Direct inversion of the NFFT

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Workshop on Mathematical Signal and Image Analysis
Raitenhaslach, April 1-3, 2019

Overview

Nonequispaced fast Fourier transform (NFFT)

Fast algorithm to evaluate a trigonometric polynomial

$$f(x) = \sum_{k=-\frac{M}{2}}^{\frac{M}{2}-1} \hat{f}_k \, \mathrm{e}^{2\pi \mathrm{i} k x}$$

at nonequispaced nodes $y_j \in \left[-\frac{1}{2}, \frac{1}{2}\right), \, j = 1, \dots, N$

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- [Nieslony, Steidl 03]: Minimization of approximation error by

$$\underset{\boldsymbol{B}\in\mathbb{R}^{N\times\sigma M}:\ \boldsymbol{B}\ (2m+1)\text{-sparse}}{\text{Minimize}} \quad \|\boldsymbol{A} - \boldsymbol{B}\boldsymbol{F}\boldsymbol{D}\|_{\mathrm{F}}^2$$

iNFFT

Given: $f_j = f(y_j)$ for $f(x) = \sum_{k=-\frac{M}{2}}^{\frac{M}{2}-1} \hat{f}_k \,\mathrm{e}^{2\pi\mathrm{i}kx}$

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various applications: MRI, solution of PDEs, …

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- iterative: [Feichtinger, Gröchenig 95]: CG algorithm, $M < N$
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- frame-theoretic: [Gelb, Song 13/14],[Davis, Gelb, Song 16], $M < N$

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Now: new direct method for the general case

Direct inversion of the NFFT
Inverse NFFT

Problems:

(1) Solve

$$\boldsymbol{A}\hat{\boldsymbol{f}} = \boldsymbol{f},$$

given: $\boldsymbol{f}$, find: $\hat{\boldsymbol{f}}$. ⇒ inverse NFFT

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given: $\boldsymbol{f}$, find: $\hat{\boldsymbol{f}}$. $\Rightarrow$ inverse NFFT

(2) Solve

$$\boldsymbol{A}^*\boldsymbol{f} = \boldsymbol{h},$$

given: $\boldsymbol{h}$, find: $\boldsymbol{f}$. $\Rightarrow$ inverse adjoint NFFT

Quadratic setting $M = N$

For given evaluations of a trigonometric polynomial

$$f_j := f(y_j) = \sum_{k=-\frac{N}{2}}^{\frac{N}{2}-1} \hat{f}_k \, \mathrm{e}^{2\pi \mathrm{i} k y_j}, \quad j = 1, \dots, N,$$

and

$$g_l := f(x_l) = \sum_{k=-\frac{N}{2}}^{\frac{N}{2}-1} \hat{f}_k \, \mathrm{e}^{2\pi \mathrm{i} k x_l}, \quad l = 1, \dots, N,$$

at different nodes $x_l, y_j \in \left[-\frac{1}{2}, \frac{1}{2}\right)$

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at different nodes $x_l, y_j \in \left[-\frac{1}{2}, \frac{1}{2}\right)$ it holds

$$g_l = a_l \sum_{j=1}^{N} f_j \, b_j \left(\frac{1}{\tan(\pi(x_l - y_j))} - \mathrm{i} \right), \quad l = 1, \dots, N,$$

with

$$a_l = \prod_{n=1}^{N} \sin(\pi(x_l - y_n)) \quad \text{and} \quad b_j = \prod_{\substack{n=1 \\ n \neq j}}^{N} \frac{1}{\sin(\pi(y_j - y_n))}, \quad l, j = 1, \dots, N.$$

Inverse NFFT - quadratic setting

Choose equispaced x_l.

$\Rightarrow$ Fast algorithm:

1. Compute g_l by means of fast summation.
2. Compute

$$\check{f}_k = \frac{1}{N} \sum_{l=1}^{N} g_l \, \mathrm{e}^{-2\pi \mathrm{i} k x_l}, \quad k = -\tfrac{N}{2}, \dots, \tfrac{N}{2} - 1,$$

   by means of an FFT.

Output: $\check{f}_k \approx \hat{f}_k$

Complexity: $\mathcal{O}(N \log N)$

Additionally: stabilization to prevent overflow/underflow in a_l and b_j

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$\Rightarrow$ inverse NFFT as well as inverse adjoint NFFT

Basic idea

Consider equispaced nodes $y_j = \frac{j}{N} \in \left[-\frac{1}{2}, \frac{1}{2}\right), \, j = -\frac{N}{2}, \dots, \frac{N}{2} - 1.$

$$\Rightarrow \quad \boldsymbol{A} = \left(\mathrm{e}^{2\pi\mathrm{i}k\frac{j}{N}}\right)_{j=-\frac{N}{2},\, k=-\frac{M}{2}}^{\frac{N}{2}-1,\ \frac{M}{2}-1} \quad \text{and} \quad \boldsymbol{A}^* = \left(\mathrm{e}^{-2\pi\mathrm{i}k\frac{j}{N}}\right)_{k=-\frac{M}{2},\, j=-\frac{N}{2}}^{\frac{M}{2}-1,\ \frac{N}{2}-1}$$

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Consider matrix products:

$$\boldsymbol{A}^*\boldsymbol{A} = N\boldsymbol{I}_M \text{ for } M \leq N$$

$$\boldsymbol{A}\boldsymbol{A}^* = M\boldsymbol{I}_N \text{ for } M \geq N \text{ with } N \mid M$$

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- $\rightsquigarrow$ approximation of the form $\boldsymbol{A}\boldsymbol{D}^*\boldsymbol{F}^*\boldsymbol{B}^* \approx M\boldsymbol{I}_N$

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- ⇝ approximation of the form $\boldsymbol{A}\boldsymbol{D}^*\boldsymbol{F}^*\boldsymbol{B}^* \approx M\boldsymbol{I}_N$
- ⇝ modification of matrix $\boldsymbol{B}$

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- ⇝ approximation of the form $\boldsymbol{A}\boldsymbol{D}^*\boldsymbol{F}^*\boldsymbol{B}^* \approx M\boldsymbol{I}_N$
- ⇝ modification of matrix $\boldsymbol{B}$
- ⇝ preserve band structure and arithmetic complexity

Inverse NFFT - underdetermined setting $M > N$

Assume

$$\boldsymbol{A}\boldsymbol{D}^*\boldsymbol{F}^*\boldsymbol{B}^* \approx M\boldsymbol{I}_N$$

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$$\boldsymbol{A}\boldsymbol{D}^*\boldsymbol{F}^*\boldsymbol{B}^* \approx M\boldsymbol{I}_N \iff \frac{1}{M}\,\boldsymbol{A}\boldsymbol{D}^*\boldsymbol{F}^*\boldsymbol{B}^*\boldsymbol{f} \approx \boldsymbol{f} \quad \forall \boldsymbol{f} \in \mathbb{C}^N$$

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$$\iff \boldsymbol{A}\check{\boldsymbol{f}} \approx \boldsymbol{f} \quad \forall \boldsymbol{f} \in \mathbb{C}^N$$

Since $\boldsymbol{A}\hat{\boldsymbol{f}} = \boldsymbol{f}$ that means $\check{\boldsymbol{f}} \approx \hat{\boldsymbol{f}}$.

$\Rightarrow$ Reconstruction of Fourier coefficients $\hat{f}_k$

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Optimization problem:

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$\Rightarrow$ inverse NFFT as well as inverse adjoint NFFT

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Numerical examples - quadratic setting

- trigonometric polynomial with arbitrary Fourier coefficients $\hat{f}_k \in [1, 100]$
- jittered equispaced nodes
- absolute and relative errors per node, $r \in \{2, \infty\}$,

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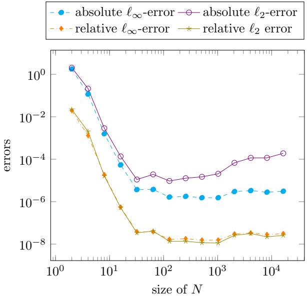

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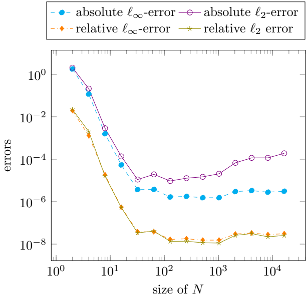

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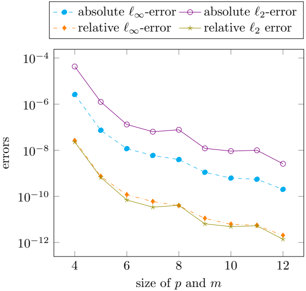

(b) $N = 1024$ and $m = p = c$, $c = 4, \dots, 12$.

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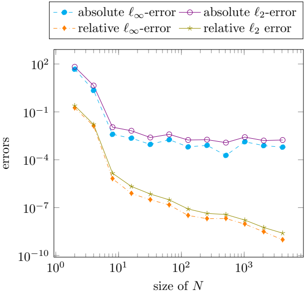

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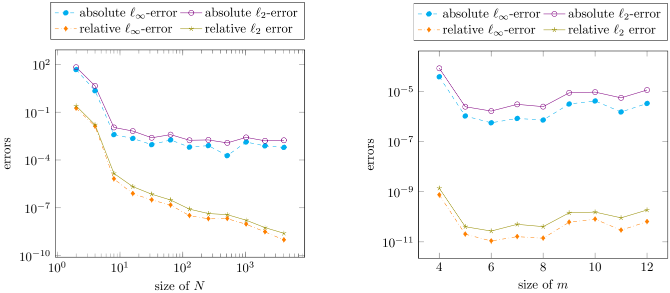

(a) $N = 2^c$, $c = 1, \dots, 12$, $M = 4N$ and $m = 4$.

(b) $N = 512$, $M = 2048$ and $m = 4, \dots, 12$.

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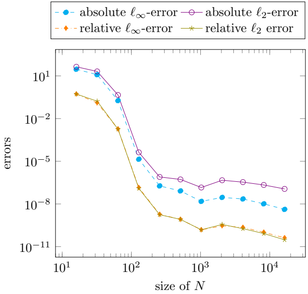

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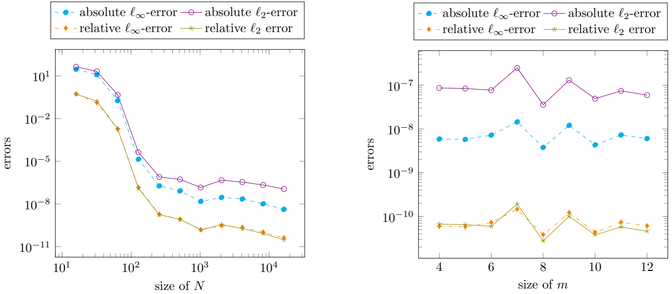

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