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Nonlinear Programming

Exercises 3

corrected on May 5

1. a) Let $f_1(x) = \frac{1}{x+2} - (x - \frac{1}{2})^4 + e^{(x+1)}, \quad x \in (-2, \infty)$. Find the descent direction of f_1 at $x_0 = 5.3$ and determine the corresponding intervals of step lengths that satisfy

   i) the Wolfe conditions for $c_1 = 0.01, \;\; \& \;\; c_2 = 0.1$.

   ii) the strong Wolfe conditions for $c_1 = 0.01, \;\; \& \;\; c_2 = 0.1$.

   iii) the Goldstein conditions for $c = 0.01$.

   b) Let $f_2(x) = e^{(x_1^2+x_2^2)} \cos(x_1^2 + x_2^2), \quad x \in \mathbb{R}^2$. Show that P=(-1,-0.5) is a descent direction of f_2 at $x_0 = (2.95, 2.95)$ and determine the corresponding intervals of step lengths that satisfy the Wolfe conditions, the strong Wolfe conditions and the Goldstein conditions, where c_1, c_2 & c are given as in 1(a).

2. **Definition:** A function $f : S \to \mathbb{R}$ with convex $S \subset \mathbb{R}^n$ is said to be **quasiconvex** on S, if for all $x_1, x_2 \in S$

   $$f(tx_1 + (1-t)x_2) \leq \max\{f(x_1), f(x_2)\}, \quad \forall t \in [0, 1].$$

   a) Show that $f \in C^1(\mathbb{R})$ is quasiconvex on $[a, b]$ if and only if either f is monotonic on $[a, b]$ or there exists $c \in (a, b)$ such that $f'(x) \leq 0, \;\; \forall x \in (a, c]$ and $f'(x) \geq 0, \;\; \forall x \in [c, b)$.

   b) Let $f : S \to \mathbb{R}$ with convex $S \subset \mathbb{R}^n$. Show that f is quasiconvex on S if and only if for all $\alpha \in \mathbb{R}$ the level set $S_\alpha = \{x \in S : f(x) \leq \alpha\}$ is convex.

   c) Define *strong* quasiconvexity and prove: For $f : \mathbb{R} \to \mathbb{R}$ continuous and strongly quasiconvex on $[a, b]$ the optimization problem

   $$\min_{a \leq x \leq b} f(x)$$

   has a unique optimal solution x^* and for $\lambda, \mu \in \mathbb{R}$ with $a \leq \lambda < \mu \leq b$,

   i) $f(\lambda) \leq f(\mu)$ implies $x^* \in [a, \mu]$,

   ii) $f(\lambda) \geq f(\mu)$ implies $x^* \in [\lambda, b]$.

3. The Golden Section search method is a line search method for minimizing a strongly quasiconvex function on $[a_1, b_1]$. In iteration k it compares function values at λ_k and μ_k defined by

   $$\begin{array}{ll} \lambda_k = a_k + (1-t)(b_k - a_k), & k = 1, 2, \ldots \\ \mu_k = a_k + t(b_k - a_k), & k = 1, 2, \ldots \end{array}$$

   where $t = \frac{-1+\sqrt{5}}{2} \approx 0.618$, and sets a_{k+1} and b_{k+1} as suggested in Ex. 2c).

   a) Show that, in every iteration,

   i) $b_k - \lambda_k = \mu_k - a_k$,

   ii) either $\lambda_{k+1} = \mu_k \;$ or $\; \mu_{k+1} = \lambda_k$.

b) Write down the algorithm for the Golden Section search method.

c) Apply the Golden Section method with $a_1 = -1$, $b_1 = 4$ to

$$\min -e^{-(x-2)^2/10} + 1$$

and list search intervals and function values at λ_k and μ_k up to $k = 10$.

4. In the Fibonacci search method the number of evaluations n is fixed in advance. It searches the minimum of a strongly quasiconvex function on $[a_1, b_1]$ by comparing, in iteration k, the function values at

$$\begin{aligned} \lambda_k &= a_k + \tfrac{F_{n-k-1}}{F_{n-k+1}}(b_k - a_k), \qquad k = 1, 2, \ldots, n-1 \\ \mu_k &= a_k + \tfrac{F_{n-k}}{F_{n-k+1}}(b_k - a_k), \qquad k = 1, 2, \ldots, n-1 \end{aligned}$$

where the Fibonacci sequence $\{F_k\}$ is defined as follows :

$$F_0 = F_1 = 1, \qquad F_{k+1} = F_k + F_{k-1}, \quad k = 1, 2, \ldots$$

a) Show that in every iteration either $\lambda_{k+1} = \mu_k$ or $\mu_{k+1} = \lambda_k$.

b) Write down the algorithm for the Fibonacci search method.

c) Apply the Fibonacci search method with $a_1 = -1$, $b_1 = 4$, $n = 11$ to

$$\min -e^{-(x-2)^2/10} + 1$$

and list search intervals and function values at λ_k and μ_k up to $k = 10$. Compare this to 3c).

5. For a differentiable strongly quasiconvex function the bisection method may be employed. It is based on the following result:
Let $f \in C^1(\mathbb{R})$ be strongly quasiconvex on $[a_1, b_1]$ and let $\lambda_k = \frac{1}{2}(a_k + b_k)$ for $k = 1, 2, \ldots$

a) Show that

i) if $f'(\lambda_k) < 0$, then a minimizer of f lies in $[\lambda_k, b_k]$.

ii) if $f'(\lambda_k) > 0$, then a minimizer of f lies in $[a_k, \lambda_k]$.

b) Write down the algorithm for the bisection method.

c) Apply the bisection method with $a_1 = -1$, $b_1 = 4$ to

$$\min -e^{-(x-2)^2/10} + 1$$

and list search intervals and function values at λ_k up to $k = 10$.