A frame-theoretical approach to the inversion of the NFFT

Melanie Kircheis

TU Chemnitz, Faculty of Mathematics

Mecklenburg Workshop
Approximation Methods and Fast Algorithms
Hasenwinkel, September 10-14, 2018

Overview

Nonequispaced fast Fourier transform

Fast algorithm to evaluate a trigonometric polynomial

$$f(x) = \sum_{k=-\frac{M}{2}}^{\frac{M}{2}-1} \hat{f}_k \, \mathrm{e}^{2\pi \mathrm{i} k x}$$

at nonequispaced nodes $x_j \in \left[-\frac{1}{2}, \frac{1}{2}\right), \; j = 1, \dots, N$

[Dutt, Rokhlin 93], [Beylkin 95],
[Potts, Steidl, Tasche 01]

Nonequispaced fast Fourier transform

Fast algorithm to evaluate a trigonometric polynomial

$$f(x) = \sum_{k=-\frac{M}{2}}^{\frac{M}{2}-1} \hat{f}_k \, \mathrm{e}^{2\pi \mathrm{i} k x}$$

at nonequispaced nodes $x_j \in \left[-\frac{1}{2}, \frac{1}{2}\right)$, $j = 1, \ldots, N$

- **Complexity:** $\mathcal{O}(M \log M + N)$

[Dutt, Rokhlin 93], [Beylkin 95],
[Potts, Steidl, Tasche 01]

Nonequispaced fast Fourier transform

Fast algorithm to evaluate a trigonometric polynomial

$$f(x) = \sum_{k=-\frac{M}{2}}^{\frac{M}{2}-1} \hat{f}_k \, \mathrm{e}^{2\pi \mathrm{i} k x}$$

at nonequispaced nodes $x_j \in \left[-\frac{1}{2}, \frac{1}{2}\right)$, $j = 1, \dots, N$

- **Complexity:** $\mathcal{O}(M \log M + N)$
- equispaced nodes x_j and $M = N \quad \Rightarrow \quad$ **FFT:** $\mathcal{O}(N \log N)$

[Dutt, Rokhlin 93], [Beylkin 95],
[Potts, Steidl, Tasche 01]

Nonequispaced fast Fourier transform

Fast algorithm to evaluate a trigonometric polynomial

$$f(x) = \sum_{k=-\frac{M}{2}}^{\frac{M}{2}-1} \hat{f}_k \, \mathrm{e}^{2\pi \mathrm{i} k x}$$

at nonequispaced nodes $x_j \in \left[-\frac{1}{2}, \frac{1}{2}\right)$, $j = 1, \dots, N$

- **Complexity:** $\mathcal{O}(M \log M + N)$
- equispaced nodes x_j and $M = N \quad \Rightarrow \quad$ **FFT:** $\mathcal{O}(N \log N)$
- Ajoint problem:

$$h_k = \sum_{j=1}^{N} f_j \, \mathrm{e}^{-2\pi \mathrm{i} k x_j}, \quad k = -\frac{M}{2}, \dots, \frac{M}{2} - 1.$$

[Dutt, Rokhlin 93], [Beylkin 95],
[Potts, Steidl, Tasche 01]

Nonequispaced fast Fourier transform

Fast algorithm to evaluate a trigonometric polynomial

$$f(x) = \sum_{k=-\frac{M}{2}}^{\frac{M}{2}-1} \hat{f}_k \,\mathrm{e}^{2\pi\mathrm{i}kx} \qquad \rightsquigarrow \qquad \boldsymbol{A} := \left(\mathrm{e}^{2\pi\mathrm{i}kx_j}\right)_{j=1,\,k=-\frac{M}{2}}^{N,\ \frac{M}{2}-1}$$

at nonequispaced nodes $x_j \in \left[-\frac{1}{2}, \frac{1}{2}\right)$, $j = 1, \dots, N$

- **Complexity:** $\mathcal{O}(M \log M + N)$
- equispaced nodes x_j and $M = N \quad \Rightarrow \quad$ **FFT:** $\mathcal{O}(N \log N)$
- Ajoint problem:

$$h_k = \sum_{j=1}^{N} f_j \,\mathrm{e}^{-2\pi\mathrm{i}kx_j}, \quad k = -\tfrac{M}{2}, \dots, \tfrac{M}{2} - 1.$$

- Factorizations: $\boldsymbol{A} \approx \underset{\substack{\nearrow\\ \text{sparse}}}{\boldsymbol{B}}\underset{\substack{\uparrow\\ \text{FFT}}}{\boldsymbol{F}}\underset{\substack{\nwarrow\\ \text{diagonal}}}{\boldsymbol{D}}$ and $\boldsymbol{A}^* \approx \boldsymbol{D}^*\boldsymbol{F}^*\boldsymbol{B}^*$

[Dutt, Rokhlin 93], [Beylkin 95],
[Potts, Steidl, Tasche 01]

Nonequispaced fast Fourier transform

Fast algorithm to evaluate a trigonometric polynomial

$$f(x) = \sum_{k=-\frac{M}{2}}^{\frac{M}{2}-1} \hat{f}_k \, \mathrm{e}^{2\pi \mathrm{i} k x} \qquad \rightsquigarrow \qquad \boldsymbol{A} := \left(\mathrm{e}^{2\pi \mathrm{i} k x_j}\right)_{j=1,\, k=-\frac{M}{2}}^{N,\ \frac{M}{2}-1}$$

at nonequispaced nodes $x_j \in \left[-\frac{1}{2}, \frac{1}{2}\right)$, $j = 1, \dots, N$

- **Complexity:** $\mathcal{O}(M \log M + N)$
- equispaced nodes x_j and $M = N \quad \Rightarrow \quad$ **FFT:** $\mathcal{O}(N \log N)$
- Ajoint problem:

$$h_k = \sum_{j=1}^{N} f_j \, \mathrm{e}^{-2\pi \mathrm{i} k x_j}, \quad k = -\frac{M}{2}, \dots, \frac{M}{2} - 1.$$

- Factorizations: $\boldsymbol{A} \approx \underset{\substack{\nearrow \\ \text{sparse}}}{\boldsymbol{B}}\underset{\substack{\uparrow \\ \text{FFT}}}{\boldsymbol{F}}\underset{\substack{\nwarrow \\ \text{diagonal}}}{\boldsymbol{D}}$ and $\boldsymbol{A}^* \approx \boldsymbol{D}^*\boldsymbol{F}^*\boldsymbol{B}^*$ [Dutt, Rokhlin 93], [Beylkin 95], [Potts, Steidl, Tasche 01]
- [Nieslony, Steidl 03]: Minimization of approximation error by

$$\operatorname*{Minimize}_{\boldsymbol{B}\in\mathbb{R}^{N\times\sigma M}:\ \boldsymbol{B}\ (2m+1)\text{-sparse}} \|\boldsymbol{A} - \boldsymbol{B}\boldsymbol{F}\boldsymbol{D}\|_{\mathrm{F}}^2$$

iNFFT

Given: $f(x_j), j = 1, \dots, N,$ for $f(x) = \sum_{k=-\frac{M}{2}}^{\frac{M}{2}-1} \hat{f}_k \,\mathrm{e}^{2\pi \mathrm{i} k x}$

Find: $\hat{f}_k \in \mathbb{C},\; k = -\frac{M}{2}, \dots, \frac{M}{2} - 1$

iNFFT

Given: $f(x_j), j = 1, \dots, N,$ for $f(x) = \sum\limits_{k=-\frac{M}{2}}^{\frac{M}{2}-1} \hat{f}_k \, \mathrm{e}^{2\pi \mathrm{i} k x}$

Find: $\hat{f}_k \in \mathbb{C},\ k = -\frac{M}{2}, \dots, \frac{M}{2} - 1$

Motivation: FFT is invertible
various applications: MRI, solution of PDEs, …

iNFFT

Given: $f(x_j), j = 1, \dots, N,$ for $f(x) = \sum_{k=-\frac{M}{2}}^{\frac{M}{2}-1} \hat{f}_k \, \mathrm{e}^{2\pi \mathrm{i} k x}$

Find: $\hat{f}_k \in \mathbb{C},\ k = -\frac{M}{2}, \dots, \frac{M}{2} - 1$

Motivation: FFT is invertible
various applications: MRI, solution of PDEs, ...

Problem: in general $N \neq M$

iNFFT

Given: $f(x_j), j = 1, \dots, N,$ for $f(x) = \displaystyle\sum_{k=-\frac{M}{2}}^{\frac{M}{2}-1} \hat{f}_k \, \mathrm{e}^{2\pi \mathrm{i} k x}$

Find: $\hat{f}_k \in \mathbb{C},\ k = -\frac{M}{2}, \dots, \frac{M}{2} - 1$

Motivation: FFT is invertible
various applications: MRI, solution of PDEs, …

Problem: in general $N \neq M$

Previous approaches:

- iterative methods: [Feichtinger, Gröchenig 95]: CG algorithm, $N > M$
  [Kunis, Potts 07]: CG algorithm and NFFT, $N < M$
- direct method for $N = M$: [Dutt, Rokhlin 93]: Lagrange interpolation and FMM
  [K. 17]: fast summation method
- frame-theoretical approach: [Gelb, Song 13/14],[Davis, Gelb, Song 16], $N > M$

iNFFT

Given: $f(x_j), j = 1, \dots, N,$ for $f(x) = \sum_{k=-\frac{M}{2}}^{\frac{M}{2}-1} \hat{f}_k \, \mathrm{e}^{2\pi \mathrm{i} k x}$

Find: $\hat{f}_k \in \mathbb{C},\ k = -\frac{M}{2}, \dots, \frac{M}{2} - 1$

Motivation: FFT is invertible
various applications: MRI, solution of PDEs, . . .

Problem: in general $N \neq M$

Previous approaches:

- iterative methods: [Feichtinger, Gröchenig 95]: CG algorithm, $N > M$
  [Kunis, Potts 07]: CG algorithm and NFFT, $N < M$
- direct method for $N = M$: [Dutt, Rokhlin 93]: Lagrange interpolation and FMM
  [K. 17]: fast summation method
- frame-theoretical approach: [Gelb, Song 13/14],[Davis, Gelb, Song 16], $N > M$

Now: new direct method for general case and connection to frame approach

Basic idea

Consider equispaced nodes

$$x_j = \tfrac{j}{N} \in \left[-\tfrac{1}{2}, \tfrac{1}{2}\right), \, j = -\tfrac{N}{2}, \dots, \tfrac{N}{2} - 1.$$

Obtain

$$\boldsymbol{A} = \left(\mathrm{e}^{2\pi \mathrm{i} k \frac{j}{N}}\right)_{j=-\frac{N}{2},\, k=-\frac{M}{2}}^{\frac{N}{2}-1,\ \frac{M}{2}-1} \quad \text{and} \quad \boldsymbol{A}^* = \left(\mathrm{e}^{-2\pi \mathrm{i} k \frac{j}{N}}\right)_{k=-\frac{M}{2},\, j=-\frac{N}{2}}^{\frac{M}{2}-1,\ \frac{N}{2}-1}.$$

Basic idea

Consider equispaced nodes

$$x_j = \tfrac{j}{N} \in \left[-\tfrac{1}{2}, \tfrac{1}{2}\right), \, j = -\tfrac{N}{2}, \dots, \tfrac{N}{2} - 1.$$

Obtain

$$\boldsymbol{A} = \left(\mathrm{e}^{2\pi \mathrm{i} k \frac{j}{N}}\right)_{j=-\frac{N}{2},\, k=-\frac{M}{2}}^{\frac{N}{2}-1,\ \frac{M}{2}-1} \quad \text{and} \quad \boldsymbol{A}^* = \left(\mathrm{e}^{-2\pi \mathrm{i} k \frac{j}{N}}\right)_{k=-\frac{M}{2},\, j=-\frac{N}{2}}^{\frac{M}{2}-1,\ \frac{N}{2}-1}.$$

Consider matrix products:

Basic idea

Consider equispaced nodes

$$x_j = \tfrac{j}{N} \in \left[-\tfrac{1}{2}, \tfrac{1}{2}\right), \, j = -\tfrac{N}{2}, \ldots, \tfrac{N}{2} - 1.$$

Obtain

$$\boldsymbol{A} = \left(\mathrm{e}^{2\pi \mathrm{i} k \frac{j}{N}}\right)_{j=-\frac{N}{2},\, k=-\frac{M}{2}}^{\frac{N}{2}-1,\ \frac{M}{2}-1} \quad \text{and} \quad \boldsymbol{A}^* = \left(\mathrm{e}^{-2\pi \mathrm{i} k \frac{j}{N}}\right)_{k=-\frac{M}{2},\, j=-\frac{N}{2}}^{\frac{M}{2}-1,\ \frac{N}{2}-1}.$$

Consider matrix products:

$$\boldsymbol{A}^*\boldsymbol{A} = N\boldsymbol{I}_M \text{ for } N \geq M$$

$$\boldsymbol{A}\boldsymbol{A}^* = M\boldsymbol{I}_N \text{ for } N \leq M \text{ with } N \mid M$$

Basic idea

Consider nonequispaced nodes

$$x_j \in \left[-\tfrac{1}{2}, \tfrac{1}{2}\right), \; j = -\tfrac{N}{2}, \dots, \tfrac{N}{2} - 1.$$

Obtain

$$\boldsymbol{A} = \left(\mathrm{e}^{2\pi\mathrm{i}kx_j}\right)_{j=-\frac{N}{2},\, k=-\frac{M}{2}}^{\frac{N}{2}-1,\ \frac{M}{2}-1} \quad \text{and} \quad \boldsymbol{A}^* = \left(\mathrm{e}^{-2\pi\mathrm{i}kx_j}\right)_{k=-\frac{M}{2},\, j=-\frac{N}{2}}^{\frac{M}{2}-1,\ \frac{N}{2}-1}.$$

Consider matrix products:

$$\boldsymbol{A}^*\boldsymbol{A} \neq N\boldsymbol{I}_M$$

$$\boldsymbol{A}\boldsymbol{A}^* \neq M\boldsymbol{I}_N$$

Basic idea

Consider nonequispaced nodes

$$x_j \in \left[-\tfrac{1}{2}, \tfrac{1}{2}\right), \ j = -\tfrac{N}{2}, \dots, \tfrac{N}{2} - 1.$$

Obtain

$$\boldsymbol{A} = \left(\mathrm{e}^{2\pi \mathrm{i} k x_j}\right)_{j=-\frac{N}{2},\, k=-\frac{M}{2}}^{\frac{N}{2}-1,\ \frac{M}{2}-1} \quad \text{and} \quad \boldsymbol{A}^* = \left(\mathrm{e}^{-2\pi \mathrm{i} k x_j}\right)_{k=-\frac{M}{2},\, j=-\frac{N}{2}}^{\frac{M}{2}-1,\ \frac{N}{2}-1}.$$

Consider matrix products:

$$\boldsymbol{A}^*\boldsymbol{A} \neq N\boldsymbol{I}_M$$

$$\boldsymbol{A}\boldsymbol{A}^* \neq M\boldsymbol{I}_N$$

Aim: $\boldsymbol{A}\boldsymbol{D}^*\boldsymbol{F}^*\underset{\nwarrow\ \text{optimize}}{\boldsymbol{B}^*} \approx M\boldsymbol{I}_N$ $\rightsquigarrow \mathcal{O}(M \log M + N)$

Inverse NFFT - basic idea

In these special cases an inversion is given.

Look for good approximation in general.

A frame-theoretical approach to the inversion of the NFFT

Inverse NFFT - basic idea

In these special cases an inversion is given.

Look for good approximation in general.

- ⇝ approximation of the form $\boldsymbol{A}\boldsymbol{D}^*\boldsymbol{F}^*\boldsymbol{B}^* \approx M\boldsymbol{I}_N$

Inverse NFFT - basic idea

In these special cases an inversion is given.

Look for good approximation in general.

- ⇝ approximation of the form $\boldsymbol{A}\boldsymbol{D}^*\boldsymbol{F}^*\boldsymbol{B}^* \approx M\boldsymbol{I}_N$
- ⇝ modification of matrix $\boldsymbol{B}$

Inverse NFFT - basic idea

In these special cases an inversion is given.

Look for good approximation in general.

- ⇝ approximation of the form $\boldsymbol{A}\boldsymbol{D}^*\boldsymbol{F}^*\boldsymbol{B}^* \approx M\boldsymbol{I}_N$
- ⇝ modification of matrix $\boldsymbol{B}$
- ⇝ preserve band structure and arithmetic complexity

Inverse NFFT - basic idea

In these special cases an inversion is given.

Look for good approximation in general.

- ⇝ approximation of the form $\boldsymbol{A}\boldsymbol{D}^*\boldsymbol{F}^*\boldsymbol{B}^* \approx M\boldsymbol{I}_N$
- ⇝ modification of matrix $\boldsymbol{B}$
- ⇝ preserve band structure and arithmetic complexity

Problems:

(1) Solve

$$\boldsymbol{A}\hat{\boldsymbol{f}} = \boldsymbol{f},$$

given: $\boldsymbol{f}$, find: $\hat{\boldsymbol{f}}$. ⇒ inverse NFFT

A frame-theoretical approach to the inversion of the NFFT
Inverse NFFT - basic idea

In these special cases an inversion is given.

Look for good approximation in general.

- ⇝ approximation of the form $\boldsymbol{A}\boldsymbol{D}^*\boldsymbol{F}^*\boldsymbol{B}^* \approx M\boldsymbol{I}_N$
- ⇝ modification of matrix $\boldsymbol{B}$
- ⇝ preserve band structure and arithmetic complexity

Problems:

(1) Solve

$$\boldsymbol{A}\hat{\boldsymbol{f}} = \boldsymbol{f},$$

given: $\boldsymbol{f}$, find: $\hat{\boldsymbol{f}}$. $\Rightarrow$ inverse NFFT

(2) Solve

$$\boldsymbol{A}^*\boldsymbol{f} = \boldsymbol{h},$$

given.: $\boldsymbol{h}$, find.: $\boldsymbol{f}$. $\Rightarrow$ inverse adjoint NFFT

Inverse NFFT - underdetermined case

Assume

$$\boldsymbol{A}\boldsymbol{D}^*\boldsymbol{F}^*\boldsymbol{B}^* \approx M\boldsymbol{I}_N$$

Inverse NFFT - underdetermined case

Assume

$$\boldsymbol{A}\boldsymbol{D}^*\boldsymbol{F}^*\boldsymbol{B}^* \approx M\boldsymbol{I}_N \iff \frac{1}{M}\boldsymbol{A}\boldsymbol{D}^*\boldsymbol{F}^*\boldsymbol{B}^*\boldsymbol{f} \approx \boldsymbol{f} \quad \forall \boldsymbol{f} \in \mathbb{C}^N$$

Inverse NFFT - underdetermined case

Assume

$$\begin{aligned}\boldsymbol{A}\boldsymbol{D}^*\boldsymbol{F}^*\boldsymbol{B}^* \approx M\boldsymbol{I}_N \quad &\Longleftrightarrow \quad \frac{1}{M}\boldsymbol{A}\boldsymbol{D}^*\boldsymbol{F}^*\boldsymbol{B}^*\boldsymbol{f} \approx \boldsymbol{f} \quad \forall \boldsymbol{f} \in \mathbb{C}^N \\ &\Longleftrightarrow \quad \boldsymbol{A}\check{\boldsymbol{f}} \approx \boldsymbol{f} \quad \forall \boldsymbol{f} \in \mathbb{C}^N\end{aligned}$$

Inverse NFFT - underdetermined case

Assume

$$\begin{aligned} \boldsymbol{A}\boldsymbol{D}^*\boldsymbol{F}^*\boldsymbol{B}^* \approx M\boldsymbol{I}_N \iff & \frac{1}{M}\,\boldsymbol{A}\boldsymbol{D}^*\boldsymbol{F}^*\boldsymbol{B}^*\boldsymbol{f} \approx \boldsymbol{f} \quad \forall \boldsymbol{f} \in \mathbb{C}^N \\ \iff & \qquad\qquad \boldsymbol{A}\check{\boldsymbol{f}} \approx \boldsymbol{f} \quad \forall \boldsymbol{f} \in \mathbb{C}^N \end{aligned}$$

Since $\boldsymbol{A}\hat{\boldsymbol{f}} = \boldsymbol{f}$ that means $\check{\boldsymbol{f}} \approx \hat{\boldsymbol{f}}$.

Inverse NFFT - underdetermined case

Assume

$$\begin{aligned} \boldsymbol{A}\boldsymbol{D}^*\boldsymbol{F}^*\boldsymbol{B}^* \approx M\boldsymbol{I}_N \iff & \frac{1}{M}\boldsymbol{A}\boldsymbol{D}^*\boldsymbol{F}^*\boldsymbol{B}^*\boldsymbol{f} \approx \boldsymbol{f} \quad \forall \boldsymbol{f} \in \mathbb{C}^N \\ \iff & \boldsymbol{A}\check{\boldsymbol{f}} \approx \boldsymbol{f} \quad \forall \boldsymbol{f} \in \mathbb{C}^N \end{aligned}$$

Since $\boldsymbol{A}\hat{\boldsymbol{f}} = \boldsymbol{f}$ that means $\check{\boldsymbol{f}} \approx \hat{\boldsymbol{f}}$.

$\Rightarrow$ Reconstruction of Fourier coefficients $\hat{f}_k$

Inverse NFFT - underdetermined case

Assume

$$\boldsymbol{A}\boldsymbol{D}^*\boldsymbol{F}^*\boldsymbol{B}^* \approx M\boldsymbol{I}_N \iff \frac{1}{M}\boldsymbol{A}\boldsymbol{D}^*\boldsymbol{F}^*\boldsymbol{B}^*\boldsymbol{f} \approx \boldsymbol{f} \quad \forall \boldsymbol{f} \in \mathbb{C}^N$$
$$\iff \boldsymbol{A}\check{\boldsymbol{f}} \approx \boldsymbol{f} \quad \forall \boldsymbol{f} \in \mathbb{C}^N$$

Since $\boldsymbol{A}\hat{\boldsymbol{f}} = \boldsymbol{f}$ that means $\check{\boldsymbol{f}} \approx \hat{\boldsymbol{f}}$.

$\Rightarrow$ Reconstruction of Fourier coefficients $\hat{f}_k$

$$\|M\boldsymbol{A}\check{\boldsymbol{f}} - M\boldsymbol{f}\|_2 \leq \|\boldsymbol{A}\boldsymbol{D}^*\boldsymbol{F}^*\boldsymbol{B}^* - M\boldsymbol{I}_N\|_{\mathrm{F}}\|\boldsymbol{f}\|_2$$

Inverse NFFT - underdetermined case

Assume

$$\boldsymbol{A}\boldsymbol{D}^*\boldsymbol{F}^*\boldsymbol{B}^* \approx M\boldsymbol{I}_N \iff \frac{1}{M}\boldsymbol{A}\boldsymbol{D}^*\boldsymbol{F}^*\boldsymbol{B}^*\boldsymbol{f} \approx \boldsymbol{f} \quad \forall \boldsymbol{f} \in \mathbb{C}^N$$

$$\iff \boldsymbol{A}\check{\boldsymbol{f}} \approx \boldsymbol{f} \quad \forall \boldsymbol{f} \in \mathbb{C}^N$$

Since $\boldsymbol{A}\hat{\boldsymbol{f}} = \boldsymbol{f}$ that means $\check{\boldsymbol{f}} \approx \hat{\boldsymbol{f}}$.

$\Rightarrow$ Reconstruction of Fourier coefficients $\hat{f}_k$

$$\|M\boldsymbol{A}\check{\boldsymbol{f}} - M\boldsymbol{f}\|_2 \leq \|\boldsymbol{A}\boldsymbol{D}^*\boldsymbol{F}^*\boldsymbol{B}^* - M\boldsymbol{I}_N\|_{\mathrm{F}}\|\boldsymbol{f}\|_2$$

Optimization problem:

$$\underset{\boldsymbol{B}\in\mathbb{R}^{N\times\sigma M}\,:\ \boldsymbol{B}\ (2m+1)\text{-sparse}}{\text{Minimize}} \quad \|\boldsymbol{A}\boldsymbol{D}^*\boldsymbol{F}^*\boldsymbol{B}^* - M\boldsymbol{I}_N\|_{\mathrm{F}}^2$$

Inverse NFFT - underdetermined case

Assume

$$\begin{aligned} \boldsymbol{A}\boldsymbol{D}^*\boldsymbol{F}^*\boldsymbol{B}^* \approx M\boldsymbol{I}_N \iff & \frac{1}{M}\,\boldsymbol{A}\boldsymbol{D}^*\boldsymbol{F}^*\boldsymbol{B}^*\boldsymbol{f} \approx \boldsymbol{f} \quad \forall \boldsymbol{f} \in \mathbb{C}^N \\ \iff & \boldsymbol{A}\check{\boldsymbol{f}} \approx \boldsymbol{f} \quad \forall \boldsymbol{f} \in \mathbb{C}^N \end{aligned}$$

Since $\boldsymbol{A}\hat{\boldsymbol{f}} = \boldsymbol{f}$ that means $\check{\boldsymbol{f}} \approx \hat{\boldsymbol{f}}$.

$\Rightarrow$ Reconstruction of Fourier coefficients $\hat{f}_k$

$$\|M\boldsymbol{A}\check{\boldsymbol{f}} - M\boldsymbol{f}\|_2 \leq \|\boldsymbol{A}\boldsymbol{D}^*\boldsymbol{F}^*\boldsymbol{B}^* - M\boldsymbol{I}_N\|_{\mathrm{F}}\|\boldsymbol{f}\|_2$$

Optimization problem:

$$\underset{\boldsymbol{B}\in\mathbb{R}^{N\times\sigma M}:\ \boldsymbol{B}\ (2m+1)\text{-sparse}}{\text{Minimize}} \ \|\boldsymbol{A}\boldsymbol{D}^*\boldsymbol{F}^*\boldsymbol{B}^* - M\boldsymbol{I}_N\|_{\mathrm{F}}^2 = \sum_{j=1}^{N} \|\boldsymbol{A}\boldsymbol{D}^*\boldsymbol{T}_j\boldsymbol{b}_j - M\boldsymbol{e}_j\|_2^2$$

Inverse NFFT - underdetermined case

Assume

$$\boldsymbol{A}\boldsymbol{D}^*\boldsymbol{F}^*\boldsymbol{B}^* \approx M\boldsymbol{I}_N \iff \frac{1}{M}\,\boldsymbol{A}\boldsymbol{D}^*\boldsymbol{F}^*\boldsymbol{B}^*\boldsymbol{f} \approx \boldsymbol{f} \quad \forall \boldsymbol{f} \in \mathbb{C}^N$$
$$\iff \boldsymbol{A}\check{\boldsymbol{f}} \approx \boldsymbol{f} \quad \forall \boldsymbol{f} \in \mathbb{C}^N$$

Since $\boldsymbol{A}\hat{\boldsymbol{f}} = \boldsymbol{f}$ that means $\check{\boldsymbol{f}} \approx \hat{\boldsymbol{f}}$.

$\Rightarrow$ Reconstruction of Fourier coefficients $\hat{f}_k$

$$\|M\boldsymbol{A}\check{\boldsymbol{f}} - M\boldsymbol{f}\|_2 \leq \|\boldsymbol{A}\boldsymbol{D}^*\boldsymbol{F}^*\boldsymbol{B}^* - M\boldsymbol{I}_N\|_{\mathrm{F}}\|\boldsymbol{f}\|_2$$

Optimization problem:

$$\underset{\boldsymbol{B}\in\mathbb{R}^{N\times\sigma M}:\ \boldsymbol{B}\ (2m+1)\text{-sparse}}{\text{Minimize}} \quad \|\boldsymbol{A}\boldsymbol{D}^*\boldsymbol{F}^*\boldsymbol{B}^* - M\boldsymbol{I}_N\|_{\mathrm{F}}^2 = \sum_{j=1}^{N} \|\boldsymbol{A}\boldsymbol{D}^*\boldsymbol{T}_j\boldsymbol{b}_j - M\boldsymbol{e}_j\|_2^2$$

$\rightsquigarrow \mathcal{O}(N^2 + M\log M)$

Inverse NFFT - underdetermined case

Assume

$$\boldsymbol{A}\boldsymbol{D}^*\boldsymbol{F}^*\boldsymbol{B}^* \approx M\boldsymbol{I}_N \iff \frac{1}{M}\boldsymbol{A}\boldsymbol{D}^*\boldsymbol{F}^*\boldsymbol{B}^*\boldsymbol{f} \approx \boldsymbol{f} \quad \forall \boldsymbol{f} \in \mathbb{C}^N$$

$$\iff \boldsymbol{A}\check{\boldsymbol{f}} \approx \boldsymbol{f} \quad \forall \boldsymbol{f} \in \mathbb{C}^N$$

Since $\boldsymbol{A}\hat{\boldsymbol{f}} = \boldsymbol{f}$ that means $\check{\boldsymbol{f}} \approx \hat{\boldsymbol{f}}$.

$\Rightarrow$ Reconstruction of Fourier coefficients $\hat{f}_k$

$$\|M\boldsymbol{A}\check{\boldsymbol{f}} - M\boldsymbol{f}\|_2 \leq \|\boldsymbol{A}\boldsymbol{D}^*\boldsymbol{F}^*\boldsymbol{B}^* - M\boldsymbol{I}_N\|_{\mathrm{F}}\|\boldsymbol{f}\|_2$$

Optimization problem:

$$\underset{\boldsymbol{B}\in\mathbb{R}^{N\times\sigma M}\,:\,\boldsymbol{B}\,(2m+1)\text{-sparse}}{\text{Minimize}} \|\boldsymbol{A}\boldsymbol{D}^*\boldsymbol{F}^*\boldsymbol{B}^* - M\boldsymbol{I}_N\|_{\mathrm{F}}^2 = \sum_{j=1}^{N} \|\boldsymbol{A}\boldsymbol{D}^*\boldsymbol{T}_j\boldsymbol{b}_j - M\boldsymbol{e}_j\|_2^2$$

$\rightsquigarrow \mathcal{O}(N^2 + M\log M)$

$\Rightarrow$ inverse NFFT as well as inverse adjoint NFFT

Inverse NFFT - overdetermined case

$$\tilde{g}(x) = \sum_{j=1}^{N} f_j \, \tilde{w}_m(x_j - x)$$

Inverse NFFT - overdetermined case

$$\tilde{g}(x) = \sum_{j=1}^{N} f_j \, \tilde{w}_m(x_j - x) \quad \Rightarrow \quad \tilde{g}\big(\tfrac{l}{\sigma M}\big) = \sum_{j=1}^{N} f_j \, \tilde{w}_m\big(x_j - \tfrac{l}{\sigma M}\big) \, ,$$

Inverse NFFT - overdetermined case

$$\tilde{g}(x) = \sum_{j=1}^{N} f_j \, \tilde{w}_m(x_j - x) \quad \Rightarrow \quad \tilde{g}\big(\tfrac{l}{\sigma M}\big) = \sum_{j=1}^{N} f_j \, \tilde{w}_m\big(x_j - \tfrac{l}{\sigma M}\big) \, ,$$

so $\tilde{\boldsymbol{g}} = \boldsymbol{B}^* \boldsymbol{f}$ and $\tilde{\boldsymbol{h}} = \boldsymbol{D}^* \boldsymbol{F}^* \tilde{\boldsymbol{g}}$.

Inverse NFFT - overdetermined case

$$\tilde{g}(x) = \sum_{j=1}^{N} f_j \, \tilde{w}_m(x_j - x) \quad \Rightarrow \quad \tilde{g}\big(\tfrac{l}{\sigma M}\big) = \sum_{j=1}^{N} f_j \, \tilde{w}_m\big(x_j - \tfrac{l}{\sigma M}\big) \, ,$$

so $\tilde{\boldsymbol{g}} = \boldsymbol{B}^* \boldsymbol{f}$ and $\tilde{\boldsymbol{h}} = \boldsymbol{D}^* \boldsymbol{F}^* \tilde{\boldsymbol{g}}$. Set $\tilde{\boldsymbol{h}} \approx \hat{\boldsymbol{f}}$.

Inverse NFFT - overdetermined case

$$\tilde{g}(x) = \sum_{j=1}^{N} f_j \, \tilde{w}_m(x_j - x) \quad \Rightarrow \quad \tilde{g}\big(\tfrac{l}{\sigma M}\big) = \sum_{j=1}^{N} f_j \, \tilde{w}_m\big(x_j - \tfrac{l}{\sigma M}\big) \, ,$$

so $\tilde{\boldsymbol{g}} = \boldsymbol{B}^* \boldsymbol{f}$ and $\tilde{\boldsymbol{h}} = \boldsymbol{D}^* \boldsymbol{F}^* \tilde{\boldsymbol{g}}$. Set $\tilde{\boldsymbol{h}} \approx \hat{\boldsymbol{f}}$.

$$\tilde{\boldsymbol{g}} = \boldsymbol{B}^* \boldsymbol{f} = \boldsymbol{B}^* \boldsymbol{A} \hat{\boldsymbol{f}} \approx \boldsymbol{B}^* \boldsymbol{A} \tilde{\boldsymbol{h}} = \boldsymbol{B}^* \boldsymbol{A} \boldsymbol{D}^* \boldsymbol{F}^* \tilde{\boldsymbol{g}}$$

Inverse NFFT - overdetermined case

$$\tilde{g}(x) = \sum_{j=1}^{N} f_j \, \tilde{w}_m(x_j - x) \quad \Rightarrow \quad \tilde{g}\big(\tfrac{l}{\sigma M}\big) = \sum_{j=1}^{N} f_j \, \tilde{w}_m\big(x_j - \tfrac{l}{\sigma M}\big) \,,$$

so $\tilde{\boldsymbol{g}} = \boldsymbol{B}^* \boldsymbol{f}$ and $\tilde{\boldsymbol{h}} = \boldsymbol{D}^* \boldsymbol{F}^* \tilde{\boldsymbol{g}}$. Set $\tilde{\boldsymbol{h}} \approx \hat{\boldsymbol{f}}$.

$$\tilde{\boldsymbol{g}} = \boldsymbol{B}^* \boldsymbol{f} = \boldsymbol{B}^* \boldsymbol{A} \hat{\boldsymbol{f}} \approx \boldsymbol{B}^* \boldsymbol{A} \tilde{\boldsymbol{h}} = \boldsymbol{B}^* \boldsymbol{A} \boldsymbol{D}^* \boldsymbol{F}^* \tilde{\boldsymbol{g}}$$

Optimization problem:

$$\underset{\boldsymbol{B} \in \mathbb{R}^{N \times \sigma M} \,:\, \boldsymbol{B}\ (2m+1)\text{-sparse}}{\text{Minimize}} \quad \|\boldsymbol{B}^* \boldsymbol{A} \boldsymbol{D}^* \boldsymbol{F}^* - \boldsymbol{I}_{\sigma M}\|_{\mathrm{F}}^2$$

Inverse NFFT - overdetermined case

$$\tilde{g}(x) = \sum_{j=1}^{N} f_j \, \tilde{w}_m(x_j - x) \quad \Rightarrow \quad \tilde{g}\big(\tfrac{l}{\sigma M}\big) = \sum_{j=1}^{N} f_j \, \tilde{w}_m\big(x_j - \tfrac{l}{\sigma M}\big) \,,$$

so $\tilde{\boldsymbol{g}} = \boldsymbol{B}^* \boldsymbol{f}$ and $\tilde{\boldsymbol{h}} = \boldsymbol{D}^* \boldsymbol{F}^* \tilde{\boldsymbol{g}}$. Set $\tilde{\boldsymbol{h}} \approx \hat{\boldsymbol{f}}$.

$$\tilde{\boldsymbol{g}} = \boldsymbol{B}^* \boldsymbol{f} = \boldsymbol{B}^* \boldsymbol{A} \hat{\boldsymbol{f}} \approx \boldsymbol{B}^* \boldsymbol{A} \tilde{\boldsymbol{h}} = \boldsymbol{B}^* \boldsymbol{A} \boldsymbol{D}^* \boldsymbol{F}^* \tilde{\boldsymbol{g}}$$

Optimization problem:

$$\underset{\boldsymbol{B} \in \mathbb{R}^{N \times \sigma M} :\ \boldsymbol{B}\ (2m+1)\text{-sparse}}{\text{Minimize}} \quad \|\boldsymbol{B}^* \boldsymbol{A} \boldsymbol{D}^* \boldsymbol{F}^* - \boldsymbol{I}_{\sigma M}\|_{\mathrm{F}}^2$$

$$\|\boldsymbol{F} \boldsymbol{D} \boldsymbol{A}^* \boldsymbol{B} - \boldsymbol{I}_{\sigma M}\|_{\mathrm{F}}^2 = \sum_{l=-\frac{\sigma M}{2}}^{\frac{\sigma M}{2}-1} \|\boldsymbol{F} \boldsymbol{D} \boldsymbol{H}_l \boldsymbol{b}_l - \boldsymbol{e}_l\|_2^2$$

Inverse NFFT - overdetermined case

$$\tilde{g}(x) = \sum_{j=1}^{N} f_j \, \tilde{w}_m(x_j - x) \quad \Rightarrow \quad \tilde{g}\big(\tfrac{l}{\sigma M}\big) = \sum_{j=1}^{N} f_j \, \tilde{w}_m\big(x_j - \tfrac{l}{\sigma M}\big) \,,$$

so $\tilde{\boldsymbol{g}} = \boldsymbol{B}^* \boldsymbol{f}$ and $\tilde{\boldsymbol{h}} = \boldsymbol{D}^* \boldsymbol{F}^* \tilde{\boldsymbol{g}}$. Set $\tilde{\boldsymbol{h}} \approx \hat{\boldsymbol{f}}$.

$$\tilde{\boldsymbol{g}} = \boldsymbol{B}^* \boldsymbol{f} = \boldsymbol{B}^* \boldsymbol{A} \hat{\boldsymbol{f}} \approx \boldsymbol{B}^* \boldsymbol{A} \tilde{\boldsymbol{h}} = \boldsymbol{B}^* \boldsymbol{A} \boldsymbol{D}^* \boldsymbol{F}^* \tilde{\boldsymbol{g}}$$

Optimization problem:

$$\underset{\boldsymbol{B} \in \mathbb{R}^{N \times \sigma M} \,:\, \boldsymbol{B}\ (2m+1)\text{-sparse}}{\text{Minimize}} \quad \|\boldsymbol{B}^* \boldsymbol{A} \boldsymbol{D}^* \boldsymbol{F}^* - \boldsymbol{I}_{\sigma M}\|_{\mathrm{F}}^2$$

$$\|\boldsymbol{F}\boldsymbol{D}\boldsymbol{A}^* \boldsymbol{B} - \boldsymbol{I}_{\sigma M}\|_{\mathrm{F}}^2 = \sum_{l=-\frac{\sigma M}{2}}^{\frac{\sigma M}{2}-1} \|\boldsymbol{F}\boldsymbol{D}\boldsymbol{H}_l \boldsymbol{b}_l - \boldsymbol{e}_l\|_2^2$$

$$\rightsquigarrow \mathcal{O}(N^2 M^2 + N^3 M)$$

Frame-theoretical approach

Approach based on

- [Gelb, Song 13] Approximating the inverse frame operator from localized frames
- [Gelb, Song 14] A frame theoretic approach to the nonuniform fast Fourier transform
- [Davis, Gelb, Song 16] A high-dimensional inverse frame operator approximation technique

$\Rightarrow$ connection to inverse NFFT

Definition

Let $\mathcal{H}$ be a Hilbert space. A sequence $\{\varphi_j\}_{j=1}^{\infty} \subset \mathcal{H}$ is called **frame** if there exist $A, B > 0$ such that

$$A\|f\|^2 \leq \sum_{j=1}^{\infty} |\langle f, \varphi_j \rangle|^2 \leq B\|f\|^2 \quad \forall f \in \mathcal{H}.$$

[Christensen 16]

Definition

Let $\mathcal{H}$ be a Hilbert space. A sequence $\{\varphi_j\}_{j=1}^{\infty} \subset \mathcal{H}$ is called **frame** if there exist $A, B > 0$ such that

$$A\|f\|^2 \leq \sum_{j=1}^{\infty} |\langle f, \varphi_j \rangle|^2 \leq B\|f\|^2 \quad \forall f \in \mathcal{H}.$$

The operator

$$S \colon \mathcal{H} \to \mathcal{H}, \quad Sf = \sum_{j=1}^{\infty} \langle f, \varphi_j \rangle \varphi_j,$$

is named the **frame operator**.

[Christensen 16]

Definition

Let $\mathcal{H}$ be a Hilbert space. A sequence $\{\varphi_j\}_{j=1}^{\infty} \subset \mathcal{H}$ is called **frame** if there exist $A, B > 0$ such that

$$A\|f\|^2 \leq \sum_{j=1}^{\infty} |\langle f, \varphi_j\rangle|^2 \leq B\|f\|^2 \quad \forall f \in \mathcal{H}.$$

The operator

$$S\colon \mathcal{H} \to \mathcal{H}, \quad Sf = \sum_{j=1}^{\infty} \langle f, \varphi_j\rangle \varphi_j,$$

is named the **frame operator**.

Let $\{\varphi_j\}_{j=1}^{\infty}$ be a frame with operator S, then

$$f = \sum_{j=1}^{\infty} \langle f, S^{-1}\varphi_j\rangle \varphi_j = \sum_{j=1}^{\infty} \langle f, \varphi_j\rangle S^{-1}\varphi_j \quad \forall f \in \mathcal{H}.$$

[Christensen 16]

Definition

Let $\mathcal{H}$ be a Hilbert space. A sequence $\{\varphi_j\}_{j=1}^{\infty} \subset \mathcal{H}$ is called **frame** if there exist $A, B > 0$ such that

$$A\|f\|^2 \leq \sum_{j=1}^{\infty} |\langle f, \varphi_j \rangle|^2 \leq B\|f\|^2 \quad \forall f \in \mathcal{H}.$$

The operator

$$S \colon \mathcal{H} \to \mathcal{H}, \quad Sf = \sum_{j=1}^{\infty} \langle f, \varphi_j \rangle \varphi_j,$$

is named the **frame operator**.

Let $\{\varphi_j\}_{j=1}^{\infty}$ be a frame with operator S, then

$$f = \sum_{j=1}^{\infty} \langle f, S^{-1}\varphi_j \rangle \varphi_j = \sum_{j=1}^{\infty} \langle f, \varphi_j \rangle S^{-1}\varphi_j \quad \forall f \in \mathcal{H}.$$

$\Rightarrow$ goal: approximate S^{-1}

[Christensen 16]

Connection to iNFFT

Frame approximation:

$$\tilde{f} := \sum_{j=1}^{N} \sum_{l=-\frac{\sigma M}{2}}^{\frac{\sigma M}{2}-1} \langle f, \varphi_j \rangle \, p_{l,j} \, \psi_l ,$$

with $\mathbf{\Phi}^\dagger =: [p_{l,j}]_{l=-\frac{\sigma M}{2},\, j=1}^{\frac{\sigma M}{2}-1,\, N}$ Moore-Penrose pseudoinverse of [Gelb, Song 13/14]

$$\mathbf{\Phi} := [\langle \varphi_j, \psi_l \rangle]_{j=1,\, l=-\frac{\sigma M}{2}}^{N,\, \frac{\sigma M}{2}-1}$$

Connection to iNFFT

Frame approximation:

$$\tilde{f} := \sum_{j=1}^{N} \sum_{l=-\frac{\sigma M}{2}}^{\frac{\sigma M}{2}-1} \langle f, \varphi_j \rangle \, p_{l,j} \, \psi_l \qquad \rightsquigarrow \qquad \tilde{\hat{f}} = \sum_{j=1}^{N} \sum_{l=-\frac{\sigma M}{2}}^{\frac{\sigma M}{2}-1} \langle \hat{f}, \varphi_j \rangle_{\ell_2} \, p_{l,j} \, \psi_l,$$

with $\boldsymbol{\Phi}^\dagger =: [p_{l,j}]_{l=-\frac{\sigma M}{2},\, j=1}^{\frac{\sigma M}{2}-1,\, N}$ Moore-Penrose pseudoinverse of [Gelb, Song 13/14]

$$\boldsymbol{\Phi} := [\langle \varphi_j, \psi_l \rangle]_{j=1,\, l=-\frac{\sigma M}{2}}^{N,\, \frac{\sigma M}{2}-1}$$

Connection to iNFFT

Frame approximation:

$$\tilde{\hat{f}} = \sum_{j=1}^{N} \sum_{l=-\frac{\sigma M}{2}}^{\frac{\sigma M}{2}-1} \langle \hat{f}, \varphi_j \rangle_{\ell_2}\, p_{l,j}\, \psi_l ,$$

with $\mathbf{\Phi}^\dagger =: [p_{l,j}]_{l=-\frac{\sigma M}{2},\, j=1}^{\frac{\sigma M}{2}-1,\ N}$ Moore-Penrose pseudoinverse of [Gelb, Song 13/14]

$$\mathbf{\Phi} := [\langle \varphi_j, \psi_l \rangle]_{j=1,\, l=-\frac{\sigma M}{2}}^{N,\ \frac{\sigma M}{2}-1}$$

⇒ Look for approximation $\tilde{h}_k \approx \tilde{\hat{f}}(k) \approx \hat{f}_k, \quad k = -\frac{M}{2}, \dots, \frac{M}{2} - 1.$

Connection to iNFFT

Choose

$$\{\varphi_j(k) := \mathrm{e}^{-2\pi \mathrm{i} k x_j},\, j \in \mathbb{N}\} \quad \text{and} \quad \left\{\psi_l(k) := \frac{\mathrm{e}^{-2\pi \mathrm{i} k l/\sigma M}}{\sigma M \hat{w}(-k)},\, l \in \mathbb{Z}\right\}.$$

Frame approximation:

$$\tilde{\hat{f}} = \sum_{j=1}^{N} \sum_{l=-\frac{\sigma M}{2}}^{\frac{\sigma M}{2}-1} \langle \hat{f}, \varphi_j \rangle_{\ell_2}\, p_{l,j}\, \psi_l,$$

with $\boldsymbol{\Phi}^\dagger =: [p_{l,j}]_{l=-\frac{\sigma M}{2},\, j=1}^{\frac{\sigma M}{2}-1,\, N}$ Moore-Penrose pseudoinverse of [Gelb, Song 13/14]

$$\boldsymbol{\Phi} := [\langle \varphi_j, \psi_l \rangle]_{j=1,\, l=-\frac{\sigma M}{2}}^{N,\, \frac{\sigma M}{2}-1}$$

$\Rightarrow$ Look for approximation $\tilde{h}_k \approx \tilde{\hat{f}}(k) \approx \hat{f}_k, \quad k = -\frac{M}{2}, \dots, \frac{M}{2} - 1.$

Connection to iNFFT

Choose

$$\{\varphi_j(k) := \mathrm{e}^{-2\pi \mathrm{i} k x_j},\, j \in \mathbb{N}\} \quad \text{and} \quad \left\{\psi_l(k) := \frac{\mathrm{e}^{-2\pi \mathrm{i} k l/\sigma M}}{\sigma M \hat{w}(-k)},\, l \in \mathbb{Z}\right\}.$$

Frame approximation:

$$\tilde{\hat{f}} = \sum_{j=1}^{N} \sum_{l=-\frac{\sigma M}{2}}^{\frac{\sigma M}{2}-1} \langle \hat{f}, \varphi_j \rangle_{\ell_2}\, p_{l,j}\, \psi_l,$$

with $\mathbf{\Phi}^\dagger =: [p_{l,j}]_{l=-\frac{\sigma M}{2},\, j=1}^{\frac{\sigma M}{2}-1,\, N}$ Moore-Penrose pseudoinverse of [Gelb, Song 13/14]

$$\mathbf{\Phi} := [\langle \varphi_j, \psi_l \rangle]_{j=1,\, l=-\frac{\sigma M}{2}}^{N,\, \frac{\sigma M}{2}-1} = \left[\sum_{k=-\frac{M}{2}}^{\frac{M}{2}-1} \frac{1}{\sigma M \hat{w}(k)}\, \mathrm{e}^{-2\pi \mathrm{i} k \left(x_j - \frac{l}{\sigma M}\right)} \right]_{j=1,\, l=-\frac{\sigma M}{2}}^{N,\, \frac{\sigma M}{2}-1}.$$

$\Rightarrow$ Look for approximation $\tilde{h}_k \approx \tilde{\hat{f}}(k) \approx \hat{f}_k, \quad k = -\frac{M}{2}, \dots, \frac{M}{2} - 1.$

Define

$$\tilde{\boldsymbol{h}} := (\tilde{h}_k)_{k=-\frac{M}{2}}^{\frac{M}{2}-1}, \qquad \tilde{\tilde{\boldsymbol{h}}} := (\tilde{\tilde{h}}_k)_{k=-\frac{M}{2}}^{\frac{M}{2}-1} = (\tilde{\hat{f}}(k))_{k=-\frac{M}{2}}^{\frac{M}{2}-1}.$$

A frame-theoretical approach to the inversion of the NFFT
Frame-theoretical approach

Define

$$\tilde{\boldsymbol{h}} := (\tilde{h}_k)_{k=-\frac{M}{2}}^{\frac{M}{2}-1}, \qquad \tilde{\tilde{\boldsymbol{h}}} := (\tilde{\tilde{h}}_k)_{k=-\frac{M}{2}}^{\frac{M}{2}-1} = (\tilde{\hat{f}}(k))_{k=-\frac{M}{2}}^{\frac{M}{2}-1}.$$

Theorem

For

$$\hat{\boldsymbol{w}} := \left(\frac{1}{\hat{w}(-k)}\right)_{k=-\frac{M}{2}}^{\frac{M}{2}-1}$$

with $\|\hat{\boldsymbol{w}}\|_2 < \infty$ *the following estimates hold.*

(i) *For* $\sigma M < N$ *we have*

$$\left\|\tilde{\boldsymbol{h}} - \tilde{\tilde{\boldsymbol{h}}}\right\|_2 \leq \frac{1}{\sqrt{\sigma M}} \|\hat{\boldsymbol{w}}\|_2 \, \|\boldsymbol{\Phi}\boldsymbol{B}^* - \boldsymbol{I}_N\|_F \, \|\boldsymbol{\Phi}^\dagger \boldsymbol{f}\|_2.$$

(ii) *For* $\sigma M > N$ *we have*

$$\left\|\tilde{\boldsymbol{h}} - \tilde{\tilde{\boldsymbol{h}}}\right\|_2 \leq \frac{1}{\sqrt{\sigma M}} \|\hat{\boldsymbol{w}}\|_2 \, \|\boldsymbol{B}^*\boldsymbol{\Phi} - \boldsymbol{I}_{\sigma M}\|_F \, \|\boldsymbol{\Phi}^\dagger \boldsymbol{f}\|_2.$$

Connection to iNFFT (II)

$\boldsymbol{\Phi B}^* = (\boldsymbol{BFDA}^*)^T$:

$$\underset{\boldsymbol{B}\in\mathbb{R}^{N\times\sigma M}\,:\;\boldsymbol{B}\ (2m+1)\text{-sparse}}{\text{Minimize}}\ \|\boldsymbol{\Phi B}^* - \boldsymbol{I}_N\|_{\mathrm{F}}^2$$

$$\Longleftrightarrow\quad \underset{\boldsymbol{B}\in\mathbb{R}^{N\times\sigma M}\,:\;\boldsymbol{B}\ (2m+1)\text{-sparse}}{\text{Minimize}}\ \|\boldsymbol{BFDA}^* - \boldsymbol{I}_N\|_{\mathrm{F}}^2$$

Connection to iNFFT (II)

$\boldsymbol{\Phi B}^* = (\boldsymbol{BFDA}^*)^T$:

$$\underset{\boldsymbol{B}\in\mathbb{R}^{N\times\sigma M}\,:\;\boldsymbol{B}\ (2m+1)\text{-sparse}}{\text{Minimize}} \quad \|\boldsymbol{\Phi B}^* - \boldsymbol{I}_N\|_{\mathrm{F}}^2$$

$$\iff \quad \underset{\boldsymbol{B}\in\mathbb{R}^{N\times\sigma M}\,:\;\boldsymbol{B}\ (2m+1)\text{-sparse}}{\text{Minimize}} \quad \|\boldsymbol{BFDA}^* - \boldsymbol{I}_N\|_{\mathrm{F}}^2$$

$\boldsymbol{B}^*\boldsymbol{\Phi} = (\boldsymbol{FDA}^*\boldsymbol{B})^T$:

$$\underset{\boldsymbol{B}\in\mathbb{R}^{N\times\sigma M}\,:\;\boldsymbol{B}\ (2m+1)\text{-sparse}}{\text{Minimize}} \quad \|\boldsymbol{B}^*\boldsymbol{\Phi} - \boldsymbol{I}_{\sigma M}\|_{\mathrm{F}}^2$$

$$\iff \quad \underset{\boldsymbol{B}\in\mathbb{R}^{N\times\sigma M}\,:\;\boldsymbol{B}\ (2m+1)\text{-sparse}}{\text{Minimize}} \quad \|\boldsymbol{FDA}^*\boldsymbol{B} - \boldsymbol{I}_{\sigma M}\|_{\mathrm{F}}^2$$

Connection to iNFFT (II)

$$\boldsymbol{\Phi}\boldsymbol{B}^* = (\boldsymbol{B}\boldsymbol{F}\boldsymbol{D}\boldsymbol{A}^*)^T :$$

$$\underset{\boldsymbol{B}\in\mathbb{R}^{N\times\sigma M}:\ \boldsymbol{B}\ (2m+1)\text{-sparse}}{\text{Minimize}} \quad \|\boldsymbol{\Phi}\boldsymbol{B}^* - \boldsymbol{I}_N\|_{\mathrm{F}}^2$$

$$\Longleftrightarrow \quad \underset{\boldsymbol{B}\in\mathbb{R}^{N\times\sigma M}:\ \boldsymbol{B}\ (2m+1)\text{-sparse}}{\text{Minimize}} \quad \|\boldsymbol{B}\boldsymbol{F}\boldsymbol{D}\boldsymbol{A}^* - \boldsymbol{I}_N\|_{\mathrm{F}}^2$$

$$\boldsymbol{B}^*\boldsymbol{\Phi} = (\boldsymbol{F}\boldsymbol{D}\boldsymbol{A}^*\boldsymbol{B})^T :$$

$$\underset{\boldsymbol{B}\in\mathbb{R}^{N\times\sigma M}:\ \boldsymbol{B}\ (2m+1)\text{-sparse}}{\text{Minimize}} \quad \|\boldsymbol{B}^*\boldsymbol{\Phi} - \boldsymbol{I}_{\sigma M}\|_{\mathrm{F}}^2$$

$$\Longleftrightarrow \quad \underset{\boldsymbol{B}\in\mathbb{R}^{N\times\sigma M}:\ \boldsymbol{B}\ (2m+1)\text{-sparse}}{\text{Minimize}} \quad \|\boldsymbol{F}\boldsymbol{D}\boldsymbol{A}^*\boldsymbol{B} - \boldsymbol{I}_{\sigma M}\|_{\mathrm{F}}^2$$

$\Rightarrow$ previously only optimization of $\boldsymbol{D}$, new fast algorithms applicable

Inverse NFFT

Trigonometric function

$$f(x) = \cos^2(\pi x^2)\ \sin(10x^2), \quad x \in \left[-\tfrac{1}{2}, \tfrac{1}{2}\right)$$

Inverse NFFT

Trigonometric function

$$f(x) = \cos^2(\pi x^2)\, \sin(10x^2), \quad x \in \left[-\tfrac{1}{2}, \tfrac{1}{2}\right)$$

Consider approximations

$$c_k(f) \approx \check{f}_k = \left(\tfrac{1}{M}\, \boldsymbol{D}^* \boldsymbol{F}^* \tilde{\boldsymbol{B}}^* \boldsymbol{f}\right)_k \quad \text{and} \quad c_k(f) \approx \check{f}_k = \left(\boldsymbol{D}^* \boldsymbol{F}^* \tilde{\boldsymbol{B}}^* \boldsymbol{f}\right)_k$$

Inverse NFFT

Trigonometric function

$$f(x) = \cos^2(\pi x^2)\ \sin(10x^2), \quad x \in \left[-\tfrac{1}{2}, \tfrac{1}{2}\right)$$

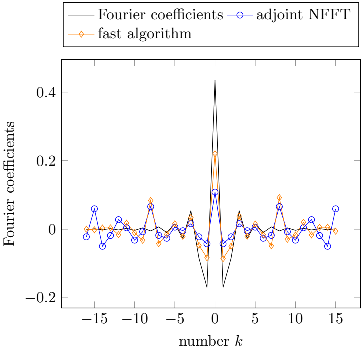
Consider approximations

$$c_k(f) \approx \check{f}_k = \left(\tfrac{1}{M}\, \boldsymbol{D}^* \boldsymbol{F}^* \tilde{\boldsymbol{B}}^* \boldsymbol{f}\right)_k \quad \text{and} \quad c_k(f) \approx \check{f}_k = \left(\boldsymbol{D}^* \boldsymbol{F}^* \tilde{\boldsymbol{B}}^* \boldsymbol{f}\right)_k$$

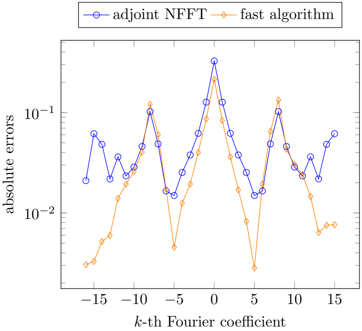
Reconstruction

- of $M = 32$ Fourier coefficients
- from function values $f(x_j)$ at jittered equispaced nodes x_j
- overdetermined as well as underdetermined
    (a) $N = 8$
    (b) $N = 128$

Inverse NFFT - underdetermined case

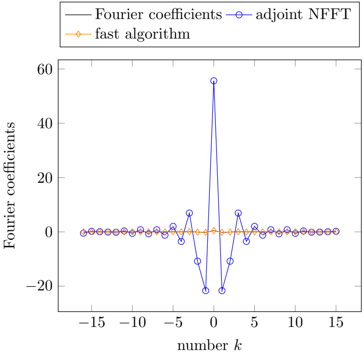


(a) reconstruction

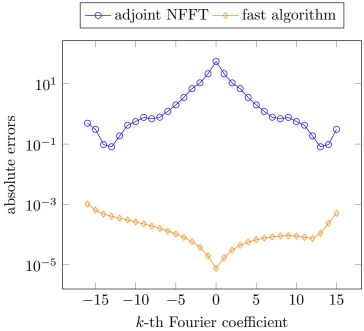


(b) pointwise errors

Inverse NFFT - overdetermined case



(a) reconstruction



(b) pointwise errors

Inverse adjoint NFFT

Consider again

$$f(x) = \cos^2(\pi x^2)\ \sin(10x^2), \quad x \in \left[-\tfrac{1}{2}, \tfrac{1}{2}\right)$$

Inverse adjoint NFFT

Consider again

$$f(x) = \cos^2(\pi x^2)\ \sin(10x^2), \quad x \in \left[-\tfrac{1}{2}, \tfrac{1}{2}\right)$$

and approximations

$$f(x_j) \approx \tilde{f}_j = \left(\tfrac{1}{M}\,\tilde{\boldsymbol{B}}\boldsymbol{F}\boldsymbol{D}\boldsymbol{f}\right)_j \quad \text{bzw.} \quad f(x_j) \approx \tilde{f}_j = \left(\tilde{\boldsymbol{B}}\boldsymbol{F}\boldsymbol{D}\boldsymbol{f}\right)_j$$

Inverse adjoint NFFT

Consider again

$$f(x) = \cos^2(\pi x^2)\ \sin(10x^2), \quad x \in \left[-\tfrac{1}{2}, \tfrac{1}{2}\right)$$

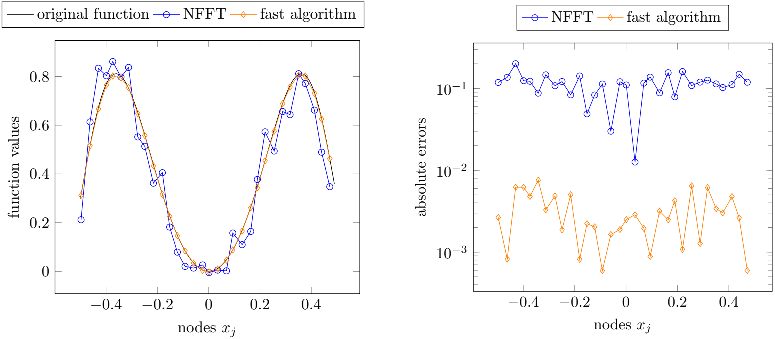
and approximations

$$f(x_j) \approx \tilde{f}_j = \left(\tfrac{1}{M}\, \tilde{\boldsymbol{B}}\boldsymbol{F}\boldsymbol{D}\boldsymbol{f}\right)_j \quad \text{bzw.} \quad f(x_j) \approx \tilde{f}_j = \left(\tilde{\boldsymbol{B}}\boldsymbol{F}\boldsymbol{D}\boldsymbol{f}\right)_j$$

Reconstruction

- of $N = 32$ function values
- from Fourier coefficients $\hat{f}_k$
- overdetermined as well as underdetermined
  - (a) $M = 8$
  - (b) $M = 128$

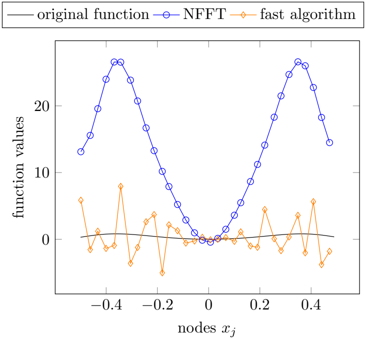
Inverse adjoint NFFT - overdetermined case



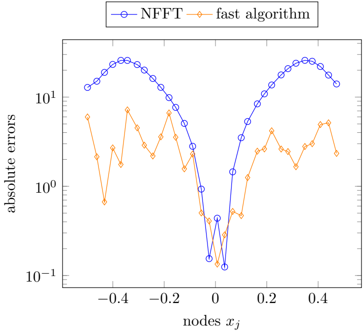
(a) reconstruction

(b) pointwise errors

Inverse adjoint NFFT - underdetermined case



(a) reconstruction



(b) pointwise errors

Summary

- new direct method for general case
- iNFFT based on factorization $\underset{\substack{\nwarrow \\ \text{optimized}}}{\boldsymbol{B}}\,\boldsymbol{F}\boldsymbol{D}$
- fast algorithms of complexity $\mathcal{O}(M \log M + N)$
- connection to frame-theoretical approach

Summary

- new direct method for general case
- iNFFT based on factorization $\underset{\substack{\nwarrow \\ \text{optimized}}}{\boldsymbol{B}}\,\boldsymbol{F}\boldsymbol{D}$
- fast algorithms of complexity $\mathcal{O}(M \log M + N)$
- connection to frame-theoretical approach

Thank you for your attention!