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Nonlinear Programming

Exercises 6

1. Suppose $\{p_0, p_1, \ldots, p_{n-1}\}$ is the set of conjugate gradient directions constructed with the linear CG-Algorithm. Show that $p_k, \quad k = 1, 2, \ldots, n-1$ are all descent directions.

2. **Definition:** A set of vectors $\{p_0, p_1, \ldots, p_{n-1}\}$ is said to be *conjugate* with respect to a symmetric positive definite matrix A if

$$p_i^T A p_j = 0, \qquad \forall i \neq j.$$

   a) Let A be an $n \times n$ symmetric positive definite matrix and $\{p_0, p_1, \ldots, p_{n-1}\}$ be a set of non-zero conjugate vectors w.r.t. A. Show that $p_k, \quad k = 1, 2, \ldots, n-1$ are linearly independent.

   b) Consider the minimization problem

$$\min_{x \in \mathbb{R}^n} \frac{1}{2} x^T A x - b^T x$$

   where A is an $n \times n$ symmetric positive definite matrix and $b \in \mathbb{R}^n$. Suppose that $\{p_0, p_1, \ldots, p_{n-1}\}$ is a set of non-zero conjugate vectors w.r.t A.

   i. Show that the solution x^* of the problem can be given as

$$x^* = \sum_{i=0}^{n-1} \left(\frac{p_i^T b}{p_i^T A p_i}\right) p_i.$$

   ii. Show that for any $x_0 \in \mathbb{R}^n$, the sequence $\{x_k\}$ generated according to

$$x_{k+1} = x_k + \alpha_k p_k, \qquad k \geq 0$$

   with

$$\alpha_k = \frac{-r_k^T p_k}{p_k^T A p_k}, \qquad r_k = A x_k - b$$

   converges to the solution x^* of the problem after n-steps.

3. Write a program, in Matlab, that implements the linear CG-Algorithm. Apply it to find a minimizer of $f(x) = \frac{1}{2} x^T A x - b^T x$, where,

   a)

$$A = \begin{bmatrix} 9 & -1 & 0 & 3 \\ -1 & 6 & 2 & 0 \\ 0 & 2 & 7 & -5 \\ 3 & 0 & -5 & 7 \end{bmatrix}, \quad b = \begin{bmatrix} 15 \\ 0 \\ -10 \\ 2 \end{bmatrix}$$

   b) Download the matrices A (sparse 1000×1000) and b ($\mathbb{R}^{1000}$) from the home-page of the course. Solve the problem and print, in each iteration, the norm of $Ax - b$ where x is the intermediate solution obtained by the linear CG-Algorithm. Compare the runtime to the commands $x = A \backslash b$ and $R = \mathrm{chol}(A)$.

4. Using the Fletcher-Reeves and Polak-Ribière Method,

   a) solve $\min_{(x,y)\in\mathbb{R}^2} x^2 + y^4 + x^2 + 2y^2 - xy - x.$

   b) find a minimizer of the Rosenbrock function $f(x, y) = 100(y - x^2)^2 + (1 - x)^2$, starting from various starting point such as (1.2, 1.2), (0,-1), (0,0.5), and (10,5).