On regularized Shannon sampling formulas with localized sampling

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joint work with
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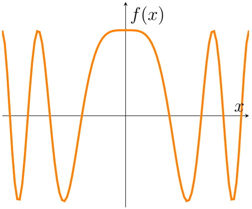
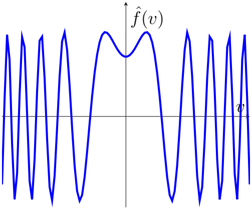
Overview

Motivation – reconstruction of functions



Fourier transform

$$\hat{f}(v) := \int_{\mathbb{R}} f(t)\, \mathrm{e}^{-2\pi \mathrm{i} t v}\, \mathrm{d}t$$



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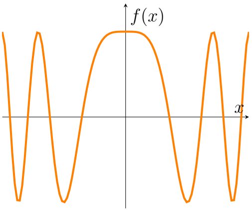

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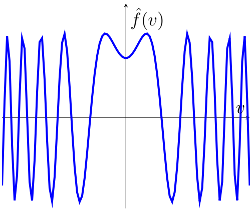
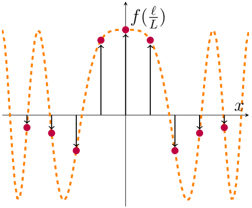
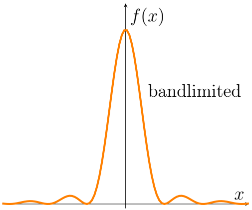




reconstruction of f

?

Motivation – reconstruction of functions



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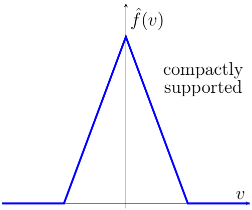
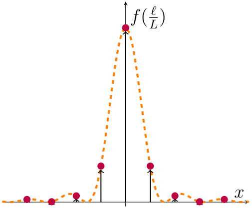
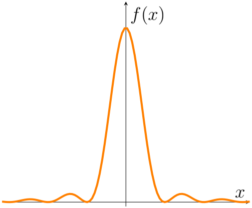




reconstruction of f

$\rightsquigarrow$ exact



Sampling theorem of Shannon–Whittaker–Kotelnikov

[Whittaker 1915], [Kotelnikov 1933], [Shannon 1949]

Let $f \in L^2(\mathbb{R})$ be bandlimited on $\left[-\frac{N}{2}, \frac{N}{2}\right]$ for some $N > 0$, i. e., its Fourier transform

$$\hat{f}(v) := \int_{\mathbb{R}} f(t)\, \mathrm{e}^{-2\pi \mathrm{i} t v}\, \mathrm{d}t$$

is supported on $\left[-\frac{N}{2}, \frac{N}{2}\right]$.
Then the function f is completely determined by its equispaced samples $f\left(\frac{\ell}{L}\right)$, $\ell \in \mathbb{Z}$, with some $L \geq N$ and it holds

$$f(t) = \sum_{\ell \in \mathbb{Z}} f\left(\tfrac{\ell}{L}\right) \operatorname{sinc}\left(L\pi\left(t - \tfrac{\ell}{L}\right)\right), \quad t \in \mathbb{R},$$

where

$$\operatorname{sinc} x := \begin{cases} \frac{\sin x}{x} & x \in \mathbb{R} \setminus \{0\}, \\ 1 & x = 0. \end{cases}$$

Localized sampling

Problem:

$$f(t) = \sum_{\ell \in \mathbb{Z}} f\big(\tfrac{\ell}{L}\big) \, \mathrm{sinc}\big(L\pi \, \big(t - \tfrac{\ell}{L}\big)\big) \,, \quad t \in \mathbb{R}$$

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Solution: truncation via localized sampling, i. e., for some $m \in \mathbb{N} \setminus \{1\}$ we consider

$$(R_{\mathrm{rect},m} f)(t) := \sum_{\ell \in \mathbb{Z}} f\left(\tfrac{\ell}{L}\right) \operatorname{sinc}\left(L\pi \left(t - \tfrac{\ell}{L}\right)\right) \mathbf{1}_{[-m/L,\, m/L]}\left(t - \tfrac{\ell}{L}\right), \quad t \in \mathbb{R} \qquad (*)$$

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Lemma (Micchelli, Xu, Zhang 09)

Let $f \in \mathcal{B}_{N/2}(\mathbb{R})$ with fixed $N \in \mathbb{N}$, $L := N(1+\lambda)$ with $\lambda \geq 0$ and $m \in \mathbb{N} \setminus \{1\}$ be given. Then it holds

$$\|f - R_{\mathrm{rect},m} f\|_{C_0(\mathbb{R})} \leq \frac{\sqrt{L}}{\pi} \sqrt{\frac{2}{m} + \frac{1}{m^2}}\, \|f\|_{L^2(\mathbb{R})}\,.$$

$\Rightarrow$ Since sinc decays slowly at infinity, $(*)$ is not a good approximation.

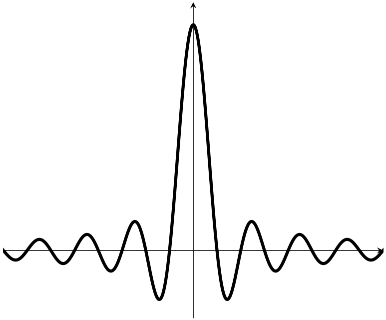
Regularized Shannon sampling formula with localized sampling

Modification: multiply sinc with a more convenient window φ, i. e.,

$$(R_{\varphi,m}f)(t) = \sum_{\ell\in\mathbb{Z}} f\big(\tfrac{\ell}{L}\big)\,\mathrm{sinc}\big(L\pi\,\big(t-\tfrac{\ell}{L}\big)\big)\,\varphi_m\big(t-\tfrac{\ell}{L}\big)\,, \quad t\in\mathbb{R}\,,$$

[Qian 03], [Lin, Zhang 17]

with $m\in\mathbb{N}\setminus\{1\}$ and $\varphi_m(x) := \varphi(x)\,\mathbf{1}_{[-m/L,\,m/L]}(x)$ with compact support



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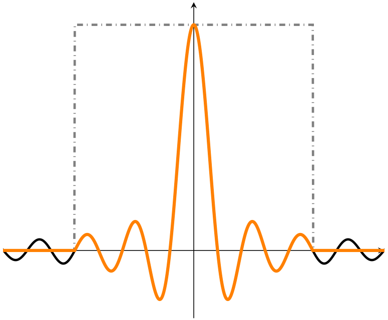

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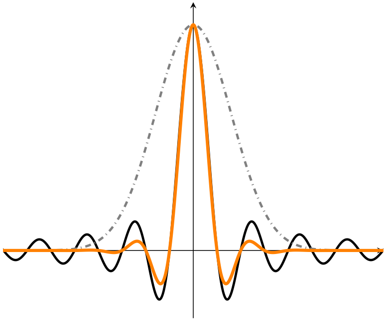

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Simplified notation:

For $t\in\big(0,\,\frac{1}{L}\big)$:

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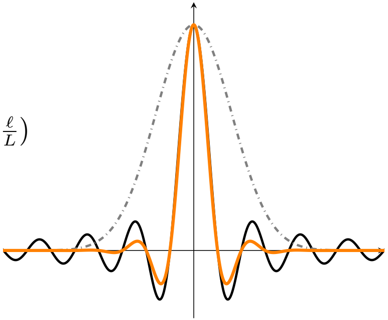

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on $\big(\frac{k}{L},\,\frac{k+1}{L}\big)$, $k\in\mathbb{Z}$

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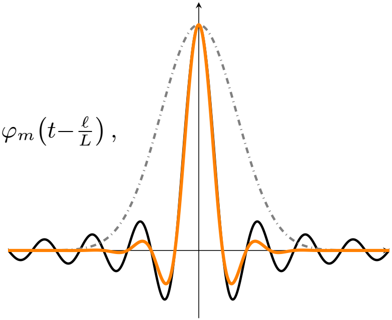

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- Gaussian window function:
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Now: propose new set of window functions φ with small support
$\rightsquigarrow$ high accuracy + fast evaluation

Window functions $\varphi : \mathbb{R} \to [0,\, 1]$

Let $L := N(1+\lambda)$, $\lambda \geq 0$, and $m \in \mathbb{N} \setminus \{1\}$ with $2m \ll L$.

We introduce a set $\Phi_{m,L}$ of window functions with the following properties:

1. $\varphi \in L^2(\mathbb{R})$ is even, positive on $(-m/L,\, m/L)$ and continuous on $\mathbb{R} \setminus \{-m/L,\, m/L\}$
2. $\varphi|_{[0,\,\infty)}$ is non-increasing with $\varphi(0) = 1$
3. the Fourier transform $\hat{\varphi}(v) := \int_{\mathbb{R}} \varphi(x)\, \mathrm{e}^{-2\pi \mathrm{i} v x}\, \mathrm{d}x$ is explicitly known

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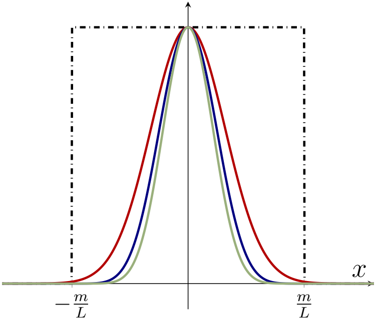

Examples:

$$\varphi_{\mathrm{rect}}(x) := \mathbf{1}_{[-m/L,\, m/L]}(x)$$

$$\varphi_{\mathrm{Gauss}}(x) := \mathrm{e}^{-x^2/(2\sigma^2)}\,,\ \sigma > 0$$

$$\varphi_{\mathrm{B}}(x) := \frac{1}{M_{2s}(0)}\, M_{2s}\left(\frac{Lxs}{m}\right)\,,\ s > 0$$

$$\varphi_{\sinh}(x) := \frac{1}{\sinh \beta}\, \sinh\left(\beta \sqrt{1 - (Lx/m)^2}\right)\,,\ \beta > 0$$

[Potts, Tasche 21]

Estimate of the uniform approximation error

Theorem (K., Potts, Tasche 22)

Let $f \in \mathcal{B}_\delta(\mathbb{R})$ *with* $\delta = \tau N$, $\tau \in (0,\, 1/2)$, $N \in \mathbb{N}$, $L = N(1+\lambda)$ *with* $\lambda \geq 0$ *and* $m \in \mathbb{N} \setminus \{1\}$. *Further let* $\varphi \in \Phi_{m,L}$ *with* $\varphi_m(x) := \varphi(x)\,\mathbf{1}_{[-m/L,\, m/L]}(x)$ *be given.*

Then it holds

$$\|f - R_{\varphi,m} f\|_{C_0(\mathbb{R})} \leq \big(E_1(m,\delta,L) + E_2(m,\delta,L)\big)\, \|f\|_{L^2(\mathbb{R})}\,,$$

where the corresponding error constants are defined by

$$E_1(m,\delta,L) := \sqrt{2\delta}\, \max_{v\in[-\delta,\delta]} \left| 1 - \int_{v-\frac{L}{2}}^{v+\frac{L}{2}} \hat{\varphi}(u)\,\mathrm{d}u \right|,$$

$$E_2(m,\delta,L) := \frac{\sqrt{2L}}{\pi m} \left(\varphi^2\big(\tfrac{m}{L}\big) + L \int_{\frac{m}{L}}^{\infty} \varphi^2(t)\,\mathrm{d}t \right)^{1/2}.$$

Proof sketch I

Only consider $t \in \left[0, \frac{1}{L}\right]$ and split the approximation error

$$f(t) - (R_{\varphi,m}f)(t) = \underbrace{f(t) - \sum_{\ell \in \mathbb{Z}} f\left(\tfrac{\ell}{L}\right) \psi\left(t - \tfrac{\ell}{L}\right)}_{\text{regularization error } e_1(t)} + \underbrace{\sum_{\ell \in \mathbb{Z}} f\left(\tfrac{\ell}{L}\right) \psi\left(t - \tfrac{\ell}{L}\right) - (R_{\varphi,m}f)(t)}_{\text{truncation error } e_{2,0}(t)}$$

with $\psi(x) := \mathrm{sinc}(L\pi x)\, \varphi(x)$.

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$$\hat{e}_1(v) = \hat{f}(v) - \left(\sum_{\ell\in\mathbb{Z}} f\left(\tfrac{\ell}{L}\right) \frac{1}{L}\,\mathrm{e}^{-2\pi\mathrm{i}v\ell/L}\right) \int_{v-L/2}^{v+L/2} \hat{\varphi}(u)\,\mathrm{d}u \qquad , \quad v \in [-\delta, \delta]$$

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Proof sketch II

(ii) Truncation error: Let $\mathcal{J}_m := \{-m+1, \dots, m\}$, then

$$e_{2,0}(t) = \sum_{\ell \in \mathbb{Z}} f\big(\tfrac{\ell}{L}\big)\, \psi\big(t - \tfrac{\ell}{L}\big) \left[1 - \mathbf{1}_{[-m/L, m/L]}\big(t - \tfrac{\ell}{L}\big)\right] = \sum_{\ell \in \mathbb{Z} \setminus \mathcal{J}_m} f\big(\tfrac{\ell}{L}\big)\, \psi\big(t - \tfrac{\ell}{L}\big)$$

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such that

$$|e_{2,0}(t)| \le \sum_{\ell \in \mathbb{Z} \setminus \mathcal{J}_m} \left|f\left(\tfrac{\ell}{L}\right)\right| \left|\mathrm{sinc}\left(L\pi \left(t - \tfrac{\ell}{L}\right)\right)\right| \varphi\left(t - \tfrac{\ell}{L}\right) \le \frac{1}{\pi m} \sum_{\ell \in \mathbb{Z} \setminus \mathcal{J}_m} \left|f\left(\tfrac{\ell}{L}\right)\right| \varphi\left(t - \tfrac{\ell}{L}\right)$$

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$$e_{2,0}(t) = \sum_{\ell \in \mathbb{Z}} f\left(\tfrac{\ell}{L}\right) \psi\left(t - \tfrac{\ell}{L}\right) \left[1 - \mathbf{1}_{[-m/L, m/L]}\left(t - \tfrac{\ell}{L}\right)\right] = \sum_{\ell \in \mathbb{Z} \setminus \mathcal{J}_m} f\left(\tfrac{\ell}{L}\right) \psi\left(t - \tfrac{\ell}{L}\right)$$

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Simplified result

This theorem can be simplified, if the window function $\varphi \in \Phi_{m,L}$

1. is continuous on $\mathbb{R}$
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Examples:

✓ B–spline

✗ Gaussian
$\rightarrow$ does not vanish on $\mathbb{R} \setminus \big[-\frac{m}{L}, \frac{m}{L}\big]$

✓ sinh-type

✗ characteristic
$\rightarrow$ not continuous on $\mathbb{R}$

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Estimate of the uniform perturbation error

Consider noisy samples $\tilde{f}_\ell := f\left(\frac{\ell}{L}\right) + \varepsilon_\ell$ with $|\varepsilon_\ell| \le \varepsilon$, $\ell \in \mathbb{Z}$, and $\varepsilon > 0$.

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Define

$$(R_{\varphi,m}\tilde{f})(t) = \sum_{\ell\in\mathbb{Z}} \tilde{f}_\ell \,\mathrm{sinc}\big(L\pi\,\big(t-\tfrac{\ell}{L}\big)\big)\,\varphi_m\big(t-\tfrac{\ell}{L}\big)\,, \quad t \in \mathbb{R}\,.$$

Theorem (K., Potts, Tasche 22)

Let $f \in \mathcal{B}_\delta(\mathbb{R})$ *with* $\delta = \tau N$, $\tau \in (0,\, 1/2)$, $N \in \mathbb{N}$, $L = N(1+\lambda)$ *with* $\lambda \ge 0$ *and* $m \in \mathbb{N} \setminus \{1\}$. *Further let* $\varphi \in \Phi_{m,L}$ *and* $\tilde{f}_\ell = f(\ell/L) + \varepsilon_\ell$, *where* $|\varepsilon_\ell| \le \varepsilon$ *for all* $\ell \in \mathbb{Z}$, *with* $\varepsilon > 0$ *be given.*

Then it holds

$$\begin{aligned} \|R_{\varphi,m}\tilde{f} - R_{\varphi,m}f\|_{C_0(\mathbb{R})} &\le \varepsilon\,\big(2 + L\,\hat{\varphi}(0)\big)\,, \\ \|f - R_{\varphi,m}\tilde{f}\|_{C_0(\mathbb{R})} &\le \|f - R_{\varphi,m}f\|_{C_0(\mathbb{R})} + \varepsilon\,\big(2 + L\,\hat{\varphi}(0)\big)\,. \end{aligned}$$

Proof sketch

Only consider $t \in \left[0,\, \frac{1}{L}\right]$ and

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By definition of Fourier transform

$$\hat{\varphi}(0) = \int_{\mathbb{R}} \varphi(t)\,\mathrm{d}t \ge \int_{-m/L}^{m/L} \varphi(t)\,\mathrm{d}t = 2\int_0^{m/L} \varphi(t)\,\mathrm{d}t\,,$$

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and therefore

$$|\tilde{e}_0(t)| \le 2\,\varepsilon \left(1 + L \int_0^{(m-1)/L} \varphi(t)\,\mathrm{d}t\right) \le 2\,\varepsilon\left(1 + \tfrac{L}{2}\,\hat{\varphi}(0)\right) = \varepsilon\left(2 + L\,\hat{\varphi}(0)\right).$$

□

Results for special window functions

$$\|R_{\varphi,m}\tilde{f} - R_{\varphi,m}f\|_{C_0(\mathbb{R})} \le \varepsilon \big(2 + L\,\hat{\varphi}(0)\big)$$

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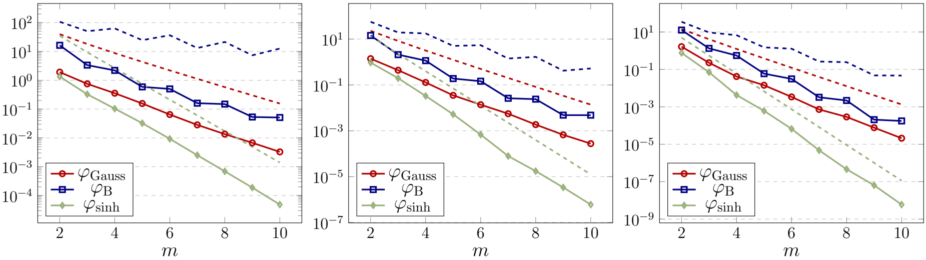
Numerical example – comparison of window functions φ



(a) $\lambda = 0.5$ (b) $\lambda = 1$ (c) $\lambda = 2$

Figure: Maximum approximation error (solid) and error constant (dashed) for $f(x) = \delta\,\mathrm{sinc}^2(\delta\pi x)$ with $N = 256$, $\tau = 0.45$, $\delta = \tau N$, as well as $m \in \{2, 3, \ldots, 10\}$, and $\lambda \in \{0.5, 1, 2\}$.

$\Rightarrow$ small $m \in \mathbb{N}$ sufficient for high precision

$\Rightarrow$ fast algorithms with $\mathcal{O}(2m)$ flops

Summary

- overcome drawbacks of Shannon series (poor convergence, non-robustness)
- proposed new window functions with compact support (B–spline, sinh-type)
- general setting: unified approach to error estimates
  unified approach to numerical robustness
- special windows: uniform approximation error $\sim \mathcal{O}(\mathrm{e}^{-m})$
  uniform perturbation error $\sim \mathcal{O}(\sqrt{m})$
- seen superiority of new sinh-type (small $m \in \mathbb{N}$ for high precision in $\mathcal{O}(2m)$ flops)

- **K., Potts, Tasche: On regularized Shannon sampling formulas with localized sampling**. arXiv:2203.09973, 2022.

Summary

- overcome drawbacks of Shannon series (poor convergence, non-robustness)
- proposed new window functions with compact support (B–spline, sinh-type)
- general setting: unified approach to error estimates
  unified approach to numerical robustness
- special windows: uniform approximation error $\sim \mathcal{O}(\mathrm{e}^{-m})$
  uniform perturbation error $\sim \mathcal{O}(\sqrt{m})$
- seen superiority of new sinh-type (small $m \in \mathbb{N}$ for high precision in $\mathcal{O}(2m)$ flops)

- K., Potts, Tasche: **On regularized Shannon sampling formulas with localized sampling**. arXiv:2203.09973, 2022.

Thank you for your attention!