Nonuniform fast Fourier transforms and the fast sinc transform

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joint work with D. Potts and M. Tasche

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Overview

Theorem (Sampling Theorem of Shannon-Whittaker-Kotelnikov)

Let $f \in L_1(\mathbb{R}) \cap C(\mathbb{R})$ be bandlimited on $[-\frac{L}{2}, \frac{L}{2}]$ for some $L > 0$, i. e., the Fourier transform of f is supported on $[-\frac{L}{2}, \frac{L}{2}]$. Then for $N \in 2\mathbb{N}$ with $N \geq L$, the function f is completely determined by its values $f\left(\frac{k}{N}\right)$, $k \in \mathbb{Z}$, and further f can be represented in the form

$$f(x) = \sum_{k\in\mathbb{Z}} f\left(\tfrac{k}{N}\right) \operatorname{sinc}\left(N\pi\left(x - \tfrac{k}{N}\right)\right), \quad x \in \mathbb{R},$$

with

$$\operatorname{sinc}(N\pi x) := \begin{cases} \frac{\sin(N\pi x)}{N\pi x} & x \in \mathbb{R} \setminus \{0\}, \\ 1 & x = 0, \end{cases}$$

where the series converges absolutely and uniformly on $\mathbb{R}$.

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$\Rightarrow$ Truncation of the series:

$$\sum_{k\in I_{L_1}} f\left(\tfrac{k}{N}\right) \operatorname{sinc}\left(N\pi\left(x - \tfrac{k}{N}\right)\right), \quad x \in \mathbb{R}$$

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$\Rightarrow$ Reconstruction of an $h\colon [-\frac{1}{2}, \frac{1}{2}] \to \mathbb{C}$ at arbitrary points $a_k \in \left[-\frac{1}{2}, \frac{1}{2}\right]$:

$$h(x) := \sum_{k\in I_{L_1}} c_k \operatorname{sinc}\left(N\pi\,(x - a_k)\right), \quad x \in \mathbb{R},\ c_k \in \mathbb{C}$$

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$\Rightarrow$ *Discrete sinc transform*:

$$h(b_\ell) = \sum_{k \in I_{L_1}} c_k \operatorname{sinc}\left(N\pi\,(b_\ell - a_k)\right), \quad b_\ell \in \left[-\tfrac{1}{2}, \tfrac{1}{2}\right], \ell \in I_{L_2}$$

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Motivation: numerous applications in signal processing and image reconstruction, e. g. MRI

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Previous approaches:

[Greengard, Lee, Inati 06]: Gauss–Legendre quadrature & nonequispaced fast Fourier transforms (NNFFTs)
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Now: approximation of sinc function by exponential sum & NNFFTs
$\Rightarrow$ error estimate

Fast Fourier transforms

Algorithms to evaluate a trigonometric polynomial

$$f(x) = \sum_{k \in \mathcal{I}_N} \hat{f}_k \, \mathrm{e}^{-2\pi \mathrm{i} k x}, \quad x \in [-\tfrac{1}{2}, \tfrac{1}{2}]$$

Notation: index set $\mathcal{I}_N := \left\{-\frac{N}{2}, \dots, \frac{N}{2} - 1\right\}$ for $N \in 2\mathbb{N}$

[Dutt, Rokhlin 93], [Beylkin 95],
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(inverse) FFT: $$f(\tfrac{j}{N}) := \sum_{k \in \mathcal{I}_N} \hat{f}_k \, \mathrm{e}^{-2\pi \mathrm{i} k \frac{j}{N}} \qquad j \in \mathcal{I}_N$$

Complexity: $\mathcal{O}(N \log N \qquad)$

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NFFT: $$f(x_j) := \sum_{k \in \mathcal{I}_N} \hat{f}_k \, \mathrm{e}^{-2\pi \mathrm{i} k x_j} \qquad x_j \in [-\tfrac{1}{2}, \tfrac{1}{2}], \; j \in \mathcal{I}_{M_1}$$

Complexity: $\mathcal{O}(N \log N + M_1 \qquad)$

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NNFFT: $f(x_j) := \sum_{k \in \mathcal{I}_{M_2}} \hat{f}_k \, \mathrm{e}^{-2\pi \mathrm{i} N v_k x_j}$ $\quad v_k, x_j \in [-\tfrac{1}{2}, \tfrac{1}{2}],\ j \in \mathcal{I}_{M_1}$

Complexity: $\mathcal{O}(N \log N + M_1 + M_2)$

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Approximation of the sinc function

Theorem (Beylkin, Monzón 02)

Let $\varepsilon > 0$ *be a given target accuracy. Then for sufficiently large* $n \in \mathbb{N}$ *with* $n \geq 4N$*, there exist constants* $w_j > 0$ *and frequencies* $z_j \in \left(-\frac{1}{2}, \frac{1}{2}\right)$*,* $j = 0, \dots, n$*, such that*

$$\left|\operatorname{sinc}(\pi N x) - \sum_{j=0}^{n} w_j \,\mathrm{e}^{-2\pi\mathrm{i}\, N\, z_j\, x}\right| \leq \varepsilon\,, \quad x \in [-1, 1]\,.$$

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In practice:

- [Beylkin, Monzón 02]: prony's method
- Now: simplify approximation procedure
  assume given n and z_j $\rightsquigarrow$ find explicit w_j $\rightsquigarrow$ show error result

Quadrature approach

$$\mathrm{sinc}(N\pi x) = \frac{1}{2}\int_{-1}^{1} \mathrm{e}^{-\pi \mathrm{i} N t x}\,\mathrm{d}t \approx \sum_{j=0}^{n} w_j\,\mathrm{e}^{-\pi \mathrm{i} N z_j x} \quad \text{for fixed} \quad x \in [-1,\, 1]\,.$$

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Clenshaw–Curtis quadrature:
Use Chebyshev points $z_j = \cos\left(\frac{j\pi}{n}\right) \in [-1,1]$, $j = 0,\dots,n$, and approximation

$$\int_{-1}^{1} f(x)\,\mathrm{d}x \approx \int_{-1}^{1} p_n(x)\,\mathrm{d}x,$$

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$$p_n(x) = \sum_{k=0}^{n} \varepsilon_n(k)^2\, a_k^{(n)}[f]\, T_k(x)$$

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with the positive coefficients

$$w_j = \begin{cases} \frac{1}{n}\,\varepsilon_n(j)^2 \sum_{k=0}^{n/2} \varepsilon_n(2k)^2 \frac{2}{1-4k^2}\,\cos\left(\frac{2kj\pi}{n}\right) & n \in 2\mathbb{N}\,, \\ \frac{1}{n}\,\varepsilon_n(j)^2 \sum_{k=0}^{(n-1)/2} \varepsilon_n(2k)^2 \frac{2}{1-4k^2}\,\cos\left(\frac{2kj\pi}{n}\right) & n \in 2\mathbb{N}+1\,, \end{cases}$$

and $\varepsilon_n(0) = \varepsilon_n(n) := \frac{\sqrt{2}}{2}$, $\varepsilon_n(k) := 1$, $k = 1, \ldots, n-1$.
Further the coefficients fulfill the condition $\sum_{k=0}^{n} w_k = 1$.

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Theorem

Let $n \in 2\mathbb{N}$ *with* $n \geq 4N$, $z_j = \cos\left(\frac{\pi j}{n}\right)$, $j = 0, \dots, n$, *and* w_j *as above. Then for all* $x \in [-1,\,1]$ *it holds*

$$\left|\mathrm{sinc}(N\pi x) - \sum_{j=0}^{n} w_j\,\mathrm{e}^{-\pi \mathrm{i} N\, z_j x}\right| \leq \frac{144}{70\,(\mathrm{e}^2-1)}\,\mathrm{e}^{-n}\,\cosh\left(\frac{\pi\,(\mathrm{e}^2-1)\,N}{2\,\mathrm{e}}\right).$$

Proof of error estimate of Clenshaw-Curtis quadrature

It holds

$$\operatorname{sinc}(N\pi x) = \frac{1}{2}\int_{-1}^{1} \mathrm{e}^{-\pi \mathrm{i} N t x}\, \mathrm{d}t = \frac{1}{2}\int_{-1}^{1} \cos(\pi N t x)\, \mathrm{d}t\,.$$

Note that $\sum_{k=0}^{n} w_k\, \mathrm{e}^{-\pi \mathrm{i} N\, z_k x} = \sum_{k=0}^{n} w_k\, \cos(\pi N\, z_k x) + 0$ by the symmetry properties $z_k = -z_{n-k}$ and $w_k = w_{n-k}$, $k = 0, \ldots, n$.

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By [Trefethen 13] we have the error estimate

$$\Big|\operatorname{sinc}(N\pi x) - \sum_{j=0}^{n} w_j\, \mathrm{e}^{-\pi \mathrm{i} N\, z_j x}\Big| \le \frac{144}{35}\, \frac{M \cdot \rho^{-n}}{\rho^2 - 1}\,,$$

if $|f(z)| \le M$ holds for f extended to the Bernstein ellipse

$$E_\rho := \Big\{ z \in \mathbb{C} :\ \operatorname{Re} z = \tfrac{1}{2}\big(\rho + \tfrac{1}{\rho}\big)\,\cos t\,,\ \operatorname{Im} z = \tfrac{1}{2}\big(\rho - \tfrac{1}{\rho}\big)\,\sin t\,,\ t \in [0,\, 2\pi) \Big\}\,.$$

with some $\rho > 1$.

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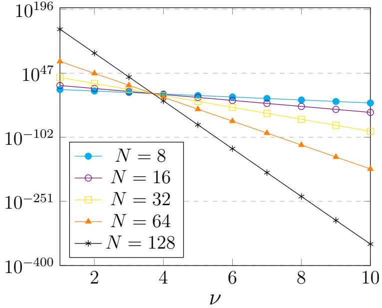
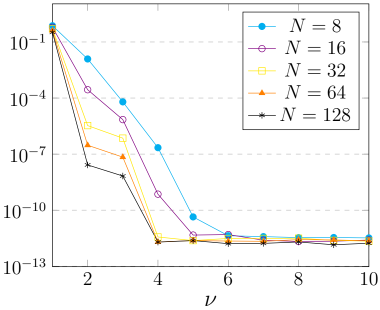
with some $\rho > 1$. For simplicity, we choose $\rho = \mathrm{e}$. Here we have

$$\left|\tfrac{1}{2}\,\cos(N\pi x z)\right| \le \tfrac{1}{2}\,\cosh\left(N\pi x\,\operatorname{Im}(z)\right) \le \tfrac{1}{2}\,\cosh\left(N\pi\,(\mathrm{e}-\mathrm{e}^{-1})/2\right)\,.$$

Error results for $n = \nu N, \nu \in \{1, \dots, 10\}$



(a) Error constant



(b) Maximum error

Fast sinc transform

Given:

$$\operatorname{sinc}(N\pi x) \approx \sum_{j=0}^{n} w_j \, \mathrm{e}^{-\pi \mathrm{i} N \, z_j x}$$

Aim:

$$h(b_\ell) = \sum_{k \in I_{L_1}} c_k \operatorname{sinc}\big(N\pi \, (b_\ell - a_k)\big) \, , \quad \ell \in I_{L_2} \, ,$$

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$$\operatorname{sinc}\big(N\pi(b_\ell - a_k)\big) \approx \sum_{j=0}^{n} w_j \, \mathrm{e}^{-\pi \mathrm{i}\, N z_j (b_\ell - a_k)}$$

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$$\Rightarrow \qquad h(b_\ell) \approx h_\ell := \sum_{k \in I_{L_1}} c_k \sum_{j=0}^{n} w_j \, \mathrm{e}^{-\pi \mathrm{i}\, N z_j (b_\ell - a_k)}$$

Fast sinc transform

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NNFFT

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$$= \sum_{j=0}^{n} w_j \cdot \tilde{g}_j \, \mathrm{e}^{\pi \mathrm{i}\, N z_j b_\ell}\,, \quad \ell \in I_{L_2}\,.$$

NNFFT $\rightsquigarrow$ multiplication by w_j

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NNFFT ⇝ multiplication by w_j ⇝ NNFFT

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Complexity: $\mathcal{O}(N \log N + L_1 + L_2 + 2n)$

Error estimate of the fast sinc transform

Consider approximation

$$\tilde{h}_\ell \approx h(b_\ell) = \sum_{k\in I_{L_1}} c_k \operatorname{sinc}\big(N\pi\,(b_\ell - a_k)\big)\,, \quad \ell \in I_{L_2}\,.$$

Then the error of the fast sinc transform can be estimated by

$$\max_{\ell\in I_{L_2}} |h(b_\ell) - \tilde{h}_\ell| \le \big(\varepsilon + 2E_{\mathrm{NNFFT}} + E_{\mathrm{NNFFT}}^2\big) \sum_{k\in I_{L_1}} |c_k|,$$

when $n \in 2\mathbb{N}$, $n \ge 4N$, is chosen such that

$$\frac{144}{70\,(\mathrm{e}^2-1)}\,\mathrm{e}^{-n}\,\cosh\left(\frac{\pi\,(\mathrm{e}^2-1)\,N}{2\,\mathrm{e}}\right) \le \varepsilon.$$

Numerical experiment

Setting:

- random nodes $a_k \in \left[-\frac{1}{2}, \frac{1}{2}\right]$, $k \in I_{L_1}$,
- equispaced points $b_\ell = \frac{\ell}{N}$ with $\ell \in I_N$,
- random coefficients $c_k \in \mathbb{C}$, $k \in I_{L_1}$,
- bandwidths $N = 2^k$, $k = 5, \ldots, 13$,
- $L_1 = \frac{N}{2}$.

Compute:

- discrete sinc transform $h(b_\ell) = \sum_{k \in I_{L_1}} c_k \operatorname{sinc}\big(N\pi\,(b_\ell - a_k)\big)\,, \quad \ell \in I_{L_2}\,,$

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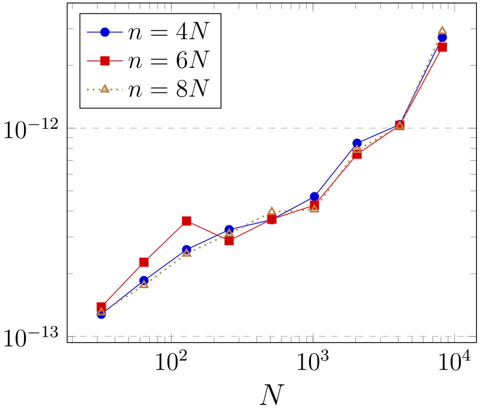
Only have to examine choice of $n \geq 4N$.

Results



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Thank you for your attention!