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Nonlinear Programming
Exercises 8

1. Verify the differentiation formulae

$$
\begin{aligned}
x_i &= L(x_j), && (8.1)\\
D_p x_i &= L'(x_j)(D_p x_j), && (8.2)\\
D_{pq} x_i &= L''(x_j)(D_p x_j)(D_q x_j) + L'(x_j) D_{pq} x_j && (8.3)
\end{aligned}
$$

for the unitary operation $x_i = L(x_j)$ where $D_p x_i$ and $D_{pq} x_i$ are defined as follows:

$$D_p x_i := (\nabla x_i)^T p = \sum_j \frac{\partial x_i}{\partial x_j} p_j, \qquad D_{pq} x_i := p^T (\nabla^2 x_i) q, \quad p, q \in \mathbb{R}^n.$$

2. Consider Theorem VI.1 in the lecture. Show that the sequence of iterates $\{x_k\}$ converges to x^* at the same rate, in the $[\nabla^2 f(x^*)]^2$- norm, as ∇f_k converges to zero.

3. a) Show that if f is differentiable and strongly convex, then

$$s_k^T y_k > 0 \tag{8.4}$$

for any vectors x_k and x_{k+1}, where $s_k = x_{k+1} - x_k$ and $y_k = \nabla f_{k+1} - \nabla f_k$.

   b) Give an example of a function of one variable satisfying $g(0) = -1$ and $g(1) = -1/4$ and show that (8.4) does not hold in this case.

4. a) Suppose A, B are nonsingular $n \times n$ matrices, R is nonsingular $r \times r$, and X, Y are $r \times n$ such that $B = A + XRY$. Show that

$$B^{-1} = A^{-1} - A^{-1} X (R^{-1} + Y A^{-1} X)^{-1} Y A^{-1}.$$

   b) Consider the matrices B_{k+1} and H_{k+1} constructed in the lecture within the DFP method. Show that $H_{k+1} = B_{k+1}^{-1}$ for each $k = 0, 1, 2, ...$

   c) Determine H_{k+1}^{-1}, where H_{k+1} is the matrix constructed in the lecture within the BFGS method.

5. Consider the matrices $B_k,\ k = 0, 1, 2, ...,$ constructed within the BFGS method ($= H_k^{-1}$ of 4(c)), and the vectors s_k and y_k defined in the Problem 3(a), above. The object of this exercise is to prove

$$\det(B_{k+1}) = \det(B_k) \frac{y_k^T s_k}{s_k^T B_k s_k}. \tag{8.5}$$

   a) First show that $\det(I + xy^T) = 1 + y^T x$, where x and y are n-vectors. Hint: Assuming that $x \neq 0$, we can find vectors $w_1, w_2, \ldots, w_{n-1}$ such that the matrix Q defined by

$$Q = [x, w_1, \ldots, w_{n-1}]$$

is nonsingular and $x = Qe_1$, where $e_1 = (1, 0, ..., 0)^T$. If we define

$$y^T Q = (z_1, ..., z_n),$$

then

$$z_1 = y^T Q e_1 = y^T Q(Q^{-1}x) = y^T x,$$

and

$$\det(I + xy^T) = \det(Q^{-1}(I + xy^T)Q) = \det(I + e_1 y^T Q).$$

b) Use a similar technique to prove that

$$\det(I + xy^T + uv^T) = (1 + y^T x)(1 + v^T u) - (x^T v)(y^T u).$$

c) Now use this relation to establish (8.5).