Nonuniform fast Fourier transforms with nonequispaced spatial and frequency data and fast sinc transforms

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In this paper we study the nonuniform fast Fourier transform with nonequispaced spatial and frequency data (NNFFT) and the fast sinc transform as its application. The computation of NNFFT is mainly based on the nonuniform fast Fourier transform with nonequispaced spatial nodes and equispaced frequencies (NFFT). The NNFFT employs two compactly supported, continuous window functions. For fixed nonharmonic bandwidth, we show that the error of the NNFFT with two sinh-type window functions has an exponential decay with respect to the truncation parameters of the used window functions. As an important application of the NNFFT, we present the fast sinc transform. The error of the fast sinc transform is estimated as well.

Key words: nonuniform fast Fourier transform, NUFFT, NNFFT, nonequispaced nodes in space and frequency domain, exponential sums, fast sinc transform, error estimates, sampling.

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1 Introduction

The *discrete Fourier transform* (DFT) can easily be generalized to arbitrary nodes in the space domain as well as in the frequency domain (see [4, 6], [13, pp. 394–397]). Let $N \in \mathbb{N}$ with $N \gg 1$ and $M_1,\, M_2 \in 2\mathbb{N}$ be given. By $\mathcal{I}_{M_1}$ we denote the index set $\{-\frac{M_1}{2},\, 1-\frac{M_1}{2},\, \ldots,\, \frac{M_1}{2}-1\}$. We consider an *exponential sum* $f:\, [-\frac{1}{2},\, \frac{1}{2}] \to \mathbb{C}$ of the

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form

$$f(x) \coloneqq \sum_{k \in \mathcal{I}_{M_1}} f_k \, \mathrm{e}^{-2\pi \mathrm{i} N v_k x}\,, \quad x \in \big[-\tfrac{1}{2},\, \tfrac{1}{2}\big]\,, \tag{1.1}$$

where $f_k \in \mathbb{C}$ are given coefficients and $v_k \in \big[-\frac{1}{2},\, \frac{1}{2}\big]$, $k \in \mathcal{I}_{M_1}$, are arbitrary nodes in the frequency domain. The parameter $N \in \mathbb{N}$ is called *nonharmonic bandwidth* of the exponential sum (1.1).
We assume that a linear combination (1.1) of exponentials with bounded frequencies is given. For arbitrary nodes $x_j \in \big[-\frac{1}{2},\, \frac{1}{2}\big]$, $j \in \mathcal{I}_{M_2}$, in the space domain, we are interested in a fast evaluation of the M_2 values

$$f(x_j) = \sum_{k \in \mathcal{I}_{M_1}} f_k \, \mathrm{e}^{-2\pi \mathrm{i} N v_k x_j}\,, \quad j \in \mathcal{I}_{M_2}\,. \tag{1.2}$$

A fast algorithm for the computation of the M_2 values (1.2) is called a *nonuniform fast Fourier transform with nonequispaced spatial and frequency data* (NNFFT) which was introduced by B. Elbel and G. Steidl in [6]. In this approach, the rapid evaluation of NNFFT is mainly based on the use of two compactly supported, continuous window functions. As in [10] this approach is also referred to as NFFT *of type* 3.
In this paper we present new error estimates for the NNFFT. Since these estimates depend exclusively on the so-called window parameters of the NNFFT, this gives rise to an appropriate parameter choice. The outline of this paper is as follows. In Section 2, we introduce the special set Ω of continuous, even functions $\omega : \mathbb{R} \to [0,\, 1]$ with the support $[-1,\, 1]$. Choosing $\omega_1,\, \omega_2 \in \Omega$, we consider two window functions

$$\varphi_1(t) = \omega_1\bigg(\frac{N_1 t}{m_1}\bigg)\,, \quad \varphi_2(t) = \omega_2\bigg(\frac{N_2 t}{m_2}\bigg)\,, \quad t \in \mathbb{R}\,,$$

where $N_1 \coloneqq \sigma_1 N \in 2\mathbb{N}$ with some oversampling factor $\sigma_1 > 1$ and where $m_1 \in \mathbb{N} \setminus \{1\}$ is a truncation parameter with $2m_1 \ll N_1$. Analogously, $N_2 \coloneqq \sigma_2\,(N_1 + 2m_1) \in 2\mathbb{N}$ is given with some oversampling factor $\sigma_2 > 1$ and $m_2 \in \mathbb{N} \setminus \{1\}$ is another truncation parameter with $2m_2 \ll \big(1 - \frac{1}{\sigma_1}\big)\, N_2$. For the fast, approximate computation of the values (1.2), we formulate the NNFFT in Algorithm 2.2. In Section 3, we derive new explicit error estimates of the NNFFT with two general window functions φ_1 and φ_2. In Section 4, we specify the result when using two sinh-type window functions. Namely, we show that for fixed nonharmonic bandwidth N of (1.1), the error of the related NNFFT has an exponential decay with respect to the truncation parameters m_1 and m_2. Numerical experiments illustrate the performance of our error estimates.
In Section 5, we study the approximation of the function $\mathrm{sinc}(N\pi x)$, $x \in [-1,\, 1]$, by an exponential sum. For given target accuracy $\varepsilon > 0$ and $n \ge 4N$, there exist coefficients $w_j > 0$ and frequencies $v_j \in (-1,\, 1)$, $j = 1\, \ldots, n$, such that for all $x \in [-1,\, 1]$,

$$\bigg|\mathrm{sinc}(N\pi x) - \sum_{j=1}^{n} w_j \, \mathrm{e}^{-\pi \mathrm{i} N v_j x}\bigg| \le \varepsilon\,.$$

In practice, we simplify the approximation procedure. Since for fixed $N \in \mathbb{N}$, it holds

$$\operatorname{sinc}(N\pi x) = \frac{1}{2} \int_{-1}^{1} \mathrm{e}^{-\pi \mathrm{i} N t x} \, \mathrm{d}t \,, \quad x \in \mathbb{R} \,,$$

we apply the Clenshaw–Curtis quadrature with Chebyshev points $z_k = \cos \frac{k\pi}{n} \in [-1,\, 1]$, $k = 0 \ldots, n$, where $n \in \mathbb{N}$ fulfills $n \geq 4N$. Then the function $\operatorname{sinc}(N\pi x)$, $x \in [-1,\, 1]$, can be approximated by the exponential sum

$$\sum_{k=0}^{n} w_k \, \mathrm{e}^{-\pi \mathrm{i} N z_k x} \tag{1.3}$$

with explicitly known coefficients $w_k > 0$ which satisfy the condition $\sum_{k=0}^{n} w_k = 1$.
An interesting signal processing application of the NNFFT is presented in the last Section 6. If a signal $h : \big[-\frac{1}{2},\, \frac{1}{2}\big] \to \mathbb{C}$ is to be reconstructed from its nonuniform samples at $a_k \in \big[-\frac{1}{2},\, \frac{1}{2}\big]$, then h is often modeled as linear combination of shifted sinc functions

$$h(x) = \sum_{k \in \mathcal{I}_{L_1}} c_k \operatorname{sinc}\big(N\pi\,(x - a_k)\big)$$

with complex coefficients c_k. Hence, we present a fast, approximate computation of the *discrete* sinc *transform* (see [7, 11])

$$h(b_\ell) = \sum_{k \in \mathcal{I}_{L_1}} c_k \operatorname{sinc}\big(N\pi\,(b_\ell - a_k)\big)\,, \quad \ell \in \mathcal{I}_{L_2}\,,$$

where $b_\ell \in \big[-\frac{1}{2},\, \frac{1}{2}\big]$ can be nonequispaced. The discrete sinc transform is motivated by numerous applications in signal processing. However, since the sinc function decays slowly, it is often avoided in favor of some more local approximation. Here we prefer the approximation of the sinc function by an exponential sum (1.3). Then we obtain the fast sinc transform in Algorithm 6.1, which is an approximate algorithm for the fast computation of the values (6.2) and applies the NNFFT twice. Besides, the error of the fast sinc transform is estimated and numerical examples are presented as well.

2 NNFFT

Now we start with the explanation of the main algorithm, the NNFFT. To this end, we firstly introduce the special set Ω, which is necessary to define required window functions $\varphi_j,\ j = 1, 2$. Since the NNFFT is mainly based on the well-known NFFT, then we proceed with a short description of the NFFT and move on to the NNFFT afterwards. This procedure is summarized in Algorithm 2.2. Note that here a parameter $a > 1$ is necessary in order to prevent aliasing artifacts, since we approximate a non-periodic function on the interval $[-1,\, 1]$ by means of a-periodic functions.
Let Ω be the set of all functions $\omega : \mathbb{R} \to [0,\, 1]$ with the following properties:

- Each function ω is even, has the support $[-1,\, 1]$, and is continuous on $\mathbb{R}$.
- Each restricted function $\omega|_{[0,\,1]}$ is decreasing with $\omega(0) = 1$.
- For each function ω its Fourier transform

$$\hat{\omega}(v) := \int_{\mathbb{R}} \omega(x)\, \mathrm{e}^{-2\pi \mathrm{i}\, vx}\, \mathrm{d}x = 2 \int_0^1 \omega(x)\, \cos(2\pi v x)\, \mathrm{d}x$$

is positive and decreasing for all $v \in \big[0,\, \frac{m_1}{2\sigma_1}\big]$, where it holds $m_1 \in \mathbb{N} \setminus \{1\}$ and $\sigma_1 \in \big[\frac{5}{4},\, 2\big]$.

Obviously, each $\omega \in \Omega$ is of bounded variation over $[-1,\, 1]$.

Example 2.1 *By B_{2m_1}, we denote the centered cardinal B-spline of even order $2m_1$ with $m_1 \in \mathbb{N}$. Thus, B_2 is the centered hat function. We consider the spline*

$$\omega_{\mathrm{B},1}(x) := \frac{1}{B_{2m_1}(0)}\, B_{2m_1}(m_1 x)\,, \quad x \in \mathbb{R}\,,$$

which has the support $[-1,\, 1]$. Its Fourier transform reads as

$$\hat{\omega}_{\mathrm{B},1}(v) = \frac{1}{m_1\, B_{2m_1}(0)} \Big(\mathrm{sinc}\, \frac{\pi v}{m_1}\Big)^{2m_1}\,, \quad v \in \mathbb{R}\,.$$

Obviously, $\hat{\omega}_{\mathrm{B},1}(v)$ is positive and decreasing for $v \in [0,\, m_1)$. Hence, the function $\omega_{\mathrm{B},1}$ belongs to the set Ω.
For $\sigma_1 > \frac{\pi}{3}$ and $\beta_1 = 3m_1$ with $m_1 \in \mathbb{N} \setminus \{1\}$, we consider

$$\omega_{\mathrm{alg},1}(x) := \begin{cases} (1-x^2)^{\beta_1 - 1/2} & x \in [-1,\, 1]\,, \\ 0 & x \in \mathbb{R} \setminus [-1,\, 1]\,. \end{cases}$$

By [12, p. 8], its Fourier transform reads as

$$\hat{\omega}_{\mathrm{alg},1}(v) = \frac{\pi\, (2\beta_1)!}{4^{\beta_1}\, \beta_1!} \cdot \begin{cases} (\pi v)^{-\beta_1}\, J_{\beta_1}(2\pi v) & v \in \mathbb{R} \setminus \{0\}\,, \\ \frac{1}{\beta_1!} & v = 0\,, \end{cases}$$

where J_{β_1} denotes the Bessel function of order β_1. *By [1, p. 370], it holds for $v \neq 0$ the equality*

$$(\pi v)^{-\beta_1}\, J_{\beta_1}(2\pi v) = \frac{1}{\beta_1!} \prod_{s=1}^{\infty} \Big(1 - \frac{4\pi^2 v^2}{j_{\beta_1,s}^2}\Big)\,,$$

where $j_{\beta_1,s}$ denotes the sth positive zero of J_{β_1}. For $\beta_1 = 3m_1$, it holds $j_{\beta_1,1} > 3m_1 + \pi - \frac{1}{2}$ (see [8]). Hence, by $\sigma_1 > \frac{\pi}{3}$ we get

$$\frac{2\pi m_1}{2\sigma_1\, j_{\beta_1,1}} < \frac{\frac{\pi}{\sigma_1}\, m_1}{3m_1 + \pi - \frac{1}{2}} < \frac{3m_1}{3m_1 + \pi - \frac{1}{2}} < 1\,.$$

Therefore, the Fourier transform $\hat{\omega}_{\mathrm{alg},1}(v)$ is positive and decreasing for $v \in \big[0,\, \frac{m_1}{2\sigma_1}\big]$. Hence, $\omega_{\mathrm{alg},1}$ belongs to the set Ω.

Let $\sigma_1 \in \big[\frac{5}{4},\, 2\big]$ *and* $m_1 \in \mathbb{N} \setminus \{1\}$ *be given. We consider the function*

$$\omega_{\mathrm{sinh},1}(x) := \begin{cases} \frac{1}{\sinh \beta_1}\, \sinh\big(\beta_1 \sqrt{1-x^2}\big) & x \in [-1,\, 1]\,, \\ 0 & x \in \mathbb{R} \setminus [-1,\, 1] \end{cases}$$

with the shape parameter

$$\beta_1 := 2\pi m_1 \Big(1 - \frac{1}{2\sigma_1}\Big)\,.$$

Then by [*12*, p. 38], *its Fourier transform reads as*

$$\hat{\omega}_{\mathrm{sinh},1}(v) = \frac{\pi \beta_1}{\sinh \beta_1} \cdot \begin{cases} (\beta_1^2 - 4\pi^2 v^2)^{-1/2}\, I_1\big(\sqrt{\beta_1^2 - 4\pi^2 v^2}\big) & |v| < m_1\big(1 - \frac{1}{2\sigma_1}\big)\,, \\ \frac{1}{2} & v = \pm m_1\big(1 - \frac{1}{2\sigma_1}\big)\,, \\ (4\pi^2 v^2 - \beta_1^2)^{-1/2}\, J_1\big(\sqrt{4\pi^2 v^2 - \beta_1^2}\big) & |v| > m_1\big(1 - \frac{1}{2\sigma_1}\big)\,, \end{cases} \tag{2.1}$$

where I_1 *and* J_1 *denote the* modified Bessel function *and the* Bessel function of first order, *respectively. Using the power series expansion of* I_1 *(see* [*1*, p. 375]*), we obtain for* $|v| < m_1\big(1 - \frac{1}{2\sigma_1}\big)$ *that*

$$(\beta_1^2 - 4\pi^2 v^2)^{-1/2}\, I_1\Big(\sqrt{\beta_1^2 - 4\pi^2 v^2}\Big) = \frac{1}{2} \sum_{k=0}^{\infty} \frac{1}{4^k k!(k+1)!}\, (\beta_1^2 - 4\pi^2 v^2)^k\,.$$

Therefore, the Fourier transform $\hat{\omega}_{\mathrm{sinh},1}(v)$ *is positive and decreasing for* $v \in \big[0,\, \frac{m_1}{2\sigma_1}\big]$, *since for* $\sigma_1 \geq \frac{5}{4}$ *it holds*

$$\frac{m_1}{2\sigma_1} < m_1\Big(1 - \frac{1}{2\sigma_1}\Big)\,.$$

Hence, $\omega_{\mathrm{sinh},1}$ *belongs to the set* Ω. □

As known (see [6, 14]), the NNFFT can mainly be computed by means of an NFFT. This is why this algorithm is briefly explained below. For fixed $N,\, M_2 \in 2\mathbb{N}$ and $N_1 := \sigma_1 N$ with $\sigma_1 > 1$, the NFFT (see [4, 5, 17] or [13, pp. 377–381]) is a fast algorithm that approximately computes the values $p(x_j)$, $j \in \mathcal{I}_{M_2}$, of any 1-periodic trigonometric polynomial

$$p(x) := \sum_{k \in \mathcal{I}_N} c_k\, \mathrm{e}^{2\pi \mathrm{i}\, kx} \tag{2.2}$$

at nonequispaced nodes $x_j \in \big[-\frac{1}{2},\, \frac{1}{2}\big]$, $j \in \mathcal{I}_{M_2}$, where $c_k \in \mathbb{C}$, $k \in \mathcal{I}_N$, are given complex coefficients. In other words, for the NFFT it holds $N = M_1 \in 2\mathbb{N}$ in (1.2).
For $\omega_1 \in \Omega$ we introduce the *window function*

$$\varphi_1(t) := \omega_1\Big(\frac{N_1 t}{m_1}\Big)\,, \quad t \in \mathbb{R}\,. \tag{2.3}$$

By construction, the window function (2.3) is even, has the support $\big[-\frac{m_1}{N_1},\, \frac{m_1}{N_1}\big]$, and is continuous on $\mathbb{R}$. Further, the restricted window function $\varphi_1|_{[0,\, m_1/N_1]}$ is decreasing with $\varphi_1(0) = 1$. Its Fourier transform

$$\hat{\varphi}_1(v) := \int_{\mathbb{R}} \varphi_1(t)\, \mathrm{e}^{-2\pi \mathrm{i}\, vt}\, \mathrm{d}t = 2 \int_0^{m_1/N_1} \varphi_1(t)\, \cos(2\pi v t)\, \mathrm{d}t$$

is positive and decreasing for $v \in \big[0,\, N_1 - \frac{N}{2}\big)$. Thus, φ_1 is of bounded variation over $\big[-\frac{1}{2},\, \frac{1}{2}\big]$.
In the following, we denote the torus $\mathbb{R}/\mathbb{Z}$ by $\mathbb{T}$ and the Banach space of continuous, 1-periodic functions by $C(\mathbb{T})$. For the window function (2.3), we denote its 1-periodization by

$$\tilde{\varphi}_1^{(1)}(x) := \sum_{k\in\mathbb{Z}} \varphi_1(x+k)\,, \quad x \in \mathbb{R}\,.$$

Using a linear combination of shifted versions of the 1-periodized window function $\tilde{\varphi}_1^{(1)}$, we construct a 1-periodic continuous function $s \in C(\mathbb{T})$ which approximates (2.2) well. Then the computation of the values $s(x_j)$, $j \in \mathcal{I}_{M_2}$, is very easy, since φ_1 has the small support $\big[-\frac{m_1}{N_1},\, \frac{m_1}{N_1}\big]$. The computational cost of NFFT is $\mathcal{O}\big(N \log N + M_2\big)$ flops, see [4, 5, 17] or [13, pp. 377–381]. The error of the NFFT (see [15]) can be estimated by

$$\begin{aligned}\max_{j\in\mathcal{I}_{M_2}} |s(x_j) - p(x_j)| \;&\le\; \|s-p\|_{C(\mathbb{T})} := \max_{x\in[-1/2,1/2]} |s(x)-p(x)| \\ &\le\; e_{\sigma_1}(\varphi_1) \sum_{n\in\mathcal{I}_N} |c_n|\,,\end{aligned}$$

where $e_{\sigma_1}(\varphi_1)$ denotes the $C(\mathbb{T})$-*error constant* defined as

$$e_{\sigma_1}(\varphi_1) = \sup_{N\in 2\mathbb{N}} e_{\sigma_1,N}(\varphi_1) \tag{2.4}$$

with

$$e_{\sigma_1,N}(\varphi_1) := \max_{n\in\mathcal{I}_N} \Bigg\| \sum_{r\in\mathbb{Z}\setminus\{0\}} \frac{\hat{\varphi}_1(n+rN_1)}{\hat{\varphi}_1(n)}\, \mathrm{e}^{2\pi\mathrm{i}\, rN_1\cdot} \Bigg\|_{C(\mathbb{T})}\,.$$

Note that the constants $e_{\sigma_1,N}(\varphi_1)$ are bounded with respect to N (see [15, Theorem 5.1]).

Now we proceed with the NNFFT. For better readability, we describe the procedure just shortly. For more detailed explanations we refer to [6]. For chosen functions $\omega_1,\, \omega_2 \in \Omega$, we form the window functions

$$\varphi_1(t) := \omega_1\Big(\frac{N_1 t}{m_1}\Big)\,, \quad \varphi_2(t) := \omega_2\Big(\frac{N_2 t}{m_2}\Big)\,, \quad t \in \mathbb{R}\,, \tag{2.5}$$

where again $N_1 := \sigma_1 N \in 2\mathbb{N}$ with some oversampling factor $\sigma_1 > 1$ and $m_1 \in \mathbb{N}\setminus\{1\}$ with $2m_1 \ll N_1$ and where $N_2 := \sigma_2\,(N_1 + 2m_1) \in 2\mathbb{N}$ with an oversampling factor $\sigma_2 > 1$ and $m_2 \in \mathbb{N}\setminus\{1\}$ with $2m_2 \le \big(1 - \frac{1}{\sigma_1}\big)\, N_2$. The second window function φ_2 has the support $\big[-\frac{m_2}{N_2},\, \frac{m_2}{N_2}\big]$. Additionally, in order to prevent aliasing, we use a-periodic functions, where we introduce the constant

$$a := 1 + \frac{2m_1}{N_1} > 1\,, \tag{2.6}$$

such that $aN_1 = N_1 + 2m_1$ and $N_2 = \sigma_2\sigma_1 a N$. Without loss of generality, we can assume that

$$v_k \in \big[-\tfrac{1}{2a},\, \tfrac{1}{2a}\big]\,. \tag{2.7}$$

If $v_k \in \big[-\frac{1}{2},\, \frac{1}{2}\big]$, then we replace the nonharmonic bandwidth N by $N^* := N + \lceil \frac{2m_1}{\sigma_1} \rceil$ and set $v_j^* := \frac{N}{N^*}\, v_j \in \big[-\frac{1}{2a},\, \frac{1}{2a}\big]$ such that $N v_j = N^* v_j^*$.
For arbitrarily given $f_k \in \mathbb{C}$, $k \in \mathcal{I}_{M_1}$, and $v_k \in \big[-\frac{1}{2a},\, \frac{1}{2a}\big]$, $k \in \mathcal{I}_{M_1}$, we introduce the compactly supported, continuous auxiliary function

$$h(t) := \sum_{k \in \mathcal{I}_{M_1}} f_k\, \varphi_1(t - v_k)\,, \quad t \in \mathbb{R}\,,$$

which has the Fourier transform

$$\begin{aligned} \hat{h}(Nx) &= \int_{\mathbb{R}} h(t)\, \mathrm{e}^{-2\pi \mathrm{i} N x t}\, \mathrm{d}t \\ &= \sum_{k \in \mathcal{I}_{M_1}} f_k \int_{\mathbb{R}} \varphi_1(t - v_k)\, \mathrm{e}^{-2\pi \mathrm{i} N x t}\, \mathrm{d}t \qquad (2.8) \\ &= \sum_{k \in \mathcal{I}_{M_1}} f_k\, \mathrm{e}^{-2\pi \mathrm{i} N v_k x}\, \hat{\varphi}_1(Nx) = f(x)\, \hat{\varphi}_1(Nx)\,, \quad x \in \mathbb{R}\,. \qquad (2.9) \end{aligned}$$

Hence, for arbitrary nodes $x_j \in \big[-\frac{1}{2},\, \frac{1}{2}\big]$, $j \in \mathcal{I}_{M_2}$, we have

$$f(x_j) = \frac{\hat{h}(N x_j)}{\hat{\varphi}_1(N x_j)}\,, \quad j \in \mathcal{I}_{M_2}\,.$$

Therefore, it remains to compute the values $\hat{h}(N x_j)$, $j \in \mathcal{I}_{M_2}$, because we can precompute the values $\hat{\varphi}_1(N x_j)$, $j \in \mathcal{I}_{M_2}$. In some cases (see Section 4), these values $\hat{\varphi}_1(N x_j)$, $j \in \mathcal{I}_{M_2}$, are explicitly known.
For arbitrary $v_k \in \big[-\frac{1}{2a},\, \frac{1}{2a}\big]$, $k \in \mathcal{I}_{M_1}$, we have $\varphi_1(t - v_k) = 0$ for all $t < -\frac{1}{2a} - \frac{m_1}{N_1} = -\frac{a}{2} + \big(\frac{1}{2} - \frac{1}{2a}\big)$ and for all $t > \frac{1}{2a} + \frac{m_1}{N_1} = \frac{a}{2} - \big(\frac{1}{2} - \frac{1}{2a}\big)$, since $\operatorname{supp} \varphi_1 = \big[-\frac{m_1}{N_1},\, \frac{m_1}{N_1}\big]$ and $\frac{1}{2} - \frac{1}{2a} > 0$. Thus, by (2.8) and

$$\operatorname{supp} \varphi_1(\cdot - v_k) \subset \Big[-\frac{a}{2},\, \frac{a}{2}\Big]\,, \quad k \in \mathcal{I}_{M_1}\,,$$

we obtain

$$\hat{h}(Nx) = \sum_{k \in \mathcal{I}_{M_1}} f_k \int_{-a/2}^{a/2} \varphi_1(t - v_k)\, \mathrm{e}^{-2\pi \mathrm{i} N x t}\, \mathrm{d}t\,, \quad x \in \mathbb{R}\,.$$

Then the rectangular quadrature rule leads to

$$s(Nx) := \sum_{k \in \mathcal{I}_{M_1}} f_k\, \frac{1}{N_1} \sum_{\ell \in \mathcal{I}_{N_1 + 2m_1}} \varphi_1\Big(\frac{\ell}{N_1} - v_k\Big)\, \mathrm{e}^{-2\pi \mathrm{i}\, \ell x/\sigma_1}\,, \quad x \in \mathbb{R}\,, \qquad (2.10)$$

which approximates $\hat{h}(Nx)$. Note that $\frac{\ell}{N_1} \in \big[-\frac{a}{2},\, \frac{a}{2}\big]$ for each $\ell \in \mathcal{I}_{N_1 + 2m_1}$ by $N_1 + 2m_1 = a N_1$. Changing the order of summations in (2.10), it follows that

$$s(Nx) = \sum_{\ell \in \mathcal{I}_{N_1 + 2m_1}} \bigg(\frac{1}{N_1} \sum_{k \in \mathcal{I}_{M_1}} f_k\, \varphi_1\Big(\frac{\ell}{N_1} - v_k\Big)\bigg)\, \mathrm{e}^{-2\pi \mathrm{i}\, \ell x/\sigma_1}\,. \qquad (2.11)$$

After computation of the inner sums

$$g_\ell := \frac{1}{N_1} \sum_{k \in \mathcal{I}_{M_1}} f_k \, \varphi_1\Big(\frac{\ell}{N_1} - v_k\Big)\,, \quad \ell \in \mathcal{I}_{N_1+2m_1}\,, \tag{2.12}$$

we arrive at the following NFFT

$$s(N\,x_j) = \sum_{\ell \in \mathcal{I}_{N_1+2m_1}} g_\ell \, \mathrm{e}^{-2\pi \mathrm{i}\, \ell x_j/\sigma_1}\,, \quad j \in \mathcal{I}_{M_2}\,.$$

If we denote the result of this NFFT (with the 1-periodization $\tilde{\varphi}_2^{(1)}$ of the second window function φ_2 and $N_2 := \sigma_2\,(N_1 + 2m_1)$) by $s_1(Nx_j)$, then $s_1(Nx_j)/\hat{\varphi}_1(Nx_j)$ is an approximate value of $f(x_j)$, $j \in \mathcal{I}_{M_2}$. Thus, the algorithm can be summarized as follows.

Algorithm 2.2 (NNFFT)
Input: Nonharmonic bandwidth $N \in \mathbb{N}$ *with* $N \gg 1$, *numbers of nodes* $M_1,\, M_2 \in 2\mathbb{N}$, $N_1 := \sigma_1 N \in 2\mathbb{N}$ *with oversampling factor* $\sigma_1 > 1$ *and truncation parameter* $m_1 \in \mathbb{N} \setminus \{1\}$ *with* $2m_1 \ll N_1$, $N_2 := \sigma_2\,(N_1 + 2m_1) \in 2\mathbb{N}$ *with oversampling factor* $\sigma_2 > 1$ *and truncation parameter* $m_2 \in \mathbb{N} \setminus \{1\}$ *with* $2m_2 \le \big(1 - \frac{1}{\sigma_1}\big)\, N_2$,
arbitrary nodes $v_k \in \big[-\frac{1}{2a},\, \frac{1}{2a}\big]$, $k \in \mathcal{I}_{M_1}$, *in the frequency domain with* $a := 1 + \frac{2m_1}{N_1}$, *arbitrary nodes* $x_j \in \big[-\frac{1}{2},\, \frac{1}{2}\big]$, $j \in \mathcal{I}_{M_2}$, *in the spatial domain as well as window functions* φ_1 *and* φ_2 *given by* (2.5).

0. *Precompute the following values:*
   (i) $\hat{\varphi}_1(Nx_j)$ *for* $j \in \mathcal{I}_{M_2}$, $\hat{\varphi}_2\big(\frac{\ell}{a}\big)$ *for* $\ell \in \mathcal{I}_{N_1+2m_1}$,
   (ii) $\varphi_1\big(\frac{\ell}{N_1} - v_k\big)$ *for* $k \in \mathcal{I}_{M_1}$ *and* $\ell \in \mathcal{I}'_{N_1+2m_1}(v_k) := \{\ell \in \mathcal{I}_{N_1+2m_1} : \big|\frac{\ell}{N_1} - v_k\big| < \frac{m_1}{N_1}\}$,
   (iii) $\varphi_2\big(\frac{x_j}{\sigma_1} - \frac{s}{N_2}\big)$ *for* $j \in \mathcal{I}_{M_2}$ *and* $s \in \mathcal{I}''_{N_2}(x_j) := \{s \in \mathcal{I}_{N_2} : \big|\frac{s}{N_2} - \frac{x_j}{\sigma_1}\big| < \frac{m_2}{N_2}\}$,
   (iv) *Further set* $\varphi_1\big(\frac{\ell}{N_1} - v_k\big) := 0$ *for* $k \in \mathcal{I}_{M_1}$ *and* $\ell \in \mathcal{I}_{N_1+2m_1} \setminus \mathcal{I}'_{N_1+2m_1}(v_k)$.
1. *For all* $\ell \in \mathcal{I}_{N_1+2m_1}$ *compute the sums* (2.12).
2. *For all* $\ell \in \mathcal{I}_{N_1+2m_1}$ *form the values*

$$\hat{g}_\ell := \frac{g_\ell}{\hat{\varphi}_2(\ell)}\,.$$

3. *For all* $s \in \mathcal{I}_{N_2}$ *compute by fast Fourier transform* (FFT) *of length* N_2

$$h_s := \frac{1}{N_2} \sum_{\ell \in \mathcal{I}_{N_1+2m_1}} \hat{g}_\ell \, \mathrm{e}^{-2\pi \mathrm{i}\, \ell s/N_2}\,.$$

4. *For all* $j \in \mathcal{I}_{M_2}$ *calculate the short sums*

$$s_1(Nx_j) := \sum_{s \in \mathcal{I}''_{N_2}(x_j)} h_s \, \varphi_2\Big(\frac{x_j}{\sigma_1} - \frac{s}{N_2}\Big)\,.$$

Output: $s_1(Nx_j)/\hat{\varphi}_1(Nx_j)$ *approximate value of* (1.2) *for* $j \in \mathcal{I}_{M_2}$.

The computational cost of the NNFFT is equal to $\mathcal{O}\big(N \log N + M_1 + M_2\big)$ flops.
In Step 4 of Algorithm 2.2 we use the assumption $2m_2 \le \big(1 - \frac{1}{\sigma_1}\big)\, N_2$ such that

$$\frac{1}{2\sigma_1} + \frac{m_2}{N_2} \le \frac{1}{2}\,.$$

Then for all $j \in \mathcal{I}_{M_2}$ and $s \in \mathcal{I}_{N_2}$, it holds

$$\tilde{\varphi}_2^{(1)}\Big(\frac{x_j}{\sigma_1} - \frac{s}{N_2}\Big) = \varphi_2\Big(\frac{x_j}{\sigma_1} - \frac{s}{N_2}\Big)\,.$$

Since we approximate a non-periodic function f on the interval $\big[-\frac{1}{2},\, \frac{1}{2}\big]$ by means of a-periodic functions on the torus $a\mathbb{T} \cong \big[-\frac{a}{2},\, \frac{a}{2}\big)$, the parameter a has to fulfill the condition $a > 1$, in order to prevent aliasing artifacts.

3 Error estimates for NNFFT

Now we study the error of the NNFFT, which is measured in the form

$$\max_{j \in \mathcal{I}_{M_2}} \left| f(x_j) - \frac{s_1(N x_j)}{\hat{\varphi}_1(N x_j)} \right|,$$

where f is a given exponential sum (1.1) and $x_j \in \big[-\frac{1}{2},\, \frac{1}{2}\big]$, $j \in \mathcal{I}_{M_2}$, are arbitrary spatial nodes. At the beginning of this section we present some technical lemmas. The main result will be Theorem 3.5.
We introduce the *a-periodization* of the given window function (2.3) by

$$\tilde{\varphi}_1^{(a)}(x) \coloneqq \sum_{\ell \in \mathbb{Z}} \varphi_1(x + a\,\ell)\,, \quad x \in \mathbb{R}\,. \tag{3.1}$$

For each $x \in \mathbb{R}$, the above series (3.1) has at most one nonzero term. This can be seen as follows: For arbitrary $x \in \mathbb{R}$ there exists a unique $\ell^* \in \mathbb{Z}$ such that $x = -a\,\ell^* + r$ with a residuum $r \in \big[-\frac{a}{2},\, \frac{a}{2}\big)$. Then $\varphi_1(x + a\,\ell^*) = \varphi_1(r)$ and hence $\varphi_1(r) > 0$ for $r \in \big(-\frac{m_1}{N_1},\, \frac{m_1}{N_1}\big)$ and $\varphi_1(r) = 0$ for $r \in \big[-\frac{a}{2},\, -\frac{m_1}{N_1}\big] \cup \big[\frac{m_1}{N_1},\, \frac{a}{2}\big)$. For each $\ell \in \mathbb{Z} \setminus \{\ell^*\}$, we have

$$\varphi_1(x + a\,\ell) = \varphi_1\big(a\,(\ell - \ell^*) + r\big) = 0\,,$$

since $\big|a\,(\ell - \ell^*) + r\big| \ge \frac{a}{2} = \frac{1}{2} + \frac{m_1}{N_1} > \frac{m_1}{N_1}$. Further it holds

$$\tilde{\varphi}_1^{(a)}(x) = \varphi_1(x)\,, \quad x \in \big[-1 - \tfrac{m_1}{N_1},\, 1 + \tfrac{m_1}{N_1}\big]\,.$$

By the construction of φ_1, the a-periodic window function (3.1) is continuous on $\mathbb{R}$ and of bounded variation over $\big[-\frac{a}{2},\, \frac{a}{2}\big]$. Then the kth Fourier coefficient of the a-periodic window function (3.1) reads as follows

$$c_k^{(a)}\Big(\tilde{\varphi}_1^{(a)}\Big) \coloneqq \frac{1}{a}\int_{-a/2}^{a/2} \tilde{\varphi}_1^{(a)}(t)\,\mathrm{e}^{-2\pi\mathrm{i}\,kt/a}\,\mathrm{d}t = \frac{1}{a}\,\hat{\varphi}_1\Big(\frac{k}{a}\Big)\,, \quad k \in \mathbb{Z}\,. \tag{3.2}$$

By the convergence theorem of Dirichlet–Jordan (see [19, Vol. 1, pp. 57–58]), the a-periodic Fourier series of (3.1) converges uniformly on $\mathbb{R}$ and it holds

$$\tilde{\varphi}_1^{(a)}(x) = \sum_{k\in\mathbb{Z}} c_k^{(a)}\Big(\tilde{\varphi}_1^{(a)}\Big)\,\mathrm{e}^{2\pi\mathrm{i}\,kx/a} = \frac{1}{a}\sum_{k\in\mathbb{Z}} \hat{\varphi}_1\Big(\frac{k}{a}\Big)\,\mathrm{e}^{2\pi\mathrm{i}\,kx/a}\,. \tag{3.3}$$

Then we have the following technical lemma.

Lemma 3.1 *Let the window function φ_1 be given by (2.3). Then for any $n \in \mathcal{I}_N$ with $N \in 2\mathbb{N}$, the series*

$$\sum_{r\in\mathbb{Z}} c_{n+r\,(N_1+2m_1)}^{(a)}\Big(\tilde{\varphi}_1^{(a)}\Big)\,\mathrm{e}^{2\pi\mathrm{i}\,(n+r\,(N_1+2m_1))x/a}$$

is uniformly convergent on $\mathbb{R}$ and has the sum

$$\frac{1}{N_1+2m_1}\sum_{\ell\in\mathcal{I}_{N_1+2m_1}} \mathrm{e}^{-2\pi\mathrm{i}\,n\ell/(N_1+2m_1)}\,\tilde{\varphi}_1^{(a)}\Big(x+\frac{\ell}{N_1}\Big)$$

which coincides with the rectangular quadrature rule of the integral

$$c_n^{(a)}\Big(\tilde{\varphi}_1^{(a)}(x+\cdot)\Big) = \frac{1}{a}\int_{-a/2}^{a/2} \tilde{\varphi}_1^{(a)}(x+s)\,\mathrm{e}^{2\pi\mathrm{i}\,ns/a}\,\mathrm{d}s = c_n^{(a)}\Big(\tilde{\varphi}_1^{(a)}\Big)\,\mathrm{e}^{2\pi\mathrm{i}\,nx/a}\,.$$

Proof. Using the uniformly convergent Fourier series (3.3), we obtain for all $n \in \mathcal{I}_N$ that

$$\mathrm{e}^{-2\pi\mathrm{i}\,nx/a}\,\tilde{\varphi}_1^{(a)}(x) = \sum_{k\in\mathbb{Z}} c_k^{(a)}\Big(\tilde{\varphi}_1^{(a)}\Big)\,\mathrm{e}^{2\pi\mathrm{i}\,(k-n)x/a} = \sum_{q\in\mathbb{Z}} c_{n+q}^{(a)}\Big(\tilde{\varphi}_1^{(a)}\Big)\,\mathrm{e}^{2\pi\mathrm{i}\,qx/a}\,.$$

Replacing x by $x + \frac{\ell}{N_1}$ with $\ell \in \mathcal{I}_{N_1+2m_1}$, we see that by $N_1 + 2m_1 = aN_1$,

$$\mathrm{e}^{-2\pi\mathrm{i}\,n(x+\ell/N_1)/a}\,\tilde{\varphi}_1^{(a)}\Big(x+\frac{\ell}{N_1}\Big) = \sum_{q\in\mathbb{Z}} c_{n+q}^{(a)}\Big(\tilde{\varphi}_1^{(a)}\Big)\,\mathrm{e}^{2\pi\mathrm{i}\,qx/a}\,\mathrm{e}^{2\pi\mathrm{i}\,q\ell/(N_1+2m_1)}\,.$$

Summing the above formulas for all $\ell \in \mathcal{I}_{N_1+2m_1}$ and applying the known formula

$$\sum_{\ell\in\mathcal{I}_{N_1+2m_1}} \mathrm{e}^{2\pi\mathrm{i}\,q\ell/(N_1+2m_1)} = \begin{cases} N_1+2m_1 & q \equiv 0 \bmod (N_1+2m_1)\,, \\ 0 & q \not\equiv 0 \bmod (N_1+2m_1)\,, \end{cases}$$

we conclude that

$$\begin{aligned}&\sum_{\ell\in\mathcal{I}_{N_1+2m_1}} \mathrm{e}^{-2\pi\mathrm{i}\,n(x+\ell/N_1)/a}\,\tilde{\varphi}_1^{(a)}\Big(x+\frac{\ell}{N_1}\Big)\\ &= (N_1+2m_1)\sum_{r\in\mathbb{Z}} c_{n+r\,(N_1+2m_1)}^{(a)}\Big(\tilde{\varphi}_1^{(a)}\Big)\,\mathrm{e}^{2\pi\mathrm{i}\,r\,(N_1+2m_1)x/a}\,.\end{aligned}$$

Obviously,

$$\frac{1}{N_1+2m_1} \sum_{\ell \in \mathcal{I}_{N_1+2m_1}} \mathrm{e}^{-2\pi \mathrm{i}\, n(x+\ell/N_1)/a}\, \tilde{\varphi}_1^{(a)}\Big(x + \frac{\ell}{N_1}\Big)$$

is the rectangular quadrature formula of the integral

$$\frac{1}{a} \int_{-a/2}^{a/2} \tilde{\varphi}_1^{(a)}(x+s)\, \mathrm{e}^{2\pi \mathrm{i}\, ns/a}\, \mathrm{d}s$$

with respect to the uniform grid $\big\{\frac{\ell}{N_1} \,:\, \ell \in \mathcal{I}_{N_1+2m_1}\big\}$ of the interval $\big[-\frac{a}{2},\, \frac{a}{2}\big]$. This completes the proof. ■

By (3.2) we obtain that for $n \in \mathcal{I}_N$,

$$\bigg|\sum_{r \in \mathbb{Z}\setminus\{0\}} \frac{c_{n+r\,(N_1+2m_1)}^{(a)}(\tilde{\varphi}_1^{(a)})}{c_n^{(a)}(\tilde{\varphi}_1^{(a)})}\, \mathrm{e}^{2\pi \mathrm{i}\, r\,(N_1+2m_1)x/a}\bigg| = \bigg|\sum_{r \in \mathbb{Z}\setminus\{0\}} \frac{\hat{\varphi}_1(n/a + rN_1)}{\hat{\varphi}_1(n/a)}\, \mathrm{e}^{2\pi \mathrm{i}\, rN_1 x/a}\bigg|\,.$$

Now we generalize the technical Lemma 3.1.

Lemma 3.2 *For arbitrary fixed* $v \in \big[-\frac{N}{2},\, \frac{N}{2}\big]$*,* $N \in \mathbb{N}$*, and given window function* (2.3)*, the function*

$$\psi_1(x) := \frac{1}{N_1} \sum_{\ell \in \mathbb{Z}} \mathrm{e}^{-2\pi \mathrm{i}\, v\ell/(N_1+2m_1)}\, \mathrm{e}^{-2\pi \mathrm{i}\, vx/a}\, \varphi_1\Big(x + \frac{\ell}{N_1}\Big) \tag{3.4}$$

is $\frac{1}{N_1}$*-periodic, continuous on* $\mathbb{R}$*, and of bounded variation over* $\big[-\frac{1}{2},\, \frac{1}{2}\big]$*. For each* $x \in \mathbb{R}$*, the corresponding* $\frac{1}{N_1}$*-periodic Fourier series converges uniformly to* $\psi_1(x)$*, i. e.,*

$$\psi_1(x) = \sum_{r \in \mathbb{Z}} \hat{\varphi}_1\Big(\frac{v}{a} + rN_1\Big)\, \mathrm{e}^{2\pi \mathrm{i}\, rN_1 x}\,.$$

Proof. The definition (3.4) of the function ψ_1 is correct, since

$$\psi_1(x) = \frac{1}{N_1} \sum_{\ell \in \mathbb{Z}_{m_1,N_1}(x)} \mathrm{e}^{-2\pi \mathrm{i}\, v\ell/(N_1+2m_1)}\, \mathrm{e}^{-2\pi \mathrm{i}\, vx/a}\, \varphi_1\Big(x + \frac{\ell}{N_1}\Big)$$

with the finite index set $\mathbb{Z}_{m_1,N_1}(x) = \{\ell \in \mathbb{Z} \,:\, |N_1 x + \ell| < m_1\}$. If $x \in \big[-\frac{1}{2},\, \frac{1}{2}\big]$, we observe that $\mathbb{Z}_{m_1,N_1}(x) \subseteq \mathcal{I}_{N_1+2m_1}$ and therefore

$$\psi_1(x) = \frac{1}{N_1} \sum_{\ell \in \mathcal{I}_{N_1+2m_1}} \mathrm{e}^{-2\pi \mathrm{i}\, v\ell/(N_1+2m_1)}\, \mathrm{e}^{-2\pi \mathrm{i}\, vx/a}\, \varphi_1\Big(x + \frac{\ell}{N_1}\Big)\,.$$

Simple calculation shows that for each $x \in \mathbb{R}$,

$$\psi_1\Big(x + \frac{1}{N_1}\Big) = \frac{1}{N_1} \sum_{\ell \in \mathbb{Z}} \mathrm{e}^{-2\pi \mathrm{i}\, v(\ell+1)/(N_1+2m_1)}\, \mathrm{e}^{-2\pi \mathrm{i}\, vx/a}\, \varphi_1\Big(x + \frac{\ell+1}{N_1}\Big) = \psi_1(x)\,.$$

By the construction of φ_1, the $\frac{1}{N_1}$-periodic function ψ_1 is continuous on $\mathbb{R}$ and of bounded variation over $\big[-\frac{1}{2},\,\frac{1}{2}\big]$. Thus, by the convergence theorem of Dirichlet–Jordan, the Fourier series of ψ_1 converges uniformly on $\mathbb{R}$ to ψ_1. The rth Fourier coefficient of ψ_1 reads as follows

$$\begin{aligned} c_r^{(1/N_1)}(\psi_1) &= N_1 \int_0^{1/N_1} \psi_1(t)\, \mathrm{e}^{-2\pi\mathrm{i}\, rN_1 t}\, \mathrm{d}t \\ &= \sum_{\ell\in\mathbb{Z}} \mathrm{e}^{-2\pi\mathrm{i}\, v\ell/(N_1+2m_1)} \int_0^{1/N_1} \mathrm{e}^{-2\pi\mathrm{i}\, vt/a}\, \varphi_1\Big(t+\frac{\ell}{N_1}\Big)\, \mathrm{d}t \\ &= \sum_{\ell\in\mathbb{Z}} \int_{\ell/N_1}^{(\ell+1)/N_1} \varphi_1(s)\, \mathrm{e}^{-2\pi\mathrm{i}\,(v/a+rN_1)\, s}\, \mathrm{d}s = \hat\varphi_1\Big(\frac{v}{a}+rN_1\Big)\,, \quad r\in\mathbb{Z}\,. \end{aligned}$$

This completes the proof. ■

From Lemma 3.2 leads immediately to the following technical result.

Corollary 3.3 *Let the window function φ_1 be given by (2.3). For all $x\in\big[-\frac{1}{2},\,\frac{1}{2}\big]$ and $w\in\big[-\frac{N}{2a},\,\frac{N}{2a}\big]$ it holds then*

$$\begin{aligned} &\sum_{r\in\mathbb{Z}\setminus\{0\}} \frac{\hat\varphi_1(w+rN_1)}{\hat\varphi_1(w)}\, \mathrm{e}^{2\pi\mathrm{i}\,(w+rN_1)\,x} \\ &= \frac{1}{N_1\,\hat\varphi_1(w)} \sum_{\ell\in\mathcal{I}_{N_1+2m_1}} \mathrm{e}^{-2\pi\mathrm{i}\, w\ell/N_1}\, \varphi_1\Big(x+\frac{\ell}{N_1}\Big) - \mathrm{e}^{2\pi\mathrm{i}\, wx}\,. \end{aligned} \tag{3.5}$$

Further, for all $w\in\big[-\frac{N}{2a},\,\frac{N}{2a}\big]$, it holds

$$\begin{aligned} &\max_{x\in[-1/2,\,1/2]} \bigg|\frac{1}{N_1\,\hat\varphi_1(w)} \sum_{\ell\in\mathcal{I}_{N_1+2m_1}} \varphi_1\Big(x+\frac{\ell}{N_1}\Big)\, \mathrm{e}^{-2\pi\mathrm{i}\, w\ell/N_1} - \mathrm{e}^{2\pi\mathrm{i}\, wx}\bigg| \\ &= \bigg\| \sum_{r\in\mathbb{Z}\setminus\{0\}} \frac{\hat\varphi_1(w+rN_1)}{\hat\varphi_1(w)}\, \mathrm{e}^{2\pi\mathrm{i}\, rN_1\,\cdot} \bigg\|_{C(\mathbb{T})}\,. \end{aligned} \tag{3.6}$$

Proof. As before, let $v\in\big[-\frac{N}{2},\,\frac{N}{2}\big]$ be given. Substituting $w:=\frac{v}{a}\in\big[-\frac{N}{2a},\,\frac{N}{2a}\big]$ and observing $N_1+2m_1=aN_1$, we obtain by Lemma 3.2 that for all $x\in\big[-\frac{1}{2},\,\frac{1}{2}\big]$ it holds,

$$\begin{aligned} &\frac{1}{N_1} \sum_{\ell\in\mathcal{I}_{N_1+2m_1}} \mathrm{e}^{-2\pi\mathrm{i}\, w\ell/N_1}\, \mathrm{e}^{-2\pi\mathrm{i}\, w\,x}\, \varphi_1\Big(x+\frac{\ell}{N_1}\Big) - \hat\varphi_1(w) \\ &= \sum_{r\in\mathbb{Z}\setminus\{0\}} \hat\varphi_1(w+rN_1)\, \mathrm{e}^{2\pi\mathrm{i}\, rN_1\, x}\,. \end{aligned}$$

Since by assumption $\hat\varphi_1(w)>0$ for all $w\in\big[-\frac{N}{2a},\,\frac{N}{2a}\big]\subset\big[-\frac{N}{2},\,\frac{N}{2}\big]$, we have

$$\frac{1}{N_1\hat\varphi_1(w)} \sum_{\ell\in\mathcal{I}_{N_1+2m_1}} \mathrm{e}^{-2\pi\mathrm{i}\, w\ell/N_1}\, \mathrm{e}^{-2\pi\mathrm{i}\, w\,x}\, \varphi_1\Big(x+\frac{\ell}{N_1}\Big) - 1$$

$$= \sum_{r \in \mathbb{Z}\setminus\{0\}} \frac{\hat\varphi_1(w + rN_1)}{\hat\varphi_1(w)}\, \mathrm{e}^{2\pi\mathrm{i}\, rN_1\, x}\,.$$

Multiplying the above equality by the exponential $\mathrm{e}^{2\pi\mathrm{i}\, w\, x}$, this results in (3.5) and (3.6). ■

We say that the window function φ_1 of the form (2.3) is *convenient for* NNFFT, if the *general* $C(\mathbb{T})$*-error constant*

$$E_{\sigma_1}(\varphi_1) := \sup_{N\in\mathbb{N}} E_{\sigma_1,N}(\varphi_1) \tag{3.7}$$

with

$$E_{\sigma_1,N}(\varphi_1) := \max_{v\in[-N/2,\,N/2]} \Big\| \sum_{r\in\mathbb{Z}\setminus\{0\}} \frac{\hat\varphi_1(v + rN_1)}{\hat\varphi_1(v)}\, \mathrm{e}^{2\pi\mathrm{i}\, rN_1\cdot} \Big\|_{C(\mathbb{T})} \tag{3.8}$$

fulfills the condition $E_{\sigma_1}(\varphi_1) \ll 1$ for conveniently chosen truncation parameter $m_1 \ge 2$ and oversampling factor $\sigma_1 > 1$. Obviously, the $C(\mathbb{T})$-error constant (2.4) is a "discrete" version of the general $C(\mathbb{T})$-error constant (3.7) with the property

$$e_{\sigma_1}(\varphi_1) \le E_{\sigma_1}(\varphi_1)\,. \tag{3.9}$$

Thus, Corollary 3.3 means that all complex exponentials $\mathrm{e}^{2\pi\mathrm{i}\, wx}$ with $w \in \big[-\frac{N}{2a},\, \frac{N}{2a}\big]$ and $x \in \big[-\frac{1}{2},\, \frac{1}{2}\big]$ can be uniformly approximated by short linear combinations of shifted window functions, cf. [4, Theorem 2.10], if φ_1 is convenient for NNFFT.

Theorem 3.4 *Let $\sigma_1 > 1$, $m_1 \in \mathbb{N}\setminus\{1\}$, and $N_1 = \sigma_1 N \in 2\mathbb{N}$ with $2m_1 \ll N_1$ be given. Let φ_1 be the scaled version* (2.3) *of $\omega_1 \in \Omega$. Assume that the Fourier transform $\hat\omega_1$ fulfills the decay condition*

$$|\hat\omega_1(v)| \le \begin{cases} c_1 & |v| \in \big[m_1\big(1 - \frac{1}{2\sigma_1}\big),\, m_1\big(1 + \frac{1}{2\sigma_1}\big)\big]\,, \\ c_2\, |v|^{-\mu} & |v| \ge m_1\big(1 + \frac{1}{2\sigma_1}\big)\,, \end{cases}$$

with certain constants $c_1 > 0$, $c_2 > 0$, and $\mu > 1$.
Then the general $C(\mathbb{T})$-error constant $E_{\sigma_1}(\varphi_1)$ of the window function (2.3) *has the upper bound*

$$E_{\sigma_1}(\varphi_1) \le \frac{1}{\hat\omega_1\big(\frac{m_1}{2\sigma_1}\big)} \bigg[2c_1 + \frac{2c_2}{(\mu - 1)\, m_1^{\mu}} \Big(1 - \frac{1}{2\sigma_1}\Big)^{1-\mu}\bigg]\,. \tag{3.10}$$

Proof. By the scaling property of the Fourier transform, we have

$$\hat\varphi_1(v) = \int_{\mathbb{R}} \varphi_1(t)\, \mathrm{e}^{-2\pi\mathrm{i}vt}\, \mathrm{d}t = \frac{m}{N_1}\, \hat\omega_1\Big(\frac{m_1 v}{N_1}\Big)\,, \quad v \in \mathbb{R}\,.$$

For all $v \in \big[-\frac{N}{2},\, \frac{N}{2}\big]$ and $r \in \mathbb{Z}\setminus\{0,\, \pm 1\}$, we obtain

$$\Big|\frac{m_1 v}{N_1} + m_1 r\Big| \ge m_1\Big(2 - \frac{1}{2\sigma_1}\Big) > m_1\Big(1 + \frac{1}{2\sigma_1}\Big)$$

and hence

$$|\hat{\varphi}_1(v+rN_1)| = \frac{m_1}{N_1}\,\Big|\hat{\omega}_1\Big(\frac{m_1 v}{N_1}+m_1 r\Big)\Big| \le \frac{m_1\, c_2}{m_1^{\mu} N_1}\,\Big|\frac{v}{N_1}+r\Big|^{-\mu}.$$

From [15, Lemma 3.1] it follows that for fixed $u = \frac{v}{N_1} \in \big[-\frac{1}{2\sigma_1},\, \frac{1}{2\sigma_1}\big]$,

$$\sum_{r\in\mathbb{Z}\setminus\{0,\pm 1\}} |u+r|^{-\mu} \le \frac{2}{\mu-1}\,\Big(1-\frac{1}{2\sigma_1}\Big)^{1-\mu}.$$

For all $v \in \big[-\frac{N}{2},\, \frac{N}{2}\big]$, we sustain

$$|\hat{\varphi}_1(v\pm N_1)| = \frac{m_1}{N_1}\,\Big|\hat{\omega}_1\Big(\frac{m_1 v}{N_1}\pm m_1\Big)\Big| \le \frac{m_1}{N_1}\,c_1\,,$$

since it holds

$$\Big|\frac{m_1 v}{N_1}\pm m_1\Big| \in \Big[m_1\Big(1-\frac{1}{2\sigma_1}\Big),\, m_1\Big(1+\frac{1}{2\sigma_1}\Big)\Big].$$

Thus, for each $v \in \big[-\frac{N}{2},\, \frac{N}{2}\big]$, we estimate the sum

$$\begin{aligned}\sum_{r\in\mathbb{Z}\setminus\{0\}} |\hat{\varphi}_1(v+rN_1)| &\le \frac{m_1}{N_1}\,\Big[\Big|\hat{\omega}_1\Big(\frac{m_1 v}{N_1}-m_1\Big)\Big| + \Big|\hat{\omega}_1\Big(\frac{m_1 v}{N_1}+m_1\Big)\Big| \\ &\quad + \sum_{k\in\mathbb{Z}\setminus\{0,\pm 1\}} \Big|\hat{\omega}_1\Big(\frac{m_1 v}{N_1}+m_1 r\Big)\Big|\Big] \\ &\le \frac{m_1}{N_1}\,\Big[2c_1 + \frac{c_2}{m_1^{\mu}} \sum_{r\in\mathbb{Z}\setminus\{0,\pm 1\}} \Big|\frac{v}{N_1}+r\Big|^{-\mu}\Big] \\ &\le \frac{m_1}{N_1}\,\Big[2c_1 + \frac{2c_2}{(\mu-1)\,m_1^{\mu}}\,\Big(1-\frac{1}{2\sigma_1}\Big)^{1-\mu}\Big]\end{aligned}$$

such that

$$\max_{v\in[-N/2,N/2]} \sum_{r\in\mathbb{Z}\setminus\{0\}} |\hat{\varphi}_1(v+rN_1)| \le \frac{m_1}{N_1}\,\Big[2c_1 + \frac{2c_2}{(\mu-1)\,m_1^{\mu}}\,\Big(1-\frac{1}{2\sigma_1}\Big)^{1-\mu}\Big].$$

Now we determine the minimum of all positive values

$$\hat{\varphi}_1(v) = \frac{m_1}{N_1}\,\hat{\omega}_1\Big(\frac{m_1 v}{N_1}\Big), \quad v \in \Big[-\frac{N}{2},\, \frac{N}{2}\Big].$$

Since $\frac{m_1\,|v|}{N_1} \le \frac{m_1}{2\sigma_1}$ for all $v \in \big[-\frac{N}{2},\, \frac{N}{2}\big]$, we obtain

$$\min_{v\in[-N/2,N/2]} \hat{\varphi}_1(v) = \frac{m_1}{N_1} \min_{v\in[-N/2,N/2]} \hat{\omega}_1\Big(\frac{m_1 v}{N_1}\Big) = \frac{m_1}{N_1}\,\hat{\omega}_1\Big(\frac{m_1}{2\sigma_1}\Big) = \hat{\varphi}_1\Big(\frac{N}{2}\Big) > 0\,.$$

Thus, we see that the constant $E_{\sigma_1,N}(\varphi_1)$ can be estimated by an upper bound which depends on m_1 and σ_1, but does not depend on N. We obtain

$$\begin{aligned} E_{\sigma_1,N}(\varphi_1) &\le \frac{1}{\hat{\varphi}_1(N/2)} \max_{v\in[-N/2,N/2]} \sum_{r\in\mathbb{Z}\setminus\{0\}} |\hat{\varphi}_1(n+rN_1)| \\ &\le \frac{1}{\hat{\omega}_1\big(\frac{m_1}{2\sigma_1}\big)} \left[2c_1 + \frac{2c_2}{(\mu-1)\,m_1^{\mu}}\left(1-\frac{1}{2\sigma_1}\right)^{1-\mu}\right]. \end{aligned}$$

Consequently, the general $C(\mathbb{T})$-error constant $E_{\sigma_1}(\varphi_1)$ has the upper bound (3.10). By (3.9), the expression (3.10) is also an upper bound of $C(\mathbb{T})$-error constant $e_{\sigma_1}(\varphi_1)$. ■

Thus, by means of these technical results we obtain the following error estimate for the NNFFT.

Theorem 3.5 *Let the nonharmonic bandwidth $N \in \mathbb{N}$ with $N \gg 1$ be given. Assume that $N_1 = \sigma_1 N \in 2\mathbb{N}$ with $\sigma_1 > 1$. For fixed $m_1 \in \mathbb{N}\setminus\{1\}$ with $2m_1 \ll N_1$, let $N_2 = \sigma_2\,(N_1 + 2m_1)$ with $\sigma_2 > 1$. For $m_2 \in \mathbb{N}\setminus\{1\}$ with $2m_2 \le \big(1-\frac{1}{\sigma_1}\big)\,N_2$, let φ_1 and φ_2 be the window functions of the form (2.5). Let $x_j \in \big[-\frac{1}{2},\,\frac{1}{2}\big]$, $j \in \mathcal{I}_{M_2}$, be arbitrary spatial nodes and let $f_k \in \mathbb{C}$, $k \in \mathcal{I}_{M_1}$, be arbitrary coefficients. Further, let $a > 1$ be the constant (2.6).*
Then for a given exponential sum (1.1) with arbitrary frequencies $v_k \in \big[-\frac{1}{2a},\,\frac{1}{2a}\big]$, $k \in \mathcal{I}_{M_1}$, the error of the NNFFT *can be estimated by*

$$\begin{aligned} \max_{j\in\mathcal{I}_{M_2}} \left|f(x_j) - \frac{s_1(Nx_j)}{\hat{\varphi}_1(Nx_j)}\right| &\le \max_{x\in[-1/2,\,1/2]} \left|f(x) - \frac{s_1(Nx)}{\hat{\varphi}_1(Nx)}\right| \\ &\le \left[E_{\sigma_1}(\varphi_1) + \frac{a}{\hat{\varphi}_1\big(\frac{N}{2}\big)}\,E_{\sigma_2}(\varphi_2)\right] \sum_{k\in\mathcal{I}_{M_1}} |f_k|\,, \end{aligned} \tag{3.11}$$

where $E_{\sigma_j}(\varphi_j)$ for $j = 1,\,2,$ are the general $C(\mathbb{T})$-error constants of the form (3.7)*.*

Proof. Now for arbitrary spatial nodes $x_j \in \big[-\frac{1}{2},\,\frac{1}{2}\big]$, $j \in \mathcal{I}_{M_2}$, we estimate the error of the NNFFT in the form

$$\max_{j\in\mathcal{I}_{M_2}} \left|f(x_j) - \frac{s_1(N\,x_j)}{\hat{\varphi}_1(Nx_j)}\right| \le \max_{j\in\mathcal{I}_{M_2}} \left|f(x_j) - \frac{s(Nx_j)}{\hat{\varphi}_1(N\,x_j)}\right| + \max_{j\in\mathcal{I}_{M_2}} \frac{|s(Nx_j) - s_1(Nx_j)|}{\hat{\varphi}_1(N\,x_j)}\,.$$

At first we consider

$$\max_{j\in\mathcal{I}_{M_2}} \left|f(x_j) - \frac{s(Nx_j)}{\hat{\varphi}_1(Nx_j)}\right| \le \max_{x\in[-1/2,\,1/2]} \left|f(x) - \frac{s(Nx)}{\hat{\varphi}_1(Nx)}\right|.$$

From (2.9) and (2.11) it follows that for all $x \in \mathbb{R}$,

$$f(x) - \frac{s(Nx)}{\hat{\varphi}_1(Nx)} = \frac{\hat{h}(Nx) - s(Nx)}{\hat{\varphi}_1(Nx)}$$

$$= \sum_{k \in \mathcal{I}_{M_1}} f_k \left[\mathrm{e}^{-2\pi \mathrm{i} N v_k x} - \frac{1}{N_1\, \hat{\varphi}_1(Nx)} \sum_{\ell \in \mathcal{I}_{N_1+2m_1}} \varphi_1\Big(\frac{\ell}{N_1} - v_k\Big)\, \mathrm{e}^{-2\pi \mathrm{i}\, \ell x/\sigma_1} \right].$$

Thus, by (2.7), (3.6), and (3.8), we obtain the estimate

$$\max_{x \in [-1/2,\, 1/2]} \left| f(x) - \frac{s(Nx)}{\hat{\varphi}_1(Nx)} \right| \le E_{\sigma_1,N}(\varphi_1) \sum_{k \in \mathcal{I}_{M_1}} |f_k| \le E_{\sigma_1}(\varphi_1) \sum_{k \in \mathcal{I}_{M_1}} |f_k|\,. \tag{3.12}$$

Now we show that for $\varphi_2(t) := \omega_2\big(\frac{N_2 t}{m_2}\big)$ and $N_2 = \sigma_2\,(N_1 + 2m_1)$ it holds

$$\max_{x \in [-1/2,\, 1/2]} |s(N\,x) - s_1(Nx)| \le E_{\sigma_2}(\varphi_2) \sum_{\ell \in \mathcal{I}_{N_1+2m_1}} |g_\ell|\,. \tag{3.13}$$

By construction, the functions s and s_1 can be represented in the form

$$\begin{aligned} s(Nx) &= \sum_{\ell \in \mathcal{I}_{N_1+2m_1}} g_\ell\, \mathrm{e}^{-2\pi \mathrm{i} \ell x/\sigma_1}\,, \\ s_1(Nx) &= \sum_{s \in \mathcal{I}_{N_2}} h_s\, \tilde{\varphi}_2^{(1)}\Big(\frac{x}{\sigma_1} - \frac{s}{N_2}\Big)\,, \quad x \in \mathbb{R}\,, \end{aligned}$$

where $\tilde{\varphi}_2^{(1)}$ denotes the 1-periodization of the second window function φ_2 and

$$h_s := \frac{1}{N_2} \sum_{\ell \in \mathcal{I}_{N_1+2m_1}} \frac{g_\ell}{\hat{\varphi}_2(\ell)}\, \mathrm{e}^{-2\pi \mathrm{i} \ell s/N_2}\,.$$

Substituting $t = \frac{x}{\sigma_1}$, it follows that

$$\begin{aligned} s(N_1 t) &= \sum_{\ell \in \mathcal{I}_{N_1+2m_1}} g_\ell\, \mathrm{e}^{-2\pi \mathrm{i} \ell t}\,, \\ s_1(N_1 t) &= \sum_{s \in \mathcal{I}_{N_2}} h_s\, \tilde{\varphi}_2^{(1)}\Big(t - \frac{s}{N_2}\Big)\,, \quad t \in \mathbb{R}\,, \end{aligned}$$

are 1-periodic functions. By [15, Lemma 2.3], we conclude

$$\max_{t \in [-1/2,\, 1/2]} |s(N_1 t) - s_1(N_1 t)| \le e_{\sigma_2}(\varphi_2) \sum_{\ell \in \mathcal{I}_{N_1+2m_1}} |g_\ell| \le E_{\sigma_2}(\varphi_2) \sum_{\ell \in \mathcal{I}_{N_1+2m_1}} |g_\ell|\,,$$

where the general $C(\mathbb{T})$-error constant $E_{\sigma_2}(\varphi_2)$ defined similar to (3.7) has an analogous property (3.9). Since $x = \sigma_1\, t$, we obtain that

$$\begin{aligned} \max_{t \in [-1/2,\, 1/2]} |s(N_1 t) - s_1(N_1 t)| &= \max_{x \in [-\sigma_1/2, \sigma_1/2]} |s(Nx) - s_1(Nx)| \\ &\le E_{\sigma_2}(\varphi_2) \sum_{\ell \in \mathcal{I}_{N_1+2m_1}} |g_\ell|\,, \end{aligned} \tag{3.14}$$

such that (3.13) is shown. Note that for $x \in \big[-\frac{1}{2},\,\frac{1}{2}\big]$ it holds

$$s_1(Nx) = \sum_{s\in\mathcal{I}''_{N_2}(x)} h_s\,\varphi_2\bigg(\frac{x}{\sigma_1} - \frac{s}{N_2}\bigg)$$

with the index set

$$\mathcal{I}''_{N_2}(x) := \bigg\{s \in \mathcal{I}_{N_2} : \bigg|\frac{s}{N_2} - \frac{x}{\sigma_1}\bigg| < \frac{m_2}{N_2}\bigg\}\,.$$

Further, by (2.6) and (2.12) it holds

$$\sum_{\ell\in\mathcal{I}_{N_1+2m_1}} |g_\ell| \;\le\; \frac{1}{N_1}\sum_{\ell\in\mathcal{I}_{N_1+2m_1}}\sum_{k\in\mathcal{I}_{M_1}} |f_k|\cdot 1 \le \frac{N_1+2m_1}{N_1}\sum_{k\in\mathcal{I}_{M_1}} |f_k| = a\sum_{k\in\mathcal{I}_{M_1}} |f_k|\,.$$

Combining this with (3.12) and (3.14) completes the proof. ■

Now it merely remains to estimate the general $C(\mathbb{T})$-error constants $E_{\sigma_j}(\varphi_j)$ for $j = 1,\,2$, and $\hat\varphi_1\big(\frac{N}{2}\big)$ in (3.11) for specific window functions.

4 Error of NNFFT with sinh-type window functions

In this section we specify the result in Theorem 3.5 for the NNFFT with two sinh-type window functions.
Let $N \in \mathbb{N}$ with $N \gg 1$ be the fixed nonharmonic bandwidth. Let $\sigma_1,\,\sigma_2 \in \big[\frac{5}{4},\,2\big]$ be given oversampling factors. Further let $N_1 = \sigma_1 N \in 2\mathbb{N}$, $m_1 \in \mathbb{N}\setminus\{1\}$ with $2m_1 \ll N_1$, and $N_2 = \sigma_2\,(N_1 + 2m_1) = \sigma_1\sigma_2 a\,N \in 2\mathbb{N}$ be given, where $a > 1$ denotes the constant (2.6). Let $m_2 \in \mathbb{N}\setminus\{1\}$ with $2m_2 \le \big(1 - \frac{1}{\sigma_1}\big)\,N_2$ be given as well.
For $j = 1,\,2$, we consider the functions

$$\omega_{\mathrm{sinh},j}(x) := \begin{cases} \frac{1}{\sinh\beta_j}\,\sinh\big(\beta_j\sqrt{1-x^2}\big) & x \in [-1,\,1]\,,\\ 0 & x \in \mathbb{R}\setminus[-1,\,1] \end{cases}$$

with the shape parameter

$$\beta_j := 2\pi m_j\bigg(1 - \frac{1}{2\sigma_j}\bigg)\,.$$

As shown in Example 2.1, both functions belong to the set Ω. By scaling, for $j = 1,\,2$, we introduce the sinh-*type window functions*

$$\varphi_{\mathrm{sinh},j}(t) := \omega_{\mathrm{sinh},j}\bigg(\frac{N_j t}{m_j}\bigg)\,,\quad t \in \mathbb{R}\,. \tag{4.1}$$

Now we show that the error of the NNFFT with two sinh-type window functions (4.1) has exponential decay with respect to the truncation parameters m_1 and m_2.

Theorem 4.1 *Let the nonharmonic bandwidth $N \in \mathbb{N}$ with $N \gg 1$ be given. Further let $N_1 = \sigma_1 N \in 2\mathbb{N}$ with $\sigma_1 \in \big[\frac{5}{4},\, 2\big]$ be given. For fixed $m_1 \in \mathbb{N} \setminus \{1\}$ with $2m_1 \ll N_1$, let $N_2 = \sigma_2\,(N_1 + 2m_1) \in 2\mathbb{N}$ with $\sigma_2 \in \big[\frac{5}{4},\, 2\big]$. For $m_2 \in \mathbb{N} \setminus \{1\}$ with $2m_2 \le \big(1 - \frac{1}{\sigma_1}\big)\, N_2$, let $\varphi_{\mathrm{sinh},1}$ and $\varphi_{\mathrm{sinh},2}$ be the* sinh*-type window functions* (4.1)*. Assume that $m_2 \ge m_1$. Let $x_j \in \big[-\frac{1}{2},\, \frac{1}{2}\big]$, $j \in \mathcal{I}_{M_2}$, be arbitrary spatial nodes and let $f_k \in \mathbb{C}$, $k \in \mathcal{I}_{M_1}$, be arbitrary coefficients. Let $a > 1$ be the constant* (2.6)*.*
Then for the exponential sum (1.1) *with arbitrary frequencies $v_k \in \big[-\frac{1}{2a},\, \frac{1}{2a}\big]$, $k \in \mathcal{I}_{M_1}$, the error of the* NNFFT *with the* sinh*-type window functions* (4.1) *can be estimated in the form*

$$\max_{j \in \mathcal{I}_{M_2}} \left| f(x_j) - \frac{s_1(N x_j)}{\hat{\varphi}_{\mathrm{sinh},1}(N x_j)} \right| \le \max_{x \in [-1/2,\, 1/2]} \left| f(x) - \frac{s_1(N x)}{\hat{\varphi}_{\mathrm{sinh},1}(N x)} \right| \le E(\varphi_{\mathrm{sinh}}) \sum_{k \in \mathcal{I}_{M_1}} |f_k|$$

with the constant

$$\begin{aligned} E(\varphi_{\mathrm{sinh}}) &:= (24 m_1^{3/2} + 10)\, \mathrm{e}^{-2\pi m_1 \sqrt{1 - 1/\sigma_1}} \\ &\quad + (24 m_2^{3/2} + 10)\, \frac{2N_1 + 4m_1}{\sqrt{2 m_1 \pi}}\, \mathrm{e}^{2\pi m_1 (1 - \sqrt{1 - 1/\sigma_1} - 1/(2\sigma_1))}\, \mathrm{e}^{-2\pi m_2 \sqrt{1 - 1/\sigma_2}}\,. \end{aligned} \tag{4.2}$$

Proof. By Theorem 3.5 we have to estimate the general $C(\mathbb{T})$-error constants $E_{\sigma_j}(\varphi_j)$, $j = 1,\, 2$, and $\hat{\varphi}_1\big(\frac{N}{2}\big)$ in (3.11) for the sinh-type window functions (4.1).
Applying Theorem 3.4, we obtain by the same technique as in [15, Theorem 5.6] that

$$E_{\sigma_j}\big(\varphi_{\mathrm{sinh},j}\big) \le (24 m_j^{3/2} + 10)\, \mathrm{e}^{-2\pi m_j \sqrt{1 - 1/\sigma_j}}\,, \quad j = 1,\, 2. \tag{4.3}$$

Now we estimate $\hat{\varphi}_{\mathrm{sinh},1}\big(\frac{N}{2}\big)$. Using the scaling property of the Fourier transform, by (2.1) we obtain

$$\begin{aligned} \hat{\varphi}_{\mathrm{sinh},1}\left(\frac{N}{2}\right) &= \frac{m_1}{N_1}\, \hat{\omega}_{\mathrm{sinh},1}\left(\frac{m_1 N}{2 N_1}\right) = \frac{m_1}{N_1}\, \hat{\omega}_{\mathrm{sinh},1}\left(\frac{m_1}{2\sigma_1}\right) \\ &= \frac{\pi m_1 \beta_1}{N_1 \sinh \beta_1} \left(\beta_1^2 - \frac{\pi^2 m_1^2}{\sigma_1^2}\right)^{-1/2} I_1\left(\sqrt{\beta_1^2 - \frac{\pi^2 m_1^2}{\sigma_1^2}}\right) \\ &= \frac{m_1 \pi}{N_1 \sinh \beta_1} \left(1 - \frac{1}{2\sigma_1}\right) \left(1 - \frac{1}{\sigma_1}\right)^{-1/2} I_1\left(2\pi m_1 \sqrt{1 - \frac{1}{\sigma_1}}\right), \end{aligned}$$

where we have used the equality

$$\left(\beta_1^2 - \frac{\pi^2 m_1^2}{\sigma_1^2}\right)^{1/2} = 2\pi m_1 \left(\left(1 - \frac{1}{2\sigma_1}\right)^2 - \frac{1}{4\sigma_1^2}\right)^{1/2} = 2\pi m_1 \sqrt{1 - \frac{1}{\sigma_1}}\,.$$

From $m_1 \ge 2$ and $\sigma_1 \ge \frac{5}{4}$, it follows that

$$2\pi m_1 \sqrt{1 - \frac{1}{\sigma_1}} \ge 4\pi \sqrt{1 - \frac{1}{\sigma_1}} \ge x_0 := \frac{4\pi}{\sqrt{5}}\,.$$

By the inequality for the modified Bessel function I_1 (see [15, Lemma 3.3]) it holds

$$I_1(x) \ge \sqrt{x_0}\, \mathrm{e}^{-x_0}\, I_1(x_0)\, x^{-1/2}\, \mathrm{e}^{x} > \frac{2}{5}\, x^{-1/2}\, \mathrm{e}^{x}\,, \quad x \ge x_0\,.$$

Thus, we obtain

$$\hat{\varphi}_{\mathrm{sinh},1}\Big(\frac{N}{2}\Big) \ge \frac{\sqrt{2m_1\pi}}{5N_1\, \sinh\beta_1}\Big(1-\frac{1}{2\sigma_1}\Big)\Big(1-\frac{1}{\sigma_1}\Big)^{-3/4}\, \mathrm{e}^{2\pi m_1\sqrt{1-1/\sigma_1}}\,.$$

By the simple inequality

$$\sinh\beta_1 < \frac{1}{2}\,\mathrm{e}^{\beta_1} = \frac{1}{2}\,\mathrm{e}^{2\pi m_1(1-1/(2\sigma_1))}\,,$$

we conclude that

$$\hat{\varphi}_{\mathrm{sinh},1}\Big(\frac{N}{2}\Big) \ge \frac{2\sqrt{2m_1\pi}}{5N_1}\Big(1-\frac{1}{2\sigma_1}\Big)\Big(1-\frac{1}{\sigma_1}\Big)^{-3/4}\, \mathrm{e}^{2\pi m_1(\sqrt{1-1/\sigma_1}-1+1/(2\sigma_1))}$$

and hence

$$\frac{a}{\hat{\varphi}_{\mathrm{sinh},1}(N/2)} \le \frac{5N_1\, a}{2\sqrt{2m_1\pi}}\Big(1-\frac{1}{2\sigma_1}\Big)^{-1}\Big(1-\frac{1}{\sigma_1}\Big)^{3/4}\, \mathrm{e}^{-2\pi m_1(\sqrt{1-1/\sigma_1}-1+1/(2\sigma_1))}\,. \qquad (4.4)$$

Applying Theorem 3.5, we estimate the error of the NNFFT with two sinh-type window functions (4.1). By (4.3) and (4.4) we obtain the inequality

$$\begin{aligned} E_{\sigma_1}(\varphi_{\mathrm{sinh},1}) \quad &+ \quad \frac{a}{\hat{\varphi}_{\mathrm{sinh},1}(N/2)}\, E_{\sigma_2}(\varphi_{\mathrm{sinh},2}) \le (24m_1^{3/2}+10)\,\mathrm{e}^{-2\pi m_1\sqrt{1-1/\sigma_1}} \\ &+ \quad (24m_2^{3/2}+10)\,\frac{2N_1 a}{\sqrt{2m_1\pi}}\,\mathrm{e}^{2\pi m_1(1-\sqrt{1-1/\sigma_1}-1/(2\sigma_1))}\,\mathrm{e}^{-2\pi m_2\sqrt{1-1/\sigma_2}}\,, \end{aligned}$$

since it holds

$$\frac{5}{2}\Big(1-\frac{1}{2\sigma_1}\Big)^{-1}\Big(1-\frac{1}{\sigma_1}\Big)^{3/4} \le \frac{5}{3}\sqrt{2} < 2\,, \quad \sigma_1 \in \big[\tfrac{5}{4},\, 2\big]\,.$$

This completes the proof. ■

Example 4.2 Now we visualize the result of Theorem 4.1. To this end, we fix $N = 1200$ and consider $m_1 \in \{2, \ldots, 8\}$ and $\sigma_1 \in \{1.25, 1.5, 2\}$. In Figure 4.1 the error bound (4.2) is depicted for several choices of $m_2 \ge m_1$ and $\sigma_2 \ge \sigma_1$. Clearly, the error bounds (4.2) decrease for increasing truncation parameters and oversampling factors, respectively. Moreover, we recognize that the results get better when choosing $\sigma_2 > \sigma_1$, cf. Figure 4.1(c), and are best for $m_2 > m_1$, cf. Figure 4.1 (a). Besides, we remark that choices $m_2 < m_1$ or $\sigma_2 < \sigma_1$ produce the same results as in the equality setting such that we omitted these tests.

Therefore, we recommend the use of truncation parameters $m_2 > m_1$ and oversampling factors $\sigma_2 \geq \sigma_1$. For the choice of m_1 and σ_1, we refer to previous works concerning the NFFT, e. g. [15, 16].

Additionally, we aim to compare these theoretical bounds with the errors obtained by the NNFFT. For this purpose, we introduce the relative error

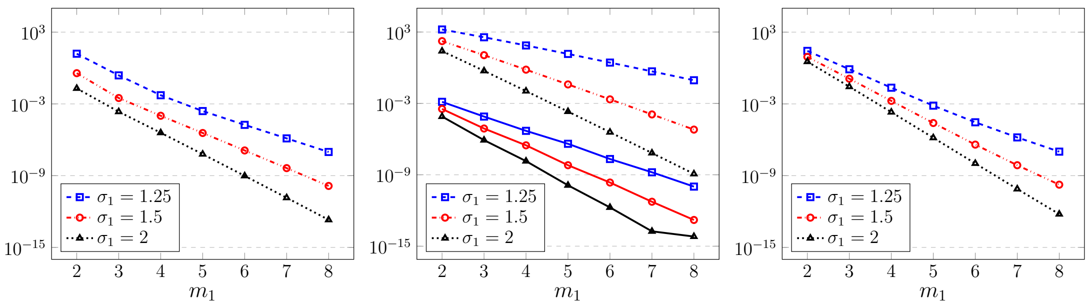
$$\Big(\sum_{k\in\mathcal{I}_{M_1}} |f_k|\Big)^{-1} \max_{j\in\mathcal{I}_{M_2}} \left| f(x_j) - \frac{s_1(Nx_j)}{\hat{\varphi}_{\mathrm{sinh},1}(Nx_j)} \right|, \tag{4.5}$$

since by Theorem 4.1 it holds

$$\Big(\sum_{k\in\mathcal{I}_{M_1}} |f_k|\Big)^{-1} \max_{j\in\mathcal{I}_{M_2}} \left| f(x_j) - \frac{s_1(Nx_j)}{\hat{\varphi}_{\mathrm{sinh},1}(Nx_j)} \right| \leq E(\varphi_{\mathrm{sinh}})\,.$$

Thus, we choose random nodes $x_j \in \big[-\frac{1}{2}, \frac{1}{2}\big]$, $j \in \mathcal{I}_{M_2}$, and $v_k \in [-\frac{1}{2a}, \frac{1}{2a}]$, $k \in \mathcal{I}_{M_1}$, with $a = 1 + \frac{2m_1}{N_1}$, as well as random coefficients $f_k \in \mathbb{C}$, $k \in \mathcal{I}_{M_1}$, and compute the values (1.2) once directly and once rapidly using the NNFFT. Due to the randomness of the given data, this test is repeated one hundred times and afterwards the maximum error over all repetitions is computed. The errors (4.5) for the parameter choice $M_1 = 2400$ and $M_2 = 1600$ are displayed in Figure 4.1 (b).

Unfortunately, the current release NFFT 3.5.3 of the software package [9] is not yet designed for the use of parameters $m_1 \neq m_2$ and $\sigma_1 \neq \sigma_2$. Therefore, we can only handle the setting $m_1 = m_2$ and $\sigma_1 = \sigma_2$ in Figure 4.1 (b). Moreover, the sinh-type window function is currently not implemented in the software package [9]. Thus, we use two standard window functions, namely the Kaiser–Bessel window functions, since it was shown in [15] that those are very much related. Since the results in Figure 4.1 show great promise, these features might be part of future releases. □



(a) $m_2 = 2m_1$ and $\sigma_2 = \sigma_1$ (b) $m_2 = m_1$ and $\sigma_2 = \sigma_1$ (c) $m_2 = m_1$ and $\sigma_2 = 2\sigma_1$

Figure 4.1: Error bound (4.2) (dashed) for the NNFFT with sinh-type window functions for $N = 1200$, $m_1 \in \{2, \ldots, 8\}$ and $\sigma_1 \in \{1.25, 1.5, 2\}$. Part (b) additionally depicts the relative error (4.5) (solid) using Kaiser-Bessel window functions.

5 Approximation of sinc function by exponential sums

Since we aim to present an interesting signal processing application of the NNFFT in the last Section 6, we now study the approximation of the function $\mathrm{sinc}(N\pi x)$, $x \in [-1,\, 1]$, by an exponential sum (1.1).
In [3] the exponential sum (1.1) is used for a local approximation of a bandlimited function F of the form

$$F(x) := \int_{-1/2}^{1/2} w(t)\, \mathrm{e}^{-2\pi\mathrm{i}Ntx}\, \mathrm{d}t\,, \quad x \in \mathbb{R}\,, \tag{5.1}$$

where $w : [-\frac{1}{2},\, \frac{1}{2}] \to [0,\, \infty)$ is an integrable function with $\int_{-1/2}^{1/2} w(t)\, \mathrm{d}t > 0$. By the substitution

$$F(x) = \frac{1}{N} \int_{-N/2}^{N/2} w\Big(-\frac{s}{N}\Big)\, \mathrm{e}^{2\pi\mathrm{i}\, sx}\, \mathrm{d}s\,,$$

we recognize that the Fourier transform of (5.1) is supported on $\big[-\frac{N}{2},\, \frac{N}{2}\big]$, i. e., the function (5.1) is bandlimited with bandwidth N. For instance, for $w(t) := 1$, $t \in \big[-\frac{1}{2},\, \frac{1}{2}\big]$, we obtain the famous bandlimited sinc function

$$F(x) = \mathrm{sinc}(\pi N x) := \begin{cases} \frac{\sin(\pi N x)}{\pi N x} & x \in \mathbb{R} \setminus \{0\}\,, \\ 1 & x = 0\,. \end{cases} \tag{5.2}$$

Now we show that the bandlimited sinc function (5.2) can be uniformly approximated on the interval $[-1,\, 1]$ by an exponential sum (1.1). We start with the uniform approximation of the sinc function on the interval $\big[-\frac{1}{2},\, \frac{1}{2}\big]$.

Theorem 5.1 *Let $\varepsilon > 0$ be a given target accuracy.*
Then for sufficiently large $n \in \mathbb{N}$ with $n \ge 2N$, there exist constants $w_j > 0$ and frequencies $v_j \in \big(-\frac{1}{2},\, \frac{1}{2}\big)$, $j = 1, \dots, n$, such that for all $x \in \big[-\frac{1}{2},\, \frac{1}{2}\big]$,

$$\Big|\mathrm{sinc}(\pi N x) - \sum_{j=1}^{n} w_j\, \mathrm{e}^{-2\pi\mathrm{i}Nv_j x}\Big| \le \varepsilon\,. \tag{5.3}$$

Proof. This result is a simple consequence of [3, Theorem 6.1]. Introducing $\nu := \frac{N}{n} \le \frac{1}{2}$, we obtain by substitution $\tau := -\frac{t}{2\nu}$ that

$$\mathrm{sinc}(\pi N x) = \int_{-1/2}^{1/2} \mathrm{e}^{-2\pi\mathrm{i}N\tau x}\, \mathrm{d}\tau = \frac{1}{2\nu} \int_{-\nu}^{\nu} \mathrm{e}^{\mathrm{i}\pi ntx}\, \mathrm{d}t\,.$$

Setting $y := nx \in \big[-\frac{n}{2},\, \frac{n}{2}\big]$, we have

$$\mathrm{sinc}(\pi\nu y) = \frac{1}{2\nu} \int_{-\nu}^{\nu} \mathrm{e}^{\mathrm{i}\pi ty}\, \mathrm{d}t\,.$$

Then from [3, Theorem 6.1] (with $d = \frac{1}{2}$), it follows the existence of $w_j > 0$ and $\Theta_j \in (-\nu,\, \nu)$, $j = 1, \ldots, n$, such that for all $y \in \big[-\frac{n}{2} - 1,\, \frac{n}{2} + 1\big]$,

$$\left| \frac{1}{2\nu} \int_{-\nu}^{\nu} \sigma(t)\, \mathrm{e}^{\mathrm{i}\pi t y}\, \mathrm{d}t - \sum_{j=1}^{n} w_j\, \mathrm{e}^{\pi \mathrm{i}\, \Theta_j y} \right| \le \varepsilon\,.$$

Hence, for all $x = \frac{y}{n} \in \big[-\frac{1}{2},\, \frac{1}{2}\big]$, we conclude that for $v_j := -\frac{\Theta_j}{2\nu} \in \big(-\frac{1}{2},\, \frac{1}{2}\big)$, $j = 1, \ldots, n$,

$$\left| \frac{1}{2\sigma} \int_{-\nu}^{\nu} \mathrm{e}^{\mathrm{i}\pi n t x}\, \mathrm{d}t - \sum_{j=1}^{n} w_j\, \mathrm{e}^{\pi \mathrm{i}\, n \Theta_j x} \right| = \left| \mathrm{sinc}(\pi N x) - \sum_{j=1}^{n} w_j\, \mathrm{e}^{-2\pi \mathrm{i} N v_j x} \right| \le \varepsilon\,.$$

This completes the proof. ■

Substituting the variable $x = \frac{t}{2}$, $t \in [-1,\, 1]$, the frequencies $v_j = \frac{z_j}{2}$, $z_j \in (-1,\, 1)$, and replacing the bandwidth N in (5.3) by $2N$, we obtain the following uniform approximation of the sinc function (5.2) on the interval $[-1,\, 1]$ (after denoting t by x and z_j by v_j again):

Corollary 5.2 *Let $\varepsilon > 0$ be a given target accuracy.*
Then for sufficiently large $n \in \mathbb{N}$ with $n \ge 4N$, there exist constants $w_j > 0$ and frequencies $v_j \in (-1,\, 1)$, $j = 1, \ldots, n$, such that (5.3) holds for all $x \in [-1,\, 1]$, i. e.,

$$\left| \mathrm{sinc}(\pi N x) - \sum_{j=1}^{n} w_j\, \mathrm{e}^{-\pi \mathrm{i} N v_j x} \right| \le \varepsilon\,, \quad x \in [-1,\, 1]\,.$$

In practice, we simplify the approximation procedure of the function $\mathrm{sinc}(N\pi x)$. Since for fixed $N \in \mathbb{N}$, it holds

$$\mathrm{sinc}(N\pi x) = \frac{1}{2} \int_{-1}^{1} \mathrm{e}^{-\pi \mathrm{i} N t x}\, \mathrm{d}t\,, \quad x \in \mathbb{R}\,,$$

the approximation on the interval $[-1, 1]$ can efficiently be realized by means of the Clenshaw–Curtis quadrature (see [18, pp. 143–153] or [13, pp. 357–364]). Using this procedure for the integrand $\frac{1}{2}\, \mathrm{e}^{-\pi \mathrm{i} N t x}$, $t \in [-1,\, 1]$, with fixed parameter $x \in [-1,\, 1]$, the Chebyshev points $z_k = \cos \frac{k\pi}{n} \in [-1, 1]$, $k = 0, \ldots, n$, and the positive coefficients

$$w_k = \begin{cases} \frac{1}{n}\, \varepsilon_n(k)^2 \sum_{j=0}^{n/2} \varepsilon_n(2j)^2 \frac{2}{1-4j^2}\, \cos \frac{2jk\pi}{n} & n \in 2\mathbb{N}\,, \\ \frac{1}{n}\, \varepsilon_n(k)^2 \sum_{j=0}^{(n-1)/2} \varepsilon_n(2j)^2 \frac{2}{1-4j^2}\, \cos \frac{2jk\pi}{n} & n \in 2\mathbb{N} + 1\,, \end{cases} \tag{5.4}$$

with $\varepsilon_n(0) = \varepsilon_n(n) := \frac{\sqrt{2}}{2}$ and $\varepsilon_n(j) := 1$, $j = 1, \ldots, n-1$ (see [13, p. 359]), we obtain

$$\mathrm{sinc}(N\pi x) = \frac{1}{2} \int_{-1}^{1} \mathrm{e}^{-\pi \mathrm{i} N t x}\, \mathrm{d}t \approx \sum_{k=0}^{n} w_k\, \mathrm{e}^{-\pi \mathrm{i} N z_k x}\,.$$

Further the coefficients fulfill the condition (see [13, p. 359])

$$\sum_{k=0}^{n} w_k = 1\,. \tag{5.5}$$

Then we receive the following error estimate.

Theorem 5.3 *Let $N \in \mathbb{N}$, $n = \nu N$ be given. Let $z_k = \cos\frac{k\pi}{n} \in [-1, 1]$, be the Chebyshev points, let w_k, $k = 0, \ldots, n$, denote the coefficients (5.4), and set $C := \frac{\pi\,(\mathrm{e}^2-1)}{2\,\mathrm{e}}$.*
Then for all $x \in [-1,\, 1]$, the approximation error of $\operatorname{sinc}(N\pi x)$ can be estimated in the form

$$\Big|\operatorname{sinc}(N\pi x) - \sum_{k=0}^{n} w_k\, \mathrm{e}^{-\pi \mathrm{i} N z_k x}\Big| \le \frac{36\,(1+\mathrm{e}^{-2CN})}{35\,(\mathrm{e}^2-1)}\, \mathrm{e}^{-N(\nu-C)}\,. \tag{5.6}$$

In other words, the error bound is exponentially decaying if $\nu > C \approx 3.69$.

Proof. Since the imaginary part of the integrand $\frac{1}{2}\,\mathrm{e}^{-\pi \mathrm{i} N t x}$, $t \in [-1,\, 1]$, is odd, it holds

$$\operatorname{sinc}(N\pi x) = \frac{1}{2}\int_{-1}^{1} \mathrm{e}^{-\pi \mathrm{i} N t x}\,\mathrm{d}t = \frac{1}{2}\int_{-1}^{1} \cos(\pi N t x)\,\mathrm{d}t\,. \tag{5.7}$$

Therefore, we apply the Clenshaw–Curtis quadrature to the analytic function $f(t, x) := \frac{1}{2}\,\cos(\pi N t x)$, $t \in [-1, 1]$, with fixed parameter $x \in [-1,\, 1]$. Note that it holds

$$\sum_{k=0}^{n} w_k\, \mathrm{e}^{-\pi \mathrm{i} N z_k x} = \sum_{k=0}^{n} w_k\, \cos(\pi N z_k x) + 0$$

by the symmetry properties of the Chebyshev points z_k and the coefficients w_k, namely $z_k = -z_{n-k}$ and $w_k = w_{n-k}$, $k = 0, \ldots, n$ (see [13, p. 359]).
By E_ρ with some $\rho > 1$, we denote the Bernstein ellipse defined by

$$E_\rho := \left\{ z \in \mathbb{C} :\ \operatorname{Re} z = \frac{1}{2}\Big(\rho + \frac{1}{\rho}\Big)\,\cos t\,,\ \operatorname{Im} z = \frac{1}{2}\Big(\rho - \frac{1}{\rho}\Big)\,\sin t\,,\ t \in [0,\, 2\pi) \right\}.$$

Then E_ρ has the foci -1 and 1. For simplicity, we choose $\rho = \mathrm{e}$.
For $z \in \mathbb{C}$ and fixed $x \in [-1,\, 1]$, it holds

$$\Big|\frac{1}{2}\,\cos(\pi N x z)\Big| \le \frac{1}{2}\,\cosh(\pi N x \operatorname{Im} z)\,.$$

For $z \in \mathbb{C}$ with $\operatorname{Re} z = 0$ we have

$$\Big|\frac{1}{2}\,\cos(\pi N x z)\Big| = \frac{1}{2}\,\cosh(\pi N x \operatorname{Im} z)\,.$$

Hence, in the interior of the Bernstein ellipse E_{e}, the integrand is bounded, since

$$\Big|\frac{1}{2}\,\cos(\pi N x z)\Big| \le \frac{1}{2}\,\cosh\frac{\pi N x\,(\mathrm{e}^2-1)}{2\,\mathrm{e}} \le \frac{1}{2}\,\cosh\frac{\pi N\,(\mathrm{e}^2-1)}{2\,\mathrm{e}}\,.$$

Therefore, by [18, p. 146] we obtain the error estimate

$$\Big|\operatorname{sinc}(N\pi x) - \sum_{k=0}^{n} w_k\, \mathrm{e}^{-\pi \mathrm{i} N z_k x}\Big| \le \frac{144}{70\,(\mathrm{e}^2-1)}\,\mathrm{e}^{-n}\,\cosh\frac{\pi\,(\mathrm{e}^2-1)\,N}{2\,\mathrm{e}}\,. \tag{5.8}$$

By defining $C := \frac{\pi\,(\mathrm{e}^2-1)}{2\,\mathrm{e}}$, the term $\mathrm{e}^{-n}\cosh(CN)$ in (5.8) can be rewritten as

$$\mathrm{e}^{-n}\cosh(CN) = \mathrm{e}^{-\nu N}\cdot\tfrac{1}{2}(\mathrm{e}^{CN}+\mathrm{e}^{-CN}) = \tfrac{1}{2}\,\mathrm{e}^{-N(\nu-C)}(1+\mathrm{e}^{-2CN})\,.$$

Thus, we end up with (5.6). This completes the proof. ■

In practice, the coefficients w_k in (5.4) can be computed by a fast algorithm, the so-called discrete cosine transform of type I (DCT–I) of length $n+1$, $n=2^t$, (see [13, Algorithm 6.28 or Algorithm 6.35]). This DCT–I uses the orthogonal cosine matrix of type I

$$\boldsymbol{C}_{n+1}^{\mathrm{I}} := \sqrt{\frac{2}{n}}\left(\varepsilon_n(j)\,\varepsilon_n(k)\,\cos\frac{jk\pi}{n}\right)_{j,k=0}^{n}.$$

Algorithm 5.4 (Fast computation of the coefficients w_k)
Input: $n=2^t$ *with* $t\in\mathbb{N}\setminus\{1\}$, $\varepsilon_n(0)=\varepsilon_n(n):=\frac{\sqrt{2}}{2}$, $\varepsilon_n(j):=1$ *for* $j=1,\dots,n-1$.

1. *Form the vector* $(a_j)_{j=0}^n$ *with* $a_{2j} := \varepsilon_n(2j)\,\frac{2}{1-4j^2}$, $j=0,\dots,n/2$ *and* $a_{2j+1}:=0$, $j=0,\dots,n/2-1$.
2. *Compute* $(\hat{a}_k)_{k=0}^n = \boldsymbol{C}_{n+1}^{\mathrm{I}}(a_j)_{j=0}^n$ *by means of* DCT − I.
3. *Form the values* $w_k := \frac{1}{\sqrt{2n}}\varepsilon_n(k)\,\hat{a}_k$, $k=0,\dots,n$.

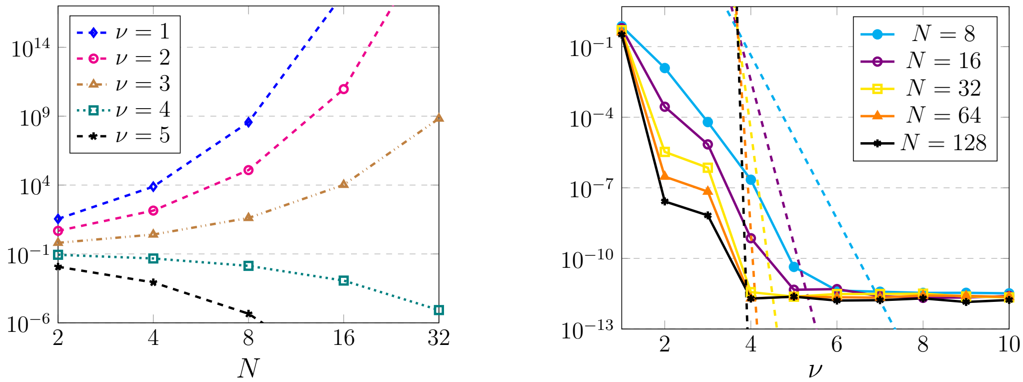
Output: w_k *in* (5.4) *for* $k=0,\dots,n$.

A similar approach can be found in [7], where a Gauss–Legendre quadrature was applied to obtain explicit coefficients w_k for given Legendre points z_k. However, the computation of the coefficients w_k using Algorithm 5.4 is more effective for large n.

Example 5.5 Now we visualize the result of Theorem 5.3. In Figure 5.1 (a) the error bound (5.6) is depicted as a function of N for several choices of $\nu\in\{1,\dots,5\}$, where $n=\nu N$. It clearly demonstrates that $\nu\geq 4$ is needed to obtain reasonable error bounds. Additionally, we compare the error constant and the maximum approximation error, cf. (5.6). To measure the accuracy we consider a fine evaluation grid $x_r=\frac{2r}{R}$, $r\in\mathcal{I}_R$, with $R\gg N$, where $R=3\cdot 10^5$ is fixed. On this grid we calculate the discrete maximum error

$$\max_{r\in\mathcal{I}_R}\left|\mathrm{sinc}(\pi N x_r) - \sum_{k=0}^{n} w_k\,\mathrm{e}^{-\pi\mathrm{i}Nz_kx_r}\right| \tag{5.9}$$

for different bandwidths $N=2^\ell$, $\ell=3,\dots,7$. For the parameter $n=\nu N$ we investigate several choices $\nu\in\{1,\dots,10\}$. We compute the coefficients w_k using Algorithm 5.4. Subsequently, the approximation to the sinc function is computed by means of the NFFT, which is possible since the x_r are equispaced. The results for both, the error bound (5.6) and the maximum error (5.9), are displayed in Figure 5.1 (b). It becomes apparent that for increasing oversampling factor ν, the maximum error (5.9) decreases to machine precision for all choices of N. Even for rather large choices of ν (up to 10) the error remains stable, so there is no worsening in terms of ν. □



(a) with respect to the bandwidth N (b) with respect to the oversampling ν

Figure 5.1: Error constant (5.6) (dashed) and maximum error (5.9) (solid) of the approximation of $\mathrm{sinc}(N\pi x)$, $x \in [-1, 1]$, for different bandwidths $N = 2^\ell$, $\ell = 1, \dots, 7$, where $n = \nu N$, $\nu \in \{1, \dots, 10\}$, and Chebyshev nodes $z_k \in [-1, 1]$, $k = 0, \dots, n$.

6 Discrete sinc transform

Finally, we present an interesting signal processing application of the NNFFT. If a signal $h : \big[-\frac{1}{2}, \frac{1}{2}\big] \to \mathbb{C}$ is to be reconstructed from its equispaced/nonequispaced samples at $a_k \in \big[-\frac{1}{2}, \frac{1}{2}\big]$, then h is often modeled as linear combination of shifted sinc functions

$$h(x) = \sum_{k \in \mathcal{I}_{L_1}} c_k \, \mathrm{sinc}\big(N\pi\,(x - a_k)\big)\,, \quad x \in \mathbb{R}\,, \tag{6.1}$$

with complex coefficients c_k. In the following, we propose a fast algorithm for the approximate computation of the *discrete* sinc *transform* (see [7, 11])

$$h(b_\ell) = \sum_{k \in \mathcal{I}_{L_1}} c_k \, \mathrm{sinc}\big(N\pi\,(b_\ell - a_k)\big)\,, \quad \ell \in \mathcal{I}_{L_2}\,, \tag{6.2}$$

where $b_\ell \in \big[-\frac{1}{2}, \frac{1}{2}\big]$ can be equispaced/nonequispaced points.

Such a function (6.1) occurs by the application of the famous sampling theorem of Shannon–Whittaker–Kotelnikov (see e. g. [13, pp. 86–88]). Let $f \in L_1(\mathbb{R}) \cap C(\mathbb{R})$ be bandlimited on $\big[-\frac{L_2}{2}, \frac{L_2}{2}\big]$ for some $L_2 > 0$, i. e., the Fourier transform of f is supported on $\big[-\frac{L_2}{2}, \frac{L_2}{2}\big]$. Then for $N \in \mathbb{N}$ with $N \geq L_2$, the function f is completely determined by its values $f\big(\frac{k}{N}\big)$, $k \in \mathbb{Z}$, and further f can be represented in the form

$$f(x) = \sum_{k \in \mathbb{Z}} f\big(\tfrac{k}{N}\big)\, \mathrm{sinc}\big(N\pi\big(x - \tfrac{k}{N}\big)\big)\,, \quad x \in \mathbb{R}\,,$$

where the series converges absolutely and uniformly on $\mathbb{R}$. By truncation of this series, we obtain the linear combination of shifted sinc functions

$$\sum_{k\in\mathcal{I}_{L_1}} f\big(\tfrac{k}{N}\big)\,\mathrm{sinc}\big(N\pi\big(x-\tfrac{k}{N}\big)\big)\,,\quad x\in\mathbb{R}\,,$$

which has the same form as (6.1), when a_k are equispaced.
Since the naive computation of (6.2) requires $\mathcal{O}(L_1\cdot L_2)$ arithmetic operations, the aim is to find a more efficient method for the evaluation of (6.2). Up to now, several approaches for a fast computation of the discrete sinc transform (6.2) are known. In [7], the discrete sinc transform (6.2) is realized by applying a Gauss–Legendre quadrature rule to the integral (5.7). The result can then be approximated by means of two NNFFT's with $\mathcal{O}((L_1+L_2)\log(L_1+L_2))$ arithmetic operations. A multilevel algorithm with $\mathcal{O}(L_2\log(1/\delta))$ arithmetic operations is presented in [11] which is most effective for equispaced points a_k and b_ℓ and, as the authors claim themselves, is only practical for rather large target evaluation accuracy $\delta>0$.
In the following, we present a new approach for a fast sinc transform (6.2), where we approximate the function $\mathrm{sinc}(N\pi x)$ by an exponential sum on the interval $[-1,1]$ by means of the Clenshaw–Curtis quadrature as described in Section 5. Let the Chebyshev points $z_j=\cos\frac{j\pi}{n}$, $j=0,\ldots,n$, and the coefficients w_j defined by (5.4) be given. Utilizing (5.8), for arbitrary $a_k,\,b_\ell\in\big[-\frac12,\,\frac12\big]$ we obtain the approximation

$$\mathrm{sinc}\big(N\pi(a_k-b_\ell)\big)\approx\sum_{j=0}^{n} w_j\,\mathrm{e}^{-\pi\mathrm{i}Nz_j(a_k-b_\ell)}=\sum_{j=0}^{n} w_j\,\mathrm{e}^{-\pi\mathrm{i}Nz_ja_k}\,\mathrm{e}^{\pi\mathrm{i}Nz_jb_\ell}\,.$$

Inserting this approximation into (6.2) yields

$$\begin{aligned} h_\ell &:= \sum_{k\in\mathcal{I}_{L_1}} c_k\sum_{j=0}^{n} w_j\,\mathrm{e}^{-\pi\mathrm{i}Nz_ja_k}\,\mathrm{e}^{\pi\mathrm{i}Nz_jb_\ell}\\ &=\sum_{j=0}^{n} w_j\Bigg(\sum_{k\in\mathcal{I}_{L_1}} c_k\,\mathrm{e}^{-\pi\mathrm{i}Nz_ja_k}\Bigg)\mathrm{e}^{\pi\mathrm{i}Nz_jb_\ell}\,,\quad \ell\in\mathcal{I}_{L_2}\,. \end{aligned}\tag{6.3}$$

If $\varepsilon>0$ denotes a target accuracy, then we choose $n=2^t$, $t\in\mathbb{N}\setminus\{1\}$ such that by Theorem 5.3 it holds

$$\frac{36\,(1+\mathrm{e}^{-2CN})}{35\,(\mathrm{e}^2-1)}\,\mathrm{e}^{-N(\nu-C)}<\varepsilon\,,\quad \nu>C=3.69\,.$$

For example, in the case $\varepsilon=10^{-8}$ we obtain $n\ge 4N$ for $N\ge 54$.
We recognize that the term inside the brackets of (6.3) is an exponential sum of the form (1.2), which can be computed by means of an NNFFT. Then the resulting outer sum is of the same form such that this can also be computed by means of an NNFFT. Thus, as in [7] we may compute the discrete sinc transform (6.2) by means of an NNFFT, a multiplication by the precomputed coefficients w_j as well as another NNFFT afterwards. Hence, the fast sinc transform, which is an application of the NNFFT, can be summarized as follows.

Algorithm 6.1 (Fast sinc transform)
Input: $N \in \mathbb{N}$, $L_1, L_2 \in 2\mathbb{N}$ *as well as* $c_k \in \mathbb{C}$, $a_k \in \big[-\frac{1}{2}, \frac{1}{2}\big]$ *for* $k \in \mathcal{I}_{L_1}$, $z_j = \cos\frac{j\pi}{n}$ *with* $j = 0, \ldots, n$ *and* $n \geq 4N$.

0. *Precompute the values* w_j, $j = 0, \ldots, n$, *by Algorithm* 5.4.
1. *For all* $j = 0, \ldots, n$, *compute by* NNFFT

$$g_j := \sum_{k \in \mathcal{I}_{L_1}} c_k \, \mathrm{e}^{-\pi \mathrm{i} N z_j a_k} \,,$$

where $\tilde{g}_j$ *is the approximate value of* g_j.
2. *For all* $j = 0, \ldots, n$, *form the products*

$$\alpha_j := w_j \cdot \tilde{g}_j \,.$$

3. *For all* $\ell \in \mathcal{I}_{L_2}$ *compute by* NNFFT

$$\hat{h}_\ell := \sum_{j=0}^{n} \alpha_j \, \mathrm{e}^{\pi \mathrm{i} N z_j b_\ell} \,, \tag{6.4}$$

where $\tilde{h}_\ell$ *is the approximate value of* $\hat{h}_\ell$.
Output: $\tilde{h}_\ell$ *approximate value of* (6.2) *for* $\ell \in \mathcal{I}_{L_2}$.

If we use the same NNFFT in both steps (with the window functions φ_j, truncation parameters m_j, and oversampling factors σ_j for $j = 1, 2$), Algorithm 6.1 requires all in all

$$\mathcal{O}(N \log N + L_1 + L_2 + 2n)$$

arithmetic operations.
Considering the discrete sinc transform (6.2), we can deal with the special sums of the form

$$h\big(\tfrac{\ell}{N}\big) = \sum_{k \in \mathcal{I}_{L_1}} c_k \operatorname{sinc}\big(N\pi\big(a_k - \tfrac{\ell}{N}\big)\big) \,, \quad \ell \in \mathcal{I}_N,$$

i. e., we are given equispaced points $b_\ell = \frac{\ell}{N}$ with $L_2 = N$. In this special case, we simply obtain an adjoint NFFT instead of the NNFFT in step 3 of Algorithm 6.1. Therefore, the computational cost of Algorithm 6.1 reduces to $\mathcal{O}(N \log N + L_1 + n)$. In the case, where $a_k = \frac{k}{L_1}$, $k \in \mathcal{I}_{L_1}$, the NNFFT in step 1 of Algorithm 6.1 naturally turns into an NFFT. Clearly, in this case the same amount of arithmetic operations is needed as in the first special case. If both sets of nodes a_k and b_ℓ are equispaced, then the computational cost reduces even more to $\mathcal{O}(N \log N + n)$. Hence, these modifications are automatically be included in our fast sinc transform.
A quite similar approach was already developed in [2] for the computation of the Coulombian interaction between punctual masses, where the main idea is using two different quadrature rules to approximate the given problem. Then the computation can be done by means of NNFFTs, i. e., they receive a 3-step method analogous to Algorithm 6.1.

Now we study the error of the fast sinc transform in Algorithm 6.1, which is measured in the form

$$\max_{\ell \in \mathcal{I}_{L_2}} |h(b_\ell) - \tilde{h}_\ell| \,. \tag{6.5}$$

Theorem 6.2 *Let $N \in \mathbb{N}$ with $N \gg 1$ and $L_1,\, L_2 \in 2\mathbb{N}$ be given. Let $N_1 = \sigma_1 N \in 2\mathbb{N}$ with $\sigma_1 > 1$. For fixed $m_1 \in \mathbb{N} \setminus \{1\}$ with $2m_1 \ll N_1$, let $N_2 = \sigma_2\,(N_1 + 2m_1)$ with $\sigma_2 > 1$. For $m_2 \in \mathbb{N} \setminus \{1\}$ with $2m_2 \le \big(1 - \frac{1}{\sigma_1}\big)\,N_2$, let φ_1 and φ_2 be the window functions of the form (2.5). Let $a_k,\, b_\ell \in \big[-\frac{1}{2},\, \frac{1}{2}\big]$ with $k \in \mathcal{I}_{L_1}$, $\ell \in \mathcal{I}_{L_2}$ be arbitrary points and let $c_k \in \mathbb{C}$, $k \in \mathcal{I}_{L_1}$, be arbitrary coefficients. Let $a > 1$ be the constant (2.6). For a given target accuracy $\varepsilon > 0$, the number $n = 2^t$, $t \in \mathbb{N} \setminus \{1\}$, is chosen such that*

$$\frac{36\,(1 + \mathrm{e}^{-2CN})}{35\,(\mathrm{e}^2 - 1)}\, \mathrm{e}^{-N(\nu - C)} < \varepsilon\,, \quad \nu > C = 3.69\,. \tag{6.6}$$

Then the error of the fast sinc *transform can be estimated by*

$$\begin{aligned}\max_{\ell \in \mathcal{I}_{L_2}} \big|h(b_\ell) - \tilde{h}_\ell\big| \;\le\; & \Bigg(\varepsilon + 2\bigg[E_{\sigma_1}(\varphi_1) + \frac{a}{\hat{\varphi}_1\big(\frac{N}{2}\big)}\, E_{\sigma_2}(\varphi_2)\bigg] \\ & + \bigg[E_{\sigma_1}(\varphi_1) + \frac{a}{\hat{\varphi}_1\big(\frac{N}{2}\big)}\, E_{\sigma_2}(\varphi_2)\bigg]^2\Bigg) \sum_{k \in \mathcal{I}_{L_1}} |c_k|\,, \end{aligned} \tag{6.7}$$

where $E_{\sigma_j}(\varphi_j)$ for $j = 1,\, 2$, are the general $C(\mathbb{T})$-error constants of the form (3.7). If it holds

$$E_{\sigma_1}(\varphi_1) + \frac{a}{\hat{\varphi}_1\big(\frac{N}{2}\big)}\, E_{\sigma_2}(\varphi_2) \le 1\,, \tag{6.8}$$

one can use the simplified estimate

$$\max_{\ell \in \mathcal{I}_{L_2}} \big|h(b_\ell) - \tilde{h}_\ell\big| \le \Bigg(\varepsilon + 3E_{\sigma_1}(\varphi_1) + \frac{3a}{\hat{\varphi}_1\big(\frac{N}{2}\big)}\, E_{\sigma_2}(\varphi_2)\Bigg) \sum_{k \in \mathcal{I}_{L_1}} |c_k|\,. \tag{6.9}$$

Proof. By (6.3), the value h_ℓ is an approximation of $h(b_\ell)$. Since $a_k,\, b_\ell \in \big[-\frac{1}{2},\, \frac{1}{2}\big]$, it holds by (5.8) and (6.6) that

$$\Bigg|\mathrm{sinc}\big(\pi N(a_k - b_\ell)\big) - \sum_{j=0}^{n} w_j\, \mathrm{e}^{-\pi \mathrm{i} N z_j (a_k - b_\ell)}\Bigg| \le \varepsilon\,.$$

Hence, we conclude that

$$|h(b_\ell) - h_\ell| \le \varepsilon \sum_{k \in \mathcal{I}_{L_1}} |c_k|\,, \quad \ell \in \mathcal{I}_{L_2}\,. \tag{6.10}$$

After step 1 of Algorithm 6.1, the error of the NNFFT (with the window functions φ_1 and φ_2) can be estimated by Theorem 3.5 in the form

$$|g_j - \tilde{g}_j| \le \bigg[E_{\sigma_1}(\varphi_1) + \frac{a}{\hat{\varphi}_1\big(\frac{N}{2}\big)}\, E_{\sigma_2}(\varphi_2)\bigg] \sum_{k \in \mathcal{I}_{L_1}} |c_k|\,, \quad j = 0, \ldots, n\,.$$

Using (5.5), step 2 of Algorithm 6.1 generates the error

$$\begin{aligned}|\hat{h}_\ell - h_\ell| &\le \sum_{j=0}^{n} w_j\,|g_j - \tilde{g}_j| \le \Big(\sum_{j=0}^{n} w_j\Big)\Big[E_{\sigma_1}(\varphi_1) + \frac{a}{\hat{\varphi}_1\big(\frac{N}{2}\big)}\,E_{\sigma_2}(\varphi_2)\Big] \sum_{k\in\mathcal{I}_{L_1}} |c_k| \\ &= \Big[E_{\sigma_1}(\varphi_1) + \frac{a}{\hat{\varphi}_1\big(\frac{N}{2}\big)}\,E_{\sigma_2}(\varphi_2)\Big] \sum_{k\in\mathcal{I}_{L_1}} |c_k|\,. \qquad (6.11)\end{aligned}$$

After step 3 of Algorithm 6.1, the error of the NNFFT (with the same window functions φ_1 and φ_2) can be estimated by Theorem 3.5 in the form

$$\big|\hat{h}_\ell - \tilde{h}_\ell\big| \le \Big[E_{\sigma_1}(\varphi_1) + \frac{a}{\hat{\varphi}_1\big(\frac{N}{2}\big)}\,E_{\sigma_2}(\varphi_2)\Big] \sum_{j=0}^{n} w_j\big|\tilde{g}_j\big|\,, \quad \ell \in \mathcal{I}_{L_2}\,.$$

Using the triangle inequality, we obtain

$$\begin{aligned}\big|\tilde{g}_j\big| &\le |g_j| + |g_j - \tilde{g}_j| \le \sum_{k\in\mathcal{I}_{L_1}} |c_k| + |g_j - \tilde{g}_j| \\ &\le \sum_{k\in\mathcal{I}_{L_1}} |c_k| + \Big[E_{\sigma_1}(\varphi_1) + \frac{a}{\hat{\varphi}_1\big(\frac{N}{2}\big)}\,E_{\sigma_2}(\varphi_2)\Big] \sum_{k\in\mathcal{I}_{L_1}} |c_k|\,, \quad j = 0, \ldots, n\end{aligned}$$

such that by (5.5)

$$\begin{aligned}\big|\hat{h}_\ell - \tilde{h}_\ell\big| &\le \Big[E_{\sigma_1}(\varphi_1) + \frac{a}{\hat{\varphi}_1\big(\frac{N}{2}\big)}\,E_{\sigma_2}(\varphi_2)\Big]\Big(\sum_{j=0}^{n} w_j\Big) \sum_{k\in\mathcal{I}_{L_1}} |c_k| \\ &\quad + \Big[E_{\sigma_1}(\varphi_1) + \frac{a}{\hat{\varphi}_1\big(\frac{N}{2}\big)}\,E_{\sigma_2}(\varphi_2)\Big]^2\Big(\sum_{j=0}^{n} w_j\Big) \sum_{k\in\mathcal{I}_{L_1}} |c_k| \\ &= \Big[E_{\sigma_1}(\varphi_1) + \frac{a}{\hat{\varphi}_1\big(\frac{N}{2}\big)}\,E_{\sigma_2}(\varphi_2)\Big] \sum_{k\in\mathcal{I}_{L_1}} |c_k| \\ &\quad + \Big[E_{\sigma_1}(\varphi_1) + \frac{a}{\hat{\varphi}_1\big(\frac{N}{2}\big)}\,E_{\sigma_2}(\varphi_2)\Big]^2 \sum_{k\in\mathcal{I}_{L_1}} |c_k|\,. \qquad (6.12)\end{aligned}$$

Thus, the error of Algorithm 6.1 can be estimated by

$$\big|h(b_\ell) - \tilde{h}_\ell\big| \le |h(b_\ell) - h_\ell| + |h_\ell - \hat{h}_\ell| + \big|\hat{h}_\ell - \tilde{h}_\ell\big|\,, \quad \ell \in \mathcal{I}_{L_2}\,.$$

From (6.10) – (6.12) it follows the estimate (6.7). If it holds (6.8), we have

$$\Big[E_{\sigma_1}(\varphi_1) + \frac{a}{\hat{\varphi}_1\big(\frac{N}{2}\big)}\,E_{\sigma_2}(\varphi_2)\Big]^2 \le E_{\sigma_1}(\varphi_1) + \frac{a}{\hat{\varphi}_1\big(\frac{N}{2}\big)}\,E_{\sigma_2}(\varphi_2)$$

and therefore the simplified estimate (6.9). ■

Thus, the error of Algorithm 6.1 for the fast sinc transform mostly depends on the target accuracy ε of the precomputation and on the general $C(\mathbb{T})$-error constants $E_{\sigma_j}(\varphi_j)$, $j = 1,\,2$, of the window functions φ_j, $j = 1,\,2$, see Theorem 3.5.

Example 6.3 Next we verify the accuracy of our fast sinc transform in Algorithm 6.1. To this end, we choose random nodes $a_k \in \left[-\frac{1}{2}, \frac{1}{2}\right]$, equispaced points $b_\ell = \frac{\ell}{N}$ with $\ell \in \mathcal{I}_N$, as well as random coefficients $c_k \in \mathbb{C}$, $k \in \mathcal{I}_{L_1}$, and compute the discrete sinc transform (6.2) directly as well as its approximation (6.4) by means of the fast sinc transform. Subsequently, we compute the maximum error (6.5). Due to the randomness of the given values this test is repeated one hundred times and afterwards the maximum error over all repetitions is computed.
In this experiment we choose different bandwidths $N = 2^k$, $k = 5, \dots, 13$, and without loss of generality we use $L_1 = \frac{N}{2}$. We apply Algorithm 6.1 using the weights w_j computed by means of Algorithm 5.4 and the Chebyshev points $z_j = \cos\frac{j\pi}{n}$, $j = 0, \dots, n$. Therefore, we only have to examine the parameter choice of $n \geq 4N$. To this end, we compare the results for several choices, namely for $n \in \{4N, 6N, 8N\}$. The appropriate results can be found in Figure 6.1. We see that for large N there is almost no difference between the different choices of n. However, we point out that a higher choice heavily increases the computational cost of Algorithm 6.1. Therefore, it is recommended to use the smallest possible choice $n = 4N$. Compared to [7] the same approximation errors are obtained, but with a more efficient precomputation of weights. □

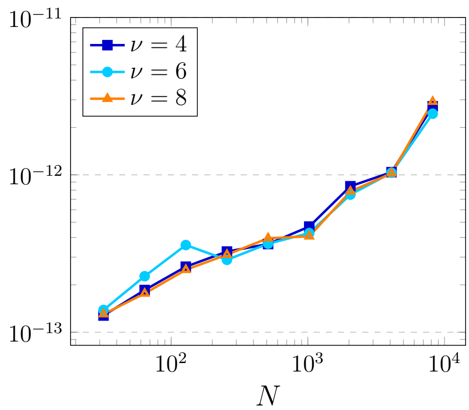


Figure 6.1: Maximum error (6.5) for several bandwidths $N = 2^k$, $k = 5, \dots, 13$,, shown for $n = \nu N$, $\nu \in \{4, 6, 8\}$, using the coefficients w_j obtained by Algorithm 5.4.

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