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Nonlinear Programming
Exercises 1

1. Compute the gradient $\nabla f_i(x)$ and Hessian matrix $\nabla^2 f_i(x)$, for $i = 1, 2, \ldots, 4$.
   - $f_1(x) = \frac{1}{x+2} - (x - \frac{1}{2})^4 + e^{x+1}, \quad x \in (-2, \infty)$.
   - $f_2(x) = e^{(x_1+x_2)} cos(x_1^2 + x_2^2), \quad x \in \mathbb{R}^2$.
   - $f_3(x) = 100(x_2 - x_1^2)^2 + (1 - x_1)^2, \quad x \in \mathbb{R}^2$. (Rosenbrock function)
   - $f_4(x) = x^T A x$, with $A^T = A \in \mathbb{R}^{n \times n}, \quad x \in \mathbb{R}^n$.

2. Given a function $f : \mathbb{R}^n \to \mathbb{R}$, the linear model of f at $\bar{x}$ is

$$f(\bar{x}) + \nabla f(\bar{x})^T (x - \bar{x})$$

   and the quadratic model of f at $\bar{x}$ is

$$f(\bar{x}) + \nabla f(\bar{x})^T (x - \bar{x}) + \frac{1}{2}(x - \bar{x})^T \nabla^2 f(\bar{x})(x - \bar{x}).$$

   Consider the functions $f_1(x)$ and $f_2(x)$ given in the problem 1.

   a) Determine the linear and quadratic model
      i) of f_1 at $\bar{x} = 0; \quad \bar{x} = 4$
      ii) of f_2 at $\bar{x} = (1.5, 1.3); \quad \bar{x} = (0.7, 1.2)$
   b) Using Matlab, plot
      i) the graph of f_1 and its linear and quadratic models at $\bar{x} = 0; \quad \bar{x} = 4$.
      ii) the graph of f_2 and its linear and quadratic models at $\bar{x} = (1.5, 1.3)$; $\bar{x} = (0.7, 1.2)$.

   (Example-files in Matlab: `www.mathematik.uni-kl.de/~helmberg/nlp_s02`).

3. Let A be a symmetric matrix of order n. Show that the following statements are equivalent:

   i) A is positive semidefinite, i.e., $v^T A v \geq 0 \ \ \forall v \in \mathbb{R}^n$.
   ii) $\lambda_i \geq 0$ where λ_i is an eigenvalue of A, $i = 1, 2, \ldots, n$.
   iii) $\exists C \in \mathbb{R}^{m \times n}$ such that $A = C^T C$ (and $rank(C) = rank(A)$).

4. Determine whether the Hessian matrix of the following functions at the given point $\bar{x}$ is positive definite, positive semidefinite, negative definite, negative semidefinite, or indefinite.

   a) $f_1(x) = x_1^2 x_2^2 + 5, \qquad (i)\ \bar{x} = (0, 0) \quad (ii)\ \bar{x} = (1, 1)$
   b) $f_2(x) = -x_1^3 + 2x_1 + x_1^2 - 6x_2 - x_2^2$,
      $(i)\ \bar{x} = (1, 0), \quad (ii)\ \bar{x} = (0, 1), \quad (iii)\ \bar{x} = (\frac{1}{3}, 3)$
   c) $f_3(x) = x_1^2 + x_2^2 - x_1 x_2 + 3, \qquad \bar{x} \in \mathbb{R}^2$
   d) $f_4(x) = e^{(x_1+x_2)} + cos(x_1^2 + x_2^2), \qquad (i)\ \bar{x} = (0, 0), \quad (ii)\ \bar{x} = (0, \sqrt{\pi})$

5. - **Definition 1. (Convex set)** A set $S \subseteq \mathbb{R}^n$ is *convex* if for any $x_1, x_2 \in S$ it holds that $\lambda x_1 + (1-\lambda)x_2 \in S$ for $0 \leq \lambda \leq 1$.

   - **Definition 2. (Convex function)** A function $f : \mathbb{R}^n \to \mathbb{R}$ is *convex* if for any $x_1, x_2 \in \mathbb{R}^n$, and $0 \leq \lambda \leq 1$, we have

$$f(\lambda x_1 + (1-\lambda)x_2) \leq \lambda f(x_1) + (1-\lambda)f(x_2).$$

   i) Prove that a local minimizer of a convex function is also a global minimizer. (*Hint: Prove by contradiction*)

   ii) Prove that the set of all minimizers of a convex function is convex.