol{x}} \colon \boldsymbol{\ell} \in \mathbb{Z}^d\}$ forms an orthonormal basis of $L_2(\mathbb{T}^d)$ [Plonka, Potts, Steidl, Tasche 18]
- $\boldsymbol{\ell} \in \mathcal{I}_M$ sufficient for trigonometric polynomials $f(\boldsymbol{x}) = \sum_{\boldsymbol{k}\in\mathcal{I}_M} \hat{f}_{\boldsymbol{k}}\, \mathrm{e}^{2\pi\mathrm{i}\boldsymbol{k}\boldsymbol{x}}$

Proof

- $\{\mathrm{e}^{2\pi\mathrm{i}\boldsymbol{\ell}\boldsymbol{x}} \colon \boldsymbol{\ell} \in \mathbb{Z}^d\}$ forms an orthonormal basis of $L_2(\mathbb{T}^d)$ [Plonka, Potts, Steidl, Tasche 18]
- $\boldsymbol{\ell} \in \mathcal{I}_M$ sufficient for trigonometric polynomials $f(\boldsymbol{x}) = \sum_{\boldsymbol{k}\in\mathcal{I}_M} \hat{f}_{\boldsymbol{k}}\, \mathrm{e}^{2\pi\mathrm{i}\boldsymbol{k}\boldsymbol{x}}$
- For fixed $\boldsymbol{\ell} \in \mathcal{I}_M$ we have

$$h_{\boldsymbol{k}}^{\mathrm{w}} = \sum_{j=1}^{N} w_j\, \mathrm{e}^{2\pi\mathrm{i}(\boldsymbol{\ell}-\boldsymbol{k})\boldsymbol{x}_j}, \quad \boldsymbol{k} \in \mathcal{I}_M,$$

and

$$\hat{f}_{\boldsymbol{k}} = c_{\boldsymbol{k}}(f) = \int_{\mathbb{T}^d} \mathrm{e}^{2\pi\mathrm{i}(\boldsymbol{\ell}-\boldsymbol{k})\boldsymbol{x}}\, \mathrm{d}\boldsymbol{x} = \delta_{\boldsymbol{\ell},\boldsymbol{k}}, \quad \boldsymbol{k} \in \mathcal{I}_M.$$

$\Rightarrow$ need to assure equality

Proof

- $\{\mathrm{e}^{2\pi\mathrm{i}\boldsymbol{\ell}\boldsymbol{x}} : \boldsymbol{\ell} \in \mathbb{Z}^d\}$ forms an orthonormal basis of $L_2(\mathbb{T}^d)$ [Plonka, Potts, Steidl, Tasche 18]
- $\boldsymbol{\ell} \in \mathcal{I}_M$ sufficient for trigonometric polynomials $f(\boldsymbol{x}) = \sum_{\boldsymbol{k}\in\mathcal{I}_M} \hat{f}_{\boldsymbol{k}}\, \mathrm{e}^{2\pi\mathrm{i}\boldsymbol{k}\boldsymbol{x}}$
- For fixed $\boldsymbol{\ell} \in \mathcal{I}_M$ we have

$$h_{\boldsymbol{k}}^{\mathrm{w}} = \sum_{j=1}^{N} w_j\, \mathrm{e}^{2\pi\mathrm{i}(\boldsymbol{\ell}-\boldsymbol{k})\boldsymbol{x}_j}, \quad \boldsymbol{k} \in \mathcal{I}_M,$$

  and

$$\hat{f}_{\boldsymbol{k}} = c_{\boldsymbol{k}}(f) = \int_{\mathbb{T}^d} \mathrm{e}^{2\pi\mathrm{i}(\boldsymbol{\ell}-\boldsymbol{k})\boldsymbol{x}}\, \mathrm{d}\boldsymbol{x} = \delta_{\boldsymbol{\ell},\boldsymbol{k}}, \quad \boldsymbol{k} \in \mathcal{I}_M.$$

  $\Rightarrow$ need to assure equality
- Since for $\boldsymbol{k}, \boldsymbol{\ell} \in \mathcal{I}_M$ we have $(\boldsymbol{\ell}-\boldsymbol{k}) \in \mathcal{I}_{\mathbf{2}M}$, this is fulfilled if

$$\sum_{j=1}^{N} w_j\, \mathrm{e}^{2\pi\mathrm{i}\boldsymbol{k}\boldsymbol{x}_j} = \delta_{\mathbf{0},\boldsymbol{k}}, \quad \boldsymbol{k} \in \mathcal{I}_{\mathbf{2}M}.$$

$$\boldsymbol{A}^T_{|\mathcal{I}_{\mathbf{2}M}|}\, \boldsymbol{w} = \boldsymbol{e}_{\mathbf{0}}$$

Proof

- $\{\mathrm{e}^{2\pi\mathrm{i}\boldsymbol{\ell}\boldsymbol{x}} \colon \boldsymbol{\ell} \in \mathbb{Z}^d\}$ forms an orthonormal basis of $L_2(\mathbb{T}^d)$ [Plonka, Potts, Steidl, Tasche 18]
- $\boldsymbol{\ell} \in \mathcal{I}_M$ sufficient for trigonometric polynomials $f(\boldsymbol{x}) = \sum_{\boldsymbol{k}\in\mathcal{I}_M} \hat{f}_{\boldsymbol{k}}\, \mathrm{e}^{2\pi\mathrm{i}\boldsymbol{k}\boldsymbol{x}}$
- For fixed $\boldsymbol{\ell} \in \mathcal{I}_M$ we have

$$h_{\boldsymbol{k}}^{\mathrm{w}} = \sum_{j=1}^{N} w_j\, \mathrm{e}^{2\pi\mathrm{i}(\boldsymbol{\ell}-\boldsymbol{k})\boldsymbol{x}_j}, \quad \boldsymbol{k} \in \mathcal{I}_M,$$

and

$$\hat{f}_{\boldsymbol{k}} = c_{\boldsymbol{k}}(f) = \int\limits_{\mathbb{T}^d} \mathrm{e}^{2\pi\mathrm{i}(\boldsymbol{\ell}-\boldsymbol{k})\boldsymbol{x}}\, \mathrm{d}\boldsymbol{x} = \delta_{\boldsymbol{\ell},\boldsymbol{k}}, \quad \boldsymbol{k} \in \mathcal{I}_M.$$

$\Rightarrow$ need to assure equality

- Since for $\boldsymbol{k}, \boldsymbol{\ell} \in \mathcal{I}_M$ we have $(\boldsymbol{\ell} - \boldsymbol{k}) \in \mathcal{I}_{\mathbf{2}M}$, this is fulfilled if

$$\sum_{j=1}^{N} w_j\, \mathrm{e}^{2\pi\mathrm{i}\boldsymbol{k}\boldsymbol{x}_j} = \delta_{\mathbf{0},\boldsymbol{k}}, \quad \boldsymbol{k} \in \mathcal{I}_{\mathbf{2}M}. \qquad \boldsymbol{A}_{|\mathcal{I}_{\mathbf{2}M}|}^T \boldsymbol{w} = \boldsymbol{e}_{\mathbf{0}}$$

$\rightsquigarrow$ necessary for exact solution: underdetermined system with $|\mathcal{I}_{\mathbf{2}M}| \le N$ [Huybrechs 09]

Computation schemes

Aim: exact solution to

$$\boldsymbol{A}^T_{|\mathcal{I}_{\mathbf{2}M}|}\,\boldsymbol{w} = \boldsymbol{e}_{\mathbf{0}} := (\delta_{\mathbf{0},\boldsymbol{k}})_{\boldsymbol{k}\in\mathcal{I}_{\mathbf{2}M}} \qquad (*)$$

Computation schemes

Aim: exact solution to

$$\boldsymbol{A}_{|\mathcal{I}_{\boldsymbol{2M}}|}^T \boldsymbol{w} = \boldsymbol{e_0} := (\delta_{\boldsymbol{0},\boldsymbol{k}})_{\boldsymbol{k}\in\mathcal{I}_{\boldsymbol{2M}}} \qquad (*)$$

Friendly setting ($|\mathcal{I}_{\boldsymbol{2M}}| \leq N$): unique solution given by normal eqs. of 2nd kind [K., Potts 23]

$$\boldsymbol{A}_{|\mathcal{I}_{\boldsymbol{2M}}|}^T \overline{\boldsymbol{A}_{|\mathcal{I}_{\boldsymbol{2M}}|}} \boldsymbol{v} = \boldsymbol{e_0}, \quad \overline{\boldsymbol{A}_{|\mathcal{I}_{\boldsymbol{2M}}|}} \boldsymbol{v} = \boldsymbol{w}$$

$\rightsquigarrow$ efficient computation: CG algorithm combined with NFFT $\mathcal{O}(|\mathcal{I}_{\boldsymbol{2M}}| \log(|\mathcal{I}_{\boldsymbol{2M}}|) + N)$

Computation schemes

Aim: exact solution to

$$\boldsymbol{A}_{|\mathcal{I}_{\mathbf{2}M}|}^{T}\,\boldsymbol{w} = \boldsymbol{e_0} := (\delta_{\mathbf{0},\boldsymbol{k}})_{\boldsymbol{k}\in\mathcal{I}_{\mathbf{2}M}} \qquad (*)$$

Friendly setting ($|\mathcal{I}_{\mathbf{2}M}| \leq N$): unique solution given by normal eqs. of 2nd kind [K., Potts 23]

$$\boldsymbol{A}_{|\mathcal{I}_{\mathbf{2}M}|}^{T}\,\overline{\boldsymbol{A}_{|\mathcal{I}_{\mathbf{2}M}|}}\,\boldsymbol{v} = \boldsymbol{e_0}, \quad \overline{\boldsymbol{A}_{|\mathcal{I}_{\mathbf{2}M}|}}\,\boldsymbol{v} = \boldsymbol{w}$$

$\rightsquigarrow$ efficient computation: CG algorithm combined with NFFT $\qquad \mathcal{O}(|\mathcal{I}_{\mathbf{2}M}|\log(|\mathcal{I}_{\mathbf{2}M}|) + N)$

Unfriendly setting ($|\mathcal{I}_{\mathbf{2}M}| > N$): no theoretical guarantee!

Computation schemes

Aim: exact solution to

$$\boldsymbol{A}_{|\mathcal{I}_{2M}|}^T \boldsymbol{w} = \boldsymbol{e_0} := (\delta_{\boldsymbol{0},\boldsymbol{k}})_{\boldsymbol{k}\in\mathcal{I}_{2M}} \qquad (*)$$

Friendly setting ($|\mathcal{I}_{2M}| \leq N$): unique solution given by normal eqs. of 2nd kind [K., Potts 23]

$$\boldsymbol{A}_{|\mathcal{I}_{2M}|}^T \overline{\boldsymbol{A}_{|\mathcal{I}_{2M}|}} \boldsymbol{v} = \boldsymbol{e_0}, \quad \overline{\boldsymbol{A}_{|\mathcal{I}_{2M}|}} \boldsymbol{v} = \boldsymbol{w}$$

⇝ efficient computation: CG algorithm combined with NFFT $\mathcal{O}(|\mathcal{I}_{2M}|\log(|\mathcal{I}_{2M}|) + N)$

Unfriendly setting ($|\mathcal{I}_{2M}| > N$): no theoretical guarantee!

- least squares solution by normal eqs. of 1st kind

$$\overline{\boldsymbol{A}_{|\mathcal{I}_{2M}|}} \boldsymbol{A}_{|\mathcal{I}_{2M}|}^T \boldsymbol{w} = \overline{\boldsymbol{A}_{|\mathcal{I}_{2M}|}} \boldsymbol{e_0}$$

[K., Potts 23]: not a good approximation...

Computation schemes

Aim: exact solution to

$$\boldsymbol{A}_{|\mathcal{I}_{2M}|}^T \boldsymbol{w} = \boldsymbol{e_0} := (\delta_{\boldsymbol{0},\boldsymbol{k}})_{\boldsymbol{k}\in\mathcal{I}_{2M}} \qquad (*)$$

Friendly setting ($|\mathcal{I}_{2M}| \le N$): unique solution given by normal eqs. of 2nd kind [K., Potts 23]

$$\boldsymbol{A}_{|\mathcal{I}_{2M}|}^T \overline{\boldsymbol{A}_{|\mathcal{I}_{2M}|}}\, \boldsymbol{v} = \boldsymbol{e_0}, \quad \overline{\boldsymbol{A}_{|\mathcal{I}_{2M}|}}\, \boldsymbol{v} = \boldsymbol{w}$$

$\rightsquigarrow$ efficient computation: CG algorithm combined with NFFT $\quad \mathcal{O}(|\mathcal{I}_{2M}|\log(|\mathcal{I}_{2M}|) + N)$

Unfriendly setting ($|\mathcal{I}_{2M}| > N$): no theoretical guarantee!

- least squares solution by normal eqs. of 1st kind

$$\overline{\boldsymbol{A}_{|\mathcal{I}_{2M}|}}\, \boldsymbol{A}_{|\mathcal{I}_{2M}|}^T \boldsymbol{w} = \overline{\boldsymbol{A}_{|\mathcal{I}_{2M}|}}\, \boldsymbol{e_0}$$

  [K., Potts 23]: not a good approximation...

- note that $(*)$ implies $\boldsymbol{A}^*\boldsymbol{W}\boldsymbol{A} = \boldsymbol{I}_{|\mathcal{I}_M|} \implies$ minimization of $\left\|\boldsymbol{A}^*\boldsymbol{W}\boldsymbol{A} - \boldsymbol{I}_{|\mathcal{I}_M|}\right\|_{\mathrm{F}}^2$
  $\rightsquigarrow$ minimizer obtained by solving $\boldsymbol{S}\boldsymbol{w} = \boldsymbol{b}$ with [Rosenfeld 98]

$$\boldsymbol{S} := \left(\left|[\boldsymbol{A}\boldsymbol{A}^*]_{j,s}\right|^2\right)_{j,s=1}^N \quad \text{and} \quad \boldsymbol{b} = |\mathcal{I}_M| \cdot \boldsymbol{1}_N$$

$\mathcal{O}(N^3)$

Analogous problem for bandlimited functions

Application: MRI (Magnetic Resonance Imaging) [Rosenfeld 98], [Greengard, Lee, Inati 06], [Eggers, K., Potts 22]

no longer discrete (trigonometric polynomials) but continuous (bandlimited functions)

Analogous problem for bandlimited functions

Application: MRI (Magnetic Resonance Imaging) [Rosenfeld 98], [Greengard, Lee, Inati 06], [Eggers, K., Potts 22]

no longer discrete (trigonometric polynomials) but continuous (bandlimited functions)

(Continuous) Fourier transform:

$$\hat{f}(\boldsymbol{v}) := \int_{\mathbb{R}^d} f(\boldsymbol{x})\, \mathrm{e}^{-2\pi \mathrm{i} \boldsymbol{v}\boldsymbol{x}}\, \mathrm{d}\boldsymbol{x}, \quad \boldsymbol{v} \in \mathbb{R}^d$$

$\rightsquigarrow$ bandlimited functions with maximum bandwidth $\boldsymbol{M}$ $\quad\Longleftrightarrow\quad$ $\operatorname{supp}(\hat{f}\,) = \left[-\frac{M}{2}, \frac{M}{2}\right)^d$

$$\Longrightarrow \quad f(\boldsymbol{x}) = \int_{\mathbb{R}^d} \hat{f}(\boldsymbol{v})\, \mathrm{e}^{2\pi \mathrm{i} \boldsymbol{v}\boldsymbol{x}}\, \mathrm{d}\boldsymbol{v} = \int_{\left[-\frac{M}{2}, \frac{M}{2}\right)^d} \hat{f}(\boldsymbol{v})\, \mathrm{e}^{2\pi \mathrm{i} \boldsymbol{v}\boldsymbol{x}_j}\, \mathrm{d}\boldsymbol{v}, \quad \boldsymbol{x} \in \mathbb{R}^d$$

Analogous problem for bandlimited functions

Application: MRI (Magnetic Resonance Imaging) [Rosenfeld 98], [Greengard, Lee, Inati 06], [Eggers, K., Potts 22]

no longer discrete (trigonometric polynomials) but continuous (bandlimited functions)

(Continuous) Fourier transform:

$$\hat{f}(\boldsymbol{v}) := \int_{\mathbb{R}^d} f(\boldsymbol{x})\, \mathrm{e}^{-2\pi\mathrm{i}\boldsymbol{v}\boldsymbol{x}}\, \mathrm{d}\boldsymbol{x}, \quad \boldsymbol{v} \in \mathbb{R}^d$$

$\rightsquigarrow$ bandlimited functions with maximum bandwidth $\boldsymbol{M}$ $\Longleftrightarrow$ $\operatorname{supp}(\hat{f}\,) = \left[-\frac{M}{2}, \frac{M}{2}\right)^d$

$$\Longrightarrow \quad f(\boldsymbol{x}) = \int_{\mathbb{R}^d} \hat{f}(\boldsymbol{v})\, \mathrm{e}^{2\pi\mathrm{i}\boldsymbol{v}\boldsymbol{x}}\, \mathrm{d}\boldsymbol{v} = \int_{\left[-\frac{M}{2}, \frac{M}{2}\right)^d} \hat{f}(\boldsymbol{v})\, \mathrm{e}^{2\pi\mathrm{i}\boldsymbol{v}\boldsymbol{x}_j}\, \mathrm{d}\boldsymbol{v}, \quad \boldsymbol{x} \in \mathbb{R}^d$$

Reconstruct: evaluations $\hat{f}(\boldsymbol{k}) \in \mathbb{C}$, $\boldsymbol{k} \in \mathcal{I}_{\boldsymbol{M}}$

Given: measurements $f(\boldsymbol{x}_j)$, $j = 1, \dots, N$

Analogous problem for bandlimited functions

Application: MRI (Magnetic Resonance Imaging) [Rosenfeld 98], [Greengard, Lee, Inati 06], [Eggers, K., Potts 22]

no longer discrete (trigonometric polynomials) but continuous (bandlimited functions)

(Continuous) Fourier transform:

$$\hat{f}(\boldsymbol{v}) := \int_{\mathbb{R}^d} f(\boldsymbol{x})\, \mathrm{e}^{-2\pi\mathrm{i}\boldsymbol{v}\boldsymbol{x}}\, \mathrm{d}\boldsymbol{x}, \quad \boldsymbol{v} \in \mathbb{R}^d$$

$\rightsquigarrow$ bandlimited functions with maximum bandwidth $\boldsymbol{M}$ $\Longleftrightarrow$ $\operatorname{supp}(\hat{f}\,) = \left[-\frac{M}{2}, \frac{M}{2}\right)^d$

$$\Longrightarrow \quad f(\boldsymbol{x}) = \int_{\mathbb{R}^d} \hat{f}(\boldsymbol{v})\, \mathrm{e}^{2\pi\mathrm{i}\boldsymbol{v}\boldsymbol{x}}\, \mathrm{d}\boldsymbol{v} = \int_{\left[-\frac{M}{2}, \frac{M}{2}\right)^d} \hat{f}(\boldsymbol{v})\, \mathrm{e}^{2\pi\mathrm{i}\boldsymbol{v}\boldsymbol{x}_j}\, \mathrm{d}\boldsymbol{v}, \quad \boldsymbol{x} \in \mathbb{R}^d$$

Reconstruct: evaluations $\hat{f}(\boldsymbol{k}) \in \mathbb{C}$, $\boldsymbol{k} \in \mathcal{I}_{\boldsymbol{M}}$

Given: measurements $f(\boldsymbol{x}_j)$, $j = 1, \dots, N$

Now: extend previous method

Reconsideration as trigonometric polynomials

Consider 1-periodization

$$\tilde{f}(\boldsymbol{x}) := \sum_{\boldsymbol{r} \in \mathbb{Z}^d} f(\boldsymbol{x} + \boldsymbol{r}) \in L_2(\mathbb{T}^d)$$

Reconsideration as trigonometric polynomials

Consider 1-periodization

$$\tilde{f}(\boldsymbol{x}) := \sum_{\boldsymbol{r}\in\mathbb{Z}^d} f(\boldsymbol{x}+\boldsymbol{r}) \in L_2(\mathbb{T}^d)$$

$\Rightarrow$ uniquely representable by absolute convergent Fourier series [Plonka, Potts, Steidl, Tasche 18]

$$\tilde{f}(\boldsymbol{x}) := \sum_{\boldsymbol{k}\in\mathbb{Z}^d} c_{\boldsymbol{k}}(\tilde{f})\,\mathrm{e}^{2\pi\mathrm{i}\boldsymbol{k}\boldsymbol{x}},$$

with Fourier coefficients

$$c_{\boldsymbol{k}}(\tilde{f}) = \int_{\mathbb{T}^d} \tilde{f}(\boldsymbol{x})\,\mathrm{e}^{-2\pi\mathrm{i}\boldsymbol{k}\boldsymbol{x}}\,\mathrm{d}\boldsymbol{x} = \int_{\mathbb{R}^d} f(\boldsymbol{x})\,\mathrm{e}^{-2\pi\mathrm{i}\boldsymbol{k}\boldsymbol{x}}\,\mathrm{d}\boldsymbol{x} = \hat{f}(\boldsymbol{k}), \quad \boldsymbol{k}\in\mathbb{Z}^d$$

Reconsideration as trigonometric polynomials

Consider 1-periodization

$$\tilde{f}(\boldsymbol{x}) := \sum_{\boldsymbol{r}\in\mathbb{Z}^d} f(\boldsymbol{x}+\boldsymbol{r}) \in L_2(\mathbb{T}^d)$$

$\Rightarrow$ uniquely representable by absolute convergent Fourier series [Plonka, Potts, Steidl, Tasche 18]

$$\tilde{f}(\boldsymbol{x}) := \sum_{\boldsymbol{k}\in\mathbb{Z}^d} c_{\boldsymbol{k}}(\tilde{f})\, \mathrm{e}^{2\pi\mathrm{i}\boldsymbol{k}\boldsymbol{x}},$$

with Fourier coefficients

$$c_{\boldsymbol{k}}(\tilde{f}) = \int_{\mathbb{T}^d} \tilde{f}(\boldsymbol{x})\, \mathrm{e}^{-2\pi\mathrm{i}\boldsymbol{k}\boldsymbol{x}}\, \mathrm{d}\boldsymbol{x} = \int_{\mathbb{R}^d} f(\boldsymbol{x})\, \mathrm{e}^{-2\pi\mathrm{i}\boldsymbol{k}\boldsymbol{x}}\, \mathrm{d}\boldsymbol{x} = \hat{f}(\boldsymbol{k}), \quad \boldsymbol{k}\in\mathbb{Z}^d$$

$\rightsquigarrow$ f bandlimited with bandwidth $\boldsymbol{M}$ $\iff$ $\mathrm{supp}(\hat{f}\,) = \left[-\frac{M}{2}, \frac{M}{2}\right)^d$ $\implies$ $\hat{f}(\boldsymbol{k}) = 0,\ \boldsymbol{k}\in\mathbb{Z}^d\setminus\mathcal{I}_{\boldsymbol{M}}$

$\implies$ $\tilde{f}$ trigonometric polynomial of degree $\boldsymbol{M}$

Reconstideration as trigonometric polynomials

Consider 1-periodization

$$\tilde{f}(\boldsymbol{x}) := \sum_{\boldsymbol{r}\in\mathbb{Z}^d} f(\boldsymbol{x}+\boldsymbol{r}) \in L_2(\mathbb{T}^d)$$

$\Rightarrow$ uniquely representable by absolute convergent Fourier series [Plonka, Potts, Steidl, Tasche 18]

$$\tilde{f}(\boldsymbol{x}) := \sum_{\boldsymbol{k}\in\mathbb{Z}^d} c_{\boldsymbol{k}}(\tilde{f})\, \mathrm{e}^{2\pi\mathrm{i}\boldsymbol{k}\boldsymbol{x}},$$

with Fourier coefficients

$$c_{\boldsymbol{k}}(\tilde{f}) = \int_{\mathbb{T}^d} \tilde{f}(\boldsymbol{x})\, \mathrm{e}^{-2\pi\mathrm{i}\boldsymbol{k}\boldsymbol{x}}\, \mathrm{d}\boldsymbol{x} = \int_{\mathbb{R}^d} f(\boldsymbol{x})\, \mathrm{e}^{-2\pi\mathrm{i}\boldsymbol{k}\boldsymbol{x}}\, \mathrm{d}\boldsymbol{x} = \hat{f}(\boldsymbol{k}), \quad \boldsymbol{k}\in\mathbb{Z}^d$$

$\rightsquigarrow$ f bandlimited with bandwidth $\boldsymbol{M}$ $\iff$ $\mathrm{supp}(\hat{f}) = \left[-\frac{M}{2}, \frac{M}{2}\right)^d$ $\implies$ $\hat{f}(\boldsymbol{k}) = 0,\ \boldsymbol{k}\in\mathbb{Z}^d\setminus\mathcal{I}_M$

$\implies$ $\tilde{f}$ trigonometric polynomial of degree $\boldsymbol{M}$

$\Rightarrow$ exact reconstruction

$$\hat{f}(\boldsymbol{k}) = c_{\boldsymbol{k}}(\tilde{f}) = \sum_{j=1}^{N} w_j\, \tilde{f}(\boldsymbol{x}_j)\, \mathrm{e}^{-2\pi\mathrm{i}\boldsymbol{k}\boldsymbol{x}_j}, \quad \boldsymbol{k}\in\mathcal{I}_M \qquad \hat{\boldsymbol{f}} = \boldsymbol{A}^*\boldsymbol{W}\tilde{\boldsymbol{f}}$$

Practical situation (e. g. MRI)

In practice: only hypothetical case !

periodization $\tilde{f}$ cannot be sampled $\iff$ f cannot be sampled on whole $\mathbb{R}^d$

Sampling: limited coverage of space

$\rightsquigarrow$ f only known on a bounded domain, w.l.o.g. for $\boldsymbol{x} \in \left[-\frac{1}{2}, \frac{1}{2}\right)^d$

Practical situation (e. g. MRI)

In practice: only hypothetical case !

periodization $\tilde{f}$ cannot be sampled $\iff$ f cannot be sampled on whole $\mathbb{R}^d$

Sampling: limited coverage of space

$\rightsquigarrow$ f only known on a bounded domain, w.l.o.g. for $\boldsymbol{x} \in \left[-\frac{1}{2}, \frac{1}{2}\right)^d$

Consequences: need to assume that f is small outside $\left[-\frac{1}{2}, \frac{1}{2}\right)^d$, such that $\tilde{f}(\boldsymbol{x}_j) \approx f(\boldsymbol{x}_j)$

$\rightsquigarrow$ have to deal with the approximation

$$\hat{f}(\boldsymbol{k}) \approx \sum_{j=1}^{N} w_j \, f(\boldsymbol{x}_j) \, \mathrm{e}^{-2\pi \mathrm{i} \boldsymbol{k} \boldsymbol{x}_j}, \quad \boldsymbol{k} \in \mathcal{I}_M$$

$$\Longrightarrow \qquad \hat{\boldsymbol{f}} = \boldsymbol{A}^* \boldsymbol{W} \tilde{\boldsymbol{f}} \approx \boldsymbol{A}^* \boldsymbol{W} \boldsymbol{f}$$

Practical situation (e. g. MRI)

In practice: only hypothetical case !

periodization $\tilde{f}$ cannot be sampled $\iff$ f cannot be sampled on whole $\mathbb{R}^d$

Sampling: limited coverage of space

$\rightsquigarrow$ f only known on a bounded domain, w.l.o.g. for $\boldsymbol{x} \in \left[-\frac{1}{2}, \frac{1}{2}\right)^d$

Consequences: need to assume that f is small outside $\left[-\frac{1}{2}, \frac{1}{2}\right)^d$, such that $\tilde{f}(\boldsymbol{x}_j) \approx f(\boldsymbol{x}_j)$

$\rightsquigarrow$ have to deal with the approximation

$$\hat{f}(\boldsymbol{k}) \approx \sum_{j=1}^{N} w_j\, f(\boldsymbol{x}_j)\, \mathrm{e}^{-2\pi \mathrm{i} \boldsymbol{k} \boldsymbol{x}_j}, \quad \boldsymbol{k} \in \mathcal{I}_M$$

$$\Longrightarrow \qquad \hat{\boldsymbol{f}} = \boldsymbol{A}^* \boldsymbol{W} \tilde{\boldsymbol{f}} \approx \boldsymbol{A}^* \boldsymbol{W} \boldsymbol{f}$$

Main cause of error: f is not known on whole $\mathbb{R}^d$

Connection to previous work

[Greengard, Lee, Inati 06]: extend approximation to whole interval

$$\hat{f}(\boldsymbol{v}) \approx \tilde{h}(\boldsymbol{v}) := \sum_{j=1}^{N} w_j \, \tilde{f}(\boldsymbol{x}_j) \, \mathrm{e}^{-2\pi \mathrm{i} \boldsymbol{v} \boldsymbol{x}_j}, \quad \boldsymbol{v} \in \left[-\tfrac{M}{2}, \tfrac{M}{2}\right)^d$$

Connection to previous work

[Greengard, Lee, Inati 06]: extend approximation to whole interval

$$\hat{f}(\boldsymbol{v}) \approx \tilde{h}(\boldsymbol{v}) := \sum_{j=1}^{N} w_j \, \tilde{f}(\boldsymbol{x}_j) \, \mathrm{e}^{-2\pi \mathrm{i} \boldsymbol{v} \boldsymbol{x}_j}, \quad \boldsymbol{v} \in \left[-\tfrac{M}{2}, \tfrac{M}{2}\right)^d$$

$\Rightarrow$ inverse Fourier transform

$$f(\boldsymbol{x}_s) = \int_{\left[-\frac{M}{2}, \frac{M}{2}\right)^d} \hat{f}(\boldsymbol{v}) \, \mathrm{e}^{2\pi \mathrm{i} \boldsymbol{v} \boldsymbol{x}_s} \, \mathrm{d}\boldsymbol{v} \approx \int_{\left[-\frac{M}{2}, \frac{M}{2}\right)^d} \tilde{h}(\boldsymbol{v}) \, \mathrm{e}^{2\pi \mathrm{i} \boldsymbol{v} \boldsymbol{x}_s} \, \mathrm{d}\boldsymbol{v} = \sum_{j=1}^{N} w_j \, \tilde{f}(\boldsymbol{x}_j) \cdot |\mathcal{I}_{\boldsymbol{M}}| \, \mathrm{sinc}\big(M\pi(\boldsymbol{x}_j - \boldsymbol{x}_s)\big)$$

Connection to previous work

[Greengard, Lee, Inati 06]: extend approximation to whole interval

$$\hat{f}(\boldsymbol{v}) \approx \tilde{h}(\boldsymbol{v}) := \sum_{j=1}^{N} w_j \, \tilde{f}(\boldsymbol{x}_j) \, \mathrm{e}^{-2\pi \mathrm{i} \boldsymbol{v} \boldsymbol{x}_j}, \quad \boldsymbol{v} \in \left[-\tfrac{M}{2}, \tfrac{M}{2}\right)^d$$

$\Rightarrow$ inverse Fourier transform

$$f(\boldsymbol{x}_s) = \int_{\left[-\frac{M}{2}, \frac{M}{2}\right)^d} \hat{f}(\boldsymbol{v}) \, \mathrm{e}^{2\pi \mathrm{i} \boldsymbol{v} \boldsymbol{x}_s} \, \mathrm{d}\boldsymbol{v} \approx \int_{\left[-\frac{M}{2}, \frac{M}{2}\right)^d} \tilde{h}(\boldsymbol{v}) \, \mathrm{e}^{2\pi \mathrm{i} \boldsymbol{v} \boldsymbol{x}_s} \, \mathrm{d}\boldsymbol{v} = \sum_{j=1}^{N} w_j \, \tilde{f}(\boldsymbol{x}_j) \cdot |\mathcal{I}_{\boldsymbol{M}}| \, \mathrm{sinc}\big(M\pi(\boldsymbol{x}_j - \boldsymbol{x}_s)\big)$$

Question: Choose w_j based on this equation

$$\boldsymbol{f} \approx |\mathcal{I}_{\boldsymbol{M}}| \cdot \boldsymbol{C}_{\mathrm{n}} \boldsymbol{W} \tilde{\boldsymbol{f}} \approx |\mathcal{I}_{\boldsymbol{M}}| \cdot \boldsymbol{C}_{\mathrm{n}} \boldsymbol{W} \boldsymbol{f},$$

?

where

$$\boldsymbol{C}_{\mathrm{n}} := \Big(\mathrm{sinc}\big(M\pi(\boldsymbol{x}_j - \boldsymbol{x}_s)\big)\Big)_{j,s=1}^{N} \in \mathbb{R}^{N \times N}$$

Connection to previous work

[Greengard, Lee, Inati 06]: extend approximation to whole interval

$$\hat{f}(\boldsymbol{v}) \approx \tilde{h}(\boldsymbol{v}) := \sum_{j=1}^{N} w_j\, \tilde{f}(\boldsymbol{x}_j)\, \mathrm{e}^{-2\pi\mathrm{i}\boldsymbol{v}\boldsymbol{x}_j}, \quad \boldsymbol{v} \in \left[-\tfrac{M}{2}, \tfrac{M}{2}\right)^d$$

$\Rightarrow$ inverse Fourier transform

$$f(\boldsymbol{x}_s) = \int_{\left[-\frac{M}{2}, \frac{M}{2}\right)^d} \hat{f}(\boldsymbol{v})\, \mathrm{e}^{2\pi\mathrm{i}\boldsymbol{v}\boldsymbol{x}_s}\, \mathrm{d}\boldsymbol{v} \approx \int_{\left[-\frac{M}{2}, \frac{M}{2}\right)^d} \tilde{h}(\boldsymbol{v})\, \mathrm{e}^{2\pi\mathrm{i}\boldsymbol{v}\boldsymbol{x}_s}\, \mathrm{d}\boldsymbol{v} = \sum_{j=1}^{N} w_j\, \tilde{f}(\boldsymbol{x}_j) \cdot |\mathcal{I}_{\boldsymbol{M}}|\, \mathrm{sinc}\big(M\pi(\boldsymbol{x}_j - \boldsymbol{x}_s)\big)$$

Question: Choose w_j based on this equation

$$\boldsymbol{f} \approx |\mathcal{I}_{\boldsymbol{M}}| \cdot \boldsymbol{C}_{\mathrm{n}} \boldsymbol{W} \tilde{\boldsymbol{f}} \approx |\mathcal{I}_{\boldsymbol{M}}| \cdot \boldsymbol{C}_{\mathrm{n}} \boldsymbol{W} \boldsymbol{f},$$

?

where

$$\boldsymbol{C}_{\mathrm{n}} := \left(\mathrm{sinc}\big(M\pi(\boldsymbol{x}_j - \boldsymbol{x}_s)\big)\right)_{j,s=1}^{N} \in \mathbb{R}^{N \times N}$$

$\rightsquigarrow$ ideally aim for $\boldsymbol{I}_N = |\mathcal{I}_{\boldsymbol{M}}| \cdot \boldsymbol{C}_{\mathrm{n}} \boldsymbol{W}$

Exact solution? main diagonal of $|\mathcal{I}_{\boldsymbol{M}}| \cdot \boldsymbol{C}_{\mathrm{n}} \boldsymbol{W} = \boldsymbol{I}_N$ reads as

$$\frac{1}{|\mathcal{I}_{\boldsymbol{M}}|} = w_j \operatorname{sinc}(0) = w_j, \quad j = 1, \dots, N,$$

i. e., need $\operatorname{sinc}(M\pi(\boldsymbol{x}_j - \boldsymbol{x}_s)) = 0,\, j \neq s$ ↯ $\boldsymbol{x}_j$ nonequispaced ⇝ only approximate solution

Exact solution? main diagonal of $|\mathcal{I}_{\boldsymbol{M}}| \cdot \boldsymbol{C}_{\mathrm{n}} \boldsymbol{W} = \boldsymbol{I}_N$ reads as

$$\frac{1}{|\mathcal{I}_{\boldsymbol{M}}|} = w_j \operatorname{sinc}(0) = w_j, \quad j = 1, \dots, N,$$

i. e., need $\operatorname{sinc}(M\pi(\boldsymbol{x}_j - \boldsymbol{x}_s)) = 0,\ j \neq s$ ↯ $\boldsymbol{x}_j$ nonequispaced ⇝ only approximate solution

Least squares problem:

$$\underset{\boldsymbol{W}=\operatorname{diag}(w_j)_{j=1}^N}{\text{Minimize}} \quad \| |\mathcal{I}_{\boldsymbol{M}}| \cdot \boldsymbol{C}_{\mathrm{n}} \boldsymbol{W} - \boldsymbol{I}_N \|_{\mathrm{F}}^2$$

Exact solution? main diagonal of $|\mathcal{I}_{\boldsymbol{M}}| \cdot \boldsymbol{C}_{\mathrm{n}} \boldsymbol{W} = \boldsymbol{I}_N$ reads as

$$\frac{1}{|\mathcal{I}_{\boldsymbol{M}}|} = w_j \operatorname{sinc}(0) = w_j, \quad j = 1, \dots, N,$$

i. e., need $\operatorname{sinc}(M\pi(\boldsymbol{x}_j - \boldsymbol{x}_s)) = 0,\, j \neq s$ ↯ $\boldsymbol{x}_j$ nonequispaced $\rightsquigarrow$ only approximate solution

Least squares problem:

$$\underset{\boldsymbol{W}=\operatorname{diag}(w_j)_{j=1}^N}{\text{Minimize}} \quad \| |\mathcal{I}_{\boldsymbol{M}}| \cdot \boldsymbol{C}_{\mathrm{n}} \boldsymbol{W} - \boldsymbol{I}_N \|_{\mathrm{F}}^2$$

$\rightsquigarrow$ consider only nonzeros

$$\| |\mathcal{I}_{\boldsymbol{M}}| \cdot \boldsymbol{C}_{\mathrm{n}} \boldsymbol{W} - \boldsymbol{I}_N \|_{\mathrm{F}}^2 = \sum_{j=1}^{N} \| |\mathcal{I}_{\boldsymbol{M}}| \cdot \boldsymbol{C}_{\mathrm{n}} \boldsymbol{w}_j - \boldsymbol{e}_j \|_2^2 = \sum_{j=1}^{N} \| |\mathcal{I}_{\boldsymbol{M}}| \cdot [\boldsymbol{C}_{\mathrm{n}}]_j \, w_j - \boldsymbol{e}_j \|_2^2$$

Exact solution? main diagonal of $|\mathcal{I}_{\boldsymbol{M}}| \cdot \boldsymbol{C}_{\mathrm{n}} \boldsymbol{W} = \boldsymbol{I}_N$ reads as

$$\frac{1}{|\mathcal{I}_{\boldsymbol{M}}|} = w_j \operatorname{sinc}(0) = w_j, \quad j = 1, \ldots, N,$$

i. e., need $\operatorname{sinc}(M\pi(\boldsymbol{x}_j - \boldsymbol{x}_s)) = 0$, $j \neq s$ $\lightning$ $\boldsymbol{x}_j$ nonequispaced $\rightsquigarrow$ only approximate solution

Least squares problem:

$$\underset{\boldsymbol{W}=\operatorname{diag}(w_j)_{j=1}^N}{\text{Minimize}} \quad \big\| |\mathcal{I}_{\boldsymbol{M}}| \cdot \boldsymbol{C}_{\mathrm{n}} \boldsymbol{W} - \boldsymbol{I}_N \big\|_{\mathrm{F}}^2$$

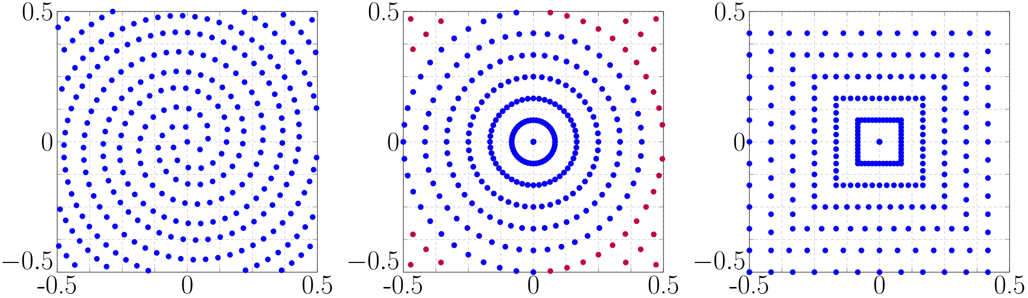
$\rightsquigarrow$ consider only nonzeros

$$\big\| |\mathcal{I}_{\boldsymbol{M}}| \cdot \boldsymbol{C}_{\mathrm{n}} \boldsymbol{W} - \boldsymbol{I}_N \big\|_{\mathrm{F}}^2 = \sum_{j=1}^N \big\| |\mathcal{I}_{\boldsymbol{M}}| \cdot \boldsymbol{C}_{\mathrm{n}} \boldsymbol{w}_j - \boldsymbol{e}_j \big\|_2^2 = \sum_{j=1}^N \big\| |\mathcal{I}_{\boldsymbol{M}}| \cdot [\boldsymbol{C}_{\mathrm{n}}]_j \, w_j - \boldsymbol{e}_j \big\|_2^2$$

$\Rightarrow$ least squares solution given by

$$w_j = \frac{1}{|\mathcal{I}_{\boldsymbol{M}}|} [\boldsymbol{C}_{\mathrm{n}}]_j^{\dagger} \boldsymbol{e}_j = \ldots = \frac{1}{|\mathcal{I}_{\boldsymbol{M}}|} \left(\sum_{s=1}^N \operatorname{sinc}^2(M\pi(\boldsymbol{x}_j - \boldsymbol{x}_s)) \right)^{-1}$$

Numerical Examples – Testing Grids



(a) Spiral grid

(b) Polar (blue) and modified polar (red) grid

(c) Linogram grid

Figure: Exemplary grids of size $R = 12$ and $T = 2R$.

Discrete problem – Shepp-Logan phantom

1. phantom data $=$ Fourier coefficients $\hat{\boldsymbol{f}} := (\hat{f}_{\boldsymbol{k}})_{\boldsymbol{k}\in\mathcal{I}_M}$ of a trigonometric polynomial
2. compute the evaluations $f(\boldsymbol{x}_j) = \sum_{\boldsymbol{k}\in\mathcal{I}_M} \hat{f}_{\boldsymbol{k}}\, \mathrm{e}^{2\pi\mathrm{i}\boldsymbol{k}\boldsymbol{x}_j}$ by means of NFFT
3. reconstruct $\tilde{h}_{\boldsymbol{k}} \approx \hat{f}_{\boldsymbol{k}},\ \boldsymbol{k} \in \mathcal{I}_M$

Discrete problem – Shepp-Logan phantom

1. phantom data $=$ Fourier coefficients $\hat{\boldsymbol{f}} := (\hat{f}_{\boldsymbol{k}})_{\boldsymbol{k}\in\mathcal{I}_M}$ of a trigonometric polynomial
2. compute the evaluations $f(\boldsymbol{x}_j) = \sum_{\boldsymbol{k}\in\mathcal{I}_M} \hat{f}_{\boldsymbol{k}}\, \mathrm{e}^{2\pi\mathrm{i}\boldsymbol{k}\boldsymbol{x}_j}$ by means of NFFT
3. reconstruct $\tilde{h}_{\boldsymbol{k}} \approx \hat{f}_{\boldsymbol{k}},\ \boldsymbol{k} \in \mathcal{I}_M$

Friendly setting ($|\mathcal{I}_{2M}| \le N$):

- linogram grid of size $R = 2M, T = 2R$
- phantom size $M \times M$
- relative errors

$$e_2 := \frac{\|\tilde{\boldsymbol{h}} - \hat{\boldsymbol{f}}\|_2}{\|\hat{\boldsymbol{f}}\|_2}$$

⇝ show exact reconstruction

Discrete problem – Shepp-Logan phantom

1. phantom data $=$ Fourier coefficients $\hat{\boldsymbol{f}} := (\hat{f}_{\boldsymbol{k}})_{\boldsymbol{k}\in\mathcal{I}_M}$ of a trigonometric polynomial
2. compute the evaluations $f(\boldsymbol{x}_j) = \sum_{\boldsymbol{k}\in\mathcal{I}_M} \hat{f}_{\boldsymbol{k}}\, \mathrm{e}^{2\pi\mathrm{i}\boldsymbol{k}\boldsymbol{x}_j}$ by means of NFFT
3. reconstruct $\tilde{h}_{\boldsymbol{k}} \approx \hat{f}_{\boldsymbol{k}},\ \boldsymbol{k} \in \mathcal{I}_M$

Friendly setting ($|\mathcal{I}_{2M}| \leq N$):

- linogram grid of size $R = 2M, T = 2R$
- phantom size $M \times M$
- relative errors

$$e_2 := \frac{\|\tilde{\boldsymbol{h}} - \hat{\boldsymbol{f}}\|_2}{\|\hat{\boldsymbol{f}}\|_2}$$

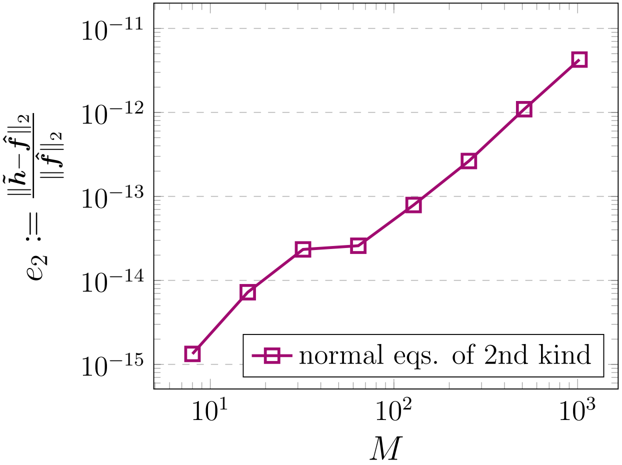
⇝ show exact reconstruction

Unfriendly setting ($|\mathcal{I}_{2M}| > N$):

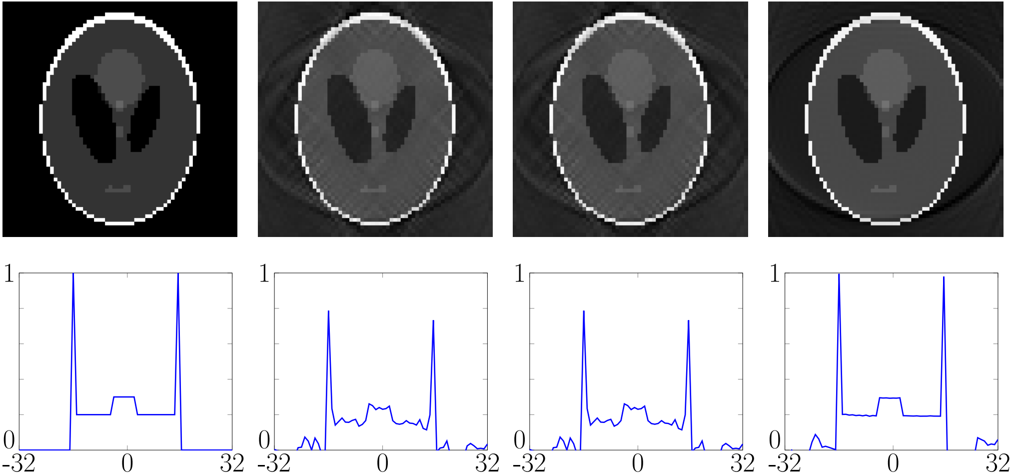
- spiral grid of size $R = M, T = 2R$
- phantom of size $M = 64$

⇝ compare presented computation schemes
  - normal eqs. of 2nd kind
  - normal eqs. of 1st kind
  - Frobenius norm minimization (small M necessary for being affordable)

Friendly setting ($|\mathcal{I}_{2M}| < N$)



Unfriendly setting ($|\mathcal{I}_{2M}| > N$)



(a) Phantom

(b) Normal eqs. of 2nd kind
$e_2 = 0.2482$

(c) Normal eqs. of 1st kind
$e_2 = 0.2482$

(d) Frobenius norm minimization
$e_2 = 0.1187$

Continuous problem – tensorized triangular pulse function

1. specify compactly supported $\hat{f}(\boldsymbol{v}) = g(v_1) \cdot g(v_2)$, with triangular pulse $g(v) := (1 - \left|\frac{v}{b}\right|) \cdot \chi_{[-b,b]}(v)$
2. compute inverse Fourier transform

$$f(\boldsymbol{x}) = \int_{\mathbb{R}^2} \hat{f}(\boldsymbol{v}) \,\mathrm{e}^{2\pi\mathrm{i}\boldsymbol{v}\boldsymbol{x}} \,\mathrm{d}\boldsymbol{v} = b^2 \operatorname{sinc}^2(b\pi\boldsymbol{x}), \quad \boldsymbol{x} \in \mathbb{R}^2$$

$\rightsquigarrow$ bandlimited with bandwidth $\boldsymbol{M}$ for all $b \in \mathbb{N}$ with $b \le \frac{M}{2}$

3. sample $f(\boldsymbol{x}_j)$ for given $\boldsymbol{x}_j \in \left[-\frac{1}{2}, \frac{1}{2}\right)^2$, $j = 1, \ldots, N$
4. reconstruct $\tilde{h}(\boldsymbol{k}) \approx \hat{f}(\boldsymbol{k})$, $\boldsymbol{k} \in \mathcal{I}_M$

Continuous problem – tensorized triangular pulse function

1. specify compactly supported $\hat{f}(\boldsymbol{v}) = g(v_1) \cdot g(v_2)$, with triangular pulse $g(v) := \left(1 - \left|\frac{v}{b}\right|\right) \cdot \chi_{[-b,b]}(v)$
2. compute inverse Fourier transform

$$f(\boldsymbol{x}) = \int_{\mathbb{R}^2} \hat{f}(\boldsymbol{v})\, \mathrm{e}^{2\pi \mathrm{i} \boldsymbol{v}\boldsymbol{x}}\, \mathrm{d}\boldsymbol{v} = b^2 \operatorname{sinc}^2(b\pi\boldsymbol{x}), \quad \boldsymbol{x} \in \mathbb{R}^2$$

$\rightsquigarrow$ bandlimited with bandwidth $\boldsymbol{M}$ for all $b \in \mathbb{N}$ with $b \leq \frac{M}{2}$

3. sample $f(\boldsymbol{x}_j)$ for given $\boldsymbol{x}_j \in \left[-\frac{1}{2}, \frac{1}{2}\right)^2$, $j = 1, \dots, N$
4. reconstruct $\tilde{h}(\boldsymbol{k}) \approx \hat{f}(\boldsymbol{k})$, $\boldsymbol{k} \in \mathcal{I}_{\boldsymbol{M}}$

Setup:

- consider $|\mathcal{I}_{\boldsymbol{2M}}| \leq N$
- $M = 32$ and $b = 12$
- modified polar grid of size $R = 2M$, $T = 2R$
- pointwise errors $|\tilde{\boldsymbol{h}} - \hat{\boldsymbol{f}}|$

Continuous problem – tensorized triangular pulse function

1. specify compactly supported $\hat{f}(\boldsymbol{v}) = g(v_1) \cdot g(v_2)$, with triangular pulse $g(v) := \left(1 - \left|\frac{v}{b}\right|\right) \cdot \chi_{[-b,b]}(v)$
2. compute inverse Fourier transform

$$f(\boldsymbol{x}) = \int_{\mathbb{R}^2} \hat{f}(\boldsymbol{v})\, \mathrm{e}^{2\pi\mathrm{i}\boldsymbol{v}\boldsymbol{x}}\, \mathrm{d}\boldsymbol{v} = b^2 \operatorname{sinc}^2(b\pi\boldsymbol{x}), \quad \boldsymbol{x} \in \mathbb{R}^2$$

$\rightsquigarrow$ bandlimited with bandwidth $\boldsymbol{M}$ for all $b \in \mathbb{N}$ with $b \leq \frac{M}{2}$

3. sample $f(\boldsymbol{x}_j)$ for given $\boldsymbol{x}_j \in \left[-\frac{1}{2}, \frac{1}{2}\right)^2, j = 1, \dots, N$
4. reconstruct $\tilde{h}(\boldsymbol{k}) \approx \hat{f}(\boldsymbol{k}), \boldsymbol{k} \in \mathcal{I}_{\boldsymbol{M}}$

Setup:

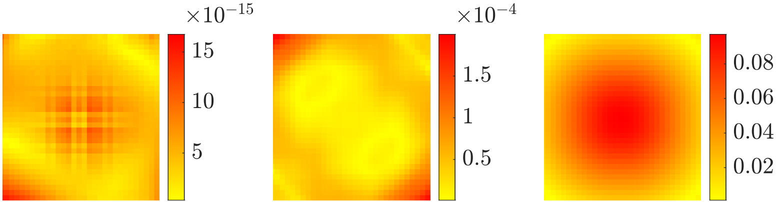
- consider $|\mathcal{I}_{\boldsymbol{2M}}| \leq N$
- $M = 32$ and $b = 12$
- modified polar grid of size $R = 2M, T = 2R$
- pointwise errors $\left|\tilde{\boldsymbol{h}} - \hat{\boldsymbol{f}}\right|$

Sampling data:

- real-world sampling $f(\boldsymbol{x}_j)$
- artificial sampling of the periodization

$$\tilde{f}(\boldsymbol{x}_j) = \sum_{\boldsymbol{k} \in \mathcal{I}_{\boldsymbol{M}}} \hat{f}(\boldsymbol{k})\, \mathrm{e}^{2\pi\mathrm{i}\boldsymbol{k}\boldsymbol{x}_j}$$

Results – real-world sampling (top) & artificial sampling (bottom)

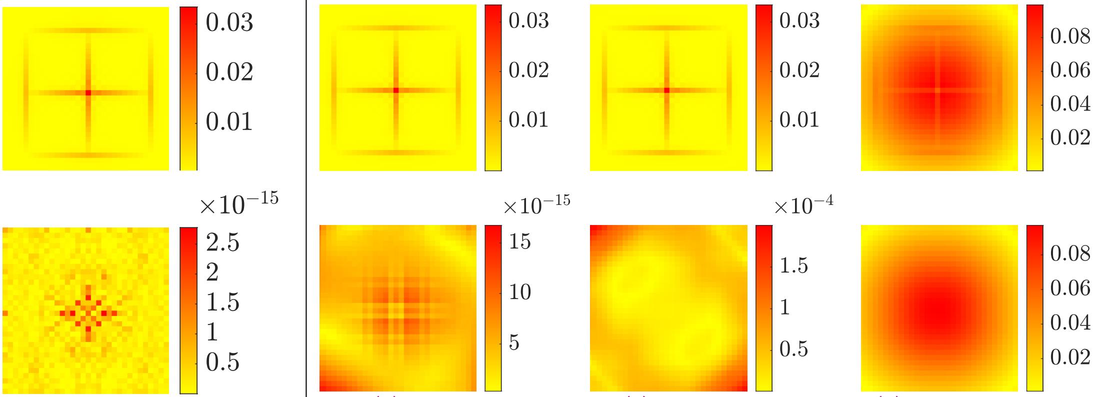


(b) normal eqs. of 2nd kind

(c) Frobenius norm minimization

(d) [Greengard, Lee, Inati 06]

Results – real-world sampling (top) & artificial sampling (bottom)



(a) equispaced setting

(b) normal eqs. of 2nd kind

(c) Frobenius norm minimization

(d) [Greengard, Lee, Inati 06]

Summary

- new direct inversion method based on sampling density compensation
- introduced for discrete problem (trigonometric polynomials)
- exact reconstruction in case $|\mathcal{I}_{\mathbf{2}M}| \leq N$
- extendable continuous problem (bandlimited functions)
- error solely occurs since f cannot be sampled on whole $\mathbb{R}^d$

- **K., Potts: Fast and direct inversion methods for the multivariate nonequispaced fast Fourier transform**. Frontiers in Applied Mathematics and Statistics 9 (2023).
- **K., Potts: Optimal density compensation factors for the reconstruction of the Fourier transform of bandlimited functions**. arXiv, 2304.00094, 2023.

Summary

- new direct inversion method based on sampling density compensation
- introduced for discrete problem (trigonometric polynomials)
- exact reconstruction in case $|\mathcal{I}_{\mathbf{2M}}| \leq N$
- extendable continuous problem (bandlimited functions)
- error solely occurs since f cannot be sampled on whole $\mathbb{R}^d$

- K., Potts: **Fast and direct inversion methods for the multivariate nonequispaced fast Fourier transform**. Frontiers in Applied Mathematics and Statistics 9 (2023).
- K., Potts: **Optimal density compensation factors for the reconstruction of the Fourier transform of bandlimited functions**. arXiv, 2304.00094, 2023.

Thank you for your attention!

NFFT – Fast computation of $f(\boldsymbol{x}_j) = \sum_{\boldsymbol{k}\in\mathcal{I}_{\boldsymbol{M}}} \hat{f}_{\boldsymbol{k}}\, \mathrm{e}^{2\pi\mathrm{i}\boldsymbol{k}\boldsymbol{x}_j}$

[Potts, Steidl, Tasche 01]

1. Set $\mathcal{O}(|\mathcal{I}_{\boldsymbol{M}}|)$

$$\hat{g}_{\boldsymbol{k}} := \begin{cases} \frac{\hat{f}_{\boldsymbol{k}}}{\hat{w}(\boldsymbol{k})} & : \boldsymbol{k} \in \mathcal{I}_{\boldsymbol{M}}, \\ 0 & : \boldsymbol{k} \in \mathcal{I}_{\boldsymbol{M_\sigma}} \setminus \mathcal{I}_{\boldsymbol{M}}. \end{cases}$$

2. Compute $\mathcal{O}(|\mathcal{I}_{\boldsymbol{M}}| \log(|\mathcal{I}_{\boldsymbol{M}}|))$

$$g_{\boldsymbol{l}} := \frac{1}{|\mathcal{I}_{\boldsymbol{M_\sigma}}|} \sum_{\boldsymbol{k}\in\mathcal{I}_{\boldsymbol{M}}} \hat{g}_{\boldsymbol{k}}\, \mathrm{e}^{2\pi\mathrm{i}\boldsymbol{k}(\boldsymbol{M}_{\boldsymbol{\sigma}}^{-1}\odot\boldsymbol{l})}, \quad \boldsymbol{l} \in \mathcal{I}_{\boldsymbol{M_\sigma}},$$

by a d-variate inverse FFT.

3. Compute $\mathcal{O}(N)$

$$\tilde{f}_j := \sum_{\boldsymbol{l}\in\mathcal{I}_{\boldsymbol{M_\sigma}}} g_{\boldsymbol{l}}\, \tilde{w}_m\big(\boldsymbol{x}_j - \boldsymbol{M}_{\boldsymbol{\sigma}}^{-1}\odot\boldsymbol{l}\big), \quad j = 1, \dots, N.$$

Output: $\tilde{f}_j \approx f_j$, $j = 1, \dots, N$. $\mathcal{O}(|\mathcal{I}_{\boldsymbol{M}}| \log(|\mathcal{I}_{\boldsymbol{M}}|) + N)$

Remark – sinc operator

Fourier series of the periodization of $g(\boldsymbol{t}) = \mathrm{e}^{2\pi\mathrm{i}\boldsymbol{t}\boldsymbol{x}}$, $\boldsymbol{t} \in \left[-\frac{M}{2}, \frac{M}{2}\right)^d$, with $\boldsymbol{x} \in \mathbb{C}^d$ fixed: [Lund, Bowers 92]

$$\mathrm{e}^{2\pi\mathrm{i}\boldsymbol{t}\boldsymbol{x}} = \sum_{\boldsymbol{\ell}\in\mathbb{Z}^d} \mathrm{e}^{2\pi\mathrm{i}\boldsymbol{t}\boldsymbol{y}_{\boldsymbol{\ell}}} \operatorname{sinc}\big(M\pi(\boldsymbol{x}-\boldsymbol{y}_{\boldsymbol{\ell}})\big), \quad \boldsymbol{x} \in \mathbb{C}^d, \tag{$\triangle$}$$

where $\boldsymbol{y}_{\boldsymbol{\ell}} := \boldsymbol{M}^{-1} \odot \boldsymbol{\ell} = \left(M^{-1}\ell_1, \dots, M^{-1}\ell_d\right)^T$, $\boldsymbol{\ell} \in \mathbb{Z}^d$, and the d-variate function $\operatorname{sinc}(\boldsymbol{x}) := \prod_{t=1}^d \operatorname{sinc}(x_t)$

Define:

$$\mathcal{C} := \Big(\operatorname{sinc}\big(M\pi(\boldsymbol{x}_j - \boldsymbol{y}_{\boldsymbol{\ell}})\big)\Big)_{j=1,\,\boldsymbol{\ell}\in\mathbb{Z}^d}^{N}$$ ... sinc operator

$$\mathcal{F} := \left(\mathrm{e}^{2\pi\mathrm{i}\boldsymbol{k}\boldsymbol{y}_{\boldsymbol{\ell}}}\right)_{\boldsymbol{\ell}\in\mathbb{Z}^d,\,\boldsymbol{k}\in\mathcal{I}_{\boldsymbol{M}}}$$ ... one-sided infinite Fourier matrix

$\Rightarrow$ point evaluations of ($\triangle$) at $\boldsymbol{x} = \boldsymbol{x}_j$, $j = 1, \dots, N$, and $\boldsymbol{t} = \boldsymbol{k} \in \mathcal{I}_{\boldsymbol{M}}$:

$$\mathcal{C}\mathcal{F} = \left(\sum_{\boldsymbol{\ell}\in\mathbb{Z}^d} \mathrm{e}^{2\pi\mathrm{i}\boldsymbol{k}\boldsymbol{y}_{\boldsymbol{\ell}}} \operatorname{sinc}\big(M\pi(\boldsymbol{x}_j - \boldsymbol{y}_{\boldsymbol{\ell}})\big)\right)_{j=1,\,\boldsymbol{k}\in\mathcal{I}_{\boldsymbol{M}}}^{N} = \boldsymbol{A}$$

Remark – equivalence of approaches [Greengard, Lee, Inati 06] and [Pipe, Menon 99]

Classical sampling theorem of Shannon-Whittaker-Kotelnikov:
Any bandlimited function $f \in L_2(\mathbb{R}^d)$ with maximum bandwidth $\boldsymbol{M}$ can be recovered from its uniform samples $f(\boldsymbol{y}_{\boldsymbol{\ell}})$, $\boldsymbol{\ell} \in \mathbb{Z}^d$, as

$$f(\boldsymbol{x}) = \sum_{\boldsymbol{\ell} \in \mathbb{Z}^d} f(\boldsymbol{y}_{\boldsymbol{\ell}}) \operatorname{sinc}\big(M\pi(\boldsymbol{x} - \boldsymbol{y}_{\boldsymbol{\ell}})\big), \quad \boldsymbol{x} \in \mathbb{R}^d.$$

$\rightsquigarrow$ application to $f(\boldsymbol{x}) = \operatorname{sinc}(M\pi(\boldsymbol{x}_j - \boldsymbol{x}))$ with j fixed $+$ evaluation at $\boldsymbol{x} = \boldsymbol{x}_s$, $s = 1, \dots, N$:

$$\operatorname{sinc}\big(M\pi(\boldsymbol{x}_j - \boldsymbol{x}_s)\big) = \sum_{\boldsymbol{\ell} \in \mathbb{Z}^d} \operatorname{sinc}\big(M\pi(\boldsymbol{x}_j - \boldsymbol{y}_{\boldsymbol{\ell}})\big) \operatorname{sinc}\big(M\pi(\boldsymbol{x}_s - \boldsymbol{y}_{\boldsymbol{\ell}})\big)$$

[Pipe, Menon 99]: restriction to finitely many $\boldsymbol{\ell} \in \mathcal{I}_M$
$\Longrightarrow$ uniform truncation of Shannon series (poor approximation)

$\Rightarrow$ equivalence of [Greengard, Lee, Inati 06] and [Pipe, Menon 99] only holds asymptotically for $|\mathcal{I}_M| \to \infty$ (slow convergence)

Continuous problem – 2nd example

Setup:

- $\hat{f}(\boldsymbol{v}) = g(v_1) \cdot g(v_2)$, with triangular pulse
  $g(v) := \left(1 - \left|\frac{v}{b}\right|\right) \cdot \chi_{[-b,b]}(v)$
- $M = 64$ and $b = 24$
- log. modified polar grids of different sizes R, $T = 2R$
- relative errors $\frac{\|\tilde{\boldsymbol{h}} - \hat{\boldsymbol{f}}\|_2}{\|\hat{\boldsymbol{f}}\|_2}$

Note for $M = 64$:

- $|\mathcal{I}_{\boldsymbol{M}}| = 4096$
- $|\mathcal{I}_{\boldsymbol{2M}}| = 16384$

$\rightsquigarrow$ consider $|\mathcal{I}_{\boldsymbol{2M}}| \leq N$ and $|\mathcal{I}_{\boldsymbol{M}}| > N$

R	N	normal eqs. of 2nd kind	Frobenius norm minimization	[Greengard, Lee, Inati 06]
40	3565	4.4908e-01	1.7608e-01	2.0475e-01
48	5145	1.0886e-01	2.0690e-02	1.5829e-01
56	7149	3.6632e-02	8.0215e-03	1.5401e-01
64	9429	2.5109e-02	4.7988e-03	1.8337e-01
72	11965	7.6871e-03	4.1096e-03	2.0633e-01
80	14909	5.5991e-03	3.8507e-03	2.1932e-01
88	18153	3.8889e-03	3.9853e-03	2.2665e-01
96	21589	4.2240e-03	3.7917e-03	2.3092e-01