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Nonlinear Programming
Exercises 9

1. When A is an $m \times n$ matrix with $m \geq n$, show that A has full column rank if and only if $A^T A$ is nonsingular.

2. Consider the least squares problem
$$\min_{x \in \mathbb{R}^n} f(x) = \frac{1}{2}\|Ax - b\|^2$$
where A is a full rank $m \times n$ matrix with $m \geq n$ and $b \in \mathbb{R}^m$. Suppose $B = A^T A$ and $c = A^T b$. If x^* and $\bar{x}$ are optimal and approximate optimal solutions to the problem, respectively, then show that
   a) $\|x^* - \bar{x}\| \leq \kappa(B)\frac{\|r\|}{\|B\|}$,
   b) $\frac{\|x^* - \bar{x}\|}{\|x^*\|} \leq \kappa(B)\frac{\|r\|}{\|c\|}$, provided $x^* \neq 0$ and $c \neq 0$,

   where $\kappa(B) = \|B\|\|B^{-1}\|$, *the condition number of* B, and $r = B\bar{x} - c$, *the residual vector.*

3. Let it be required to determine parameter values $x_i (i = 1, 2, 3)$ such that the formula
$$\phi(x; t) = \frac{x_1 x_3 t_1}{1 + x_1 t_1 + x_2 t_2}$$
fits the data given in the table below in the least squares sense.

t_1	t_2	y
1.0	1.0	0.126
2.0	1.0	0.219
1.0	2.0	0.076
2.0	2.0	0.126
1.0	0.0	0.186

   where each y is an observed value at the corresponding (t_1, t_2). Formulate the problem of finding the maximum liklihood estimates of parameters x_1, x_2, and x_3 assuming errors are i.i.d. according to the normal distribution. Solve the problem using the Gauss-Newton method.

4. Find a solution to the following nonlinear system using Newton's method.
$$\begin{aligned} &3x_1 - \cos(x_2 x_3) - 1/2 = 0 \\ &x_1^2 - 81(x_2 + 0.1)^2 + \sin(x_3) + 1.06 = 0 \\ &e^{-x_1 x_2} + 20x_3 + (10\pi - 3)/3 = 0 \end{aligned}$$

5. For the problem
$$\min_{x \in \Omega} (x_1 - 2)^2 + (x_2 - 6)^2$$
where $\Omega = \{(x_1, x_2) \in \mathbb{R}^2 : \ x_1 - x_2 \geq -3, \ x_1 + x_2 \leq 6, \ x_1 - 3x_2 \leq 0, \ x_1 \geq 0, x_2 \geq 0\}$ do the following:

a) Illustrate the problem graphically.

b) Verify that for the point $\bar{x} = (1.5, 4.5)$ the optimalitiy condition $\nabla f(\bar{x})^T(x - \bar{x}) \geq 0$ is satisfied for all $x \in \Omega$.

c) Verify that the condition is not satisfied for the point $\bar{x} = (0, 3)$.

d) What is the graphical(geometric) interpretation of the optimality condition?