TECHNISCHE UNIVERSITÄT CHEMNITZ

Exercise sheet 7 – released on 21.01.2020

Exercise 7.1 (written Exercise)[4 Points]
Let $A \in \mathbb{R}^{n\times n}$ and all eigenvalues $\lambda_1, \ldots, \lambda_n$ of A satisfy $\lambda_i \in \mathbb{C}^- := \{\lambda \in \mathbb{C} : \mathrm{Re}(\lambda) < 0\}$. Then, for every matrix $Q \in \mathbb{R}^{n\times n}$, the *Lyapunov equation*

$$AP + PA^\intercal = Q$$

does have a unique solution $P = -\int_0^\infty e^{At} Q e^{A^\intercal t}\,\mathrm{d}t \in \mathbb{R}^{n\times n}$.

Exercise 7.2 (voting Exercise)
Let $[A, B, C, D] \in \Sigma_{n,m,p}$ be asymptotically stable. Given an invertible matrix $T \in \mathbb{R}^{n\times n}$ with associated transformed system $\left[\hat{A}, \hat{B}, \hat{C}, \hat{D}\right] := [TAT^{-1}, TB, CT^{-1}, D]$, then

(a) P is the controllability Gramian of $[A, B, C, D] \;\Leftrightarrow\; \hat{P} := TPT^\intercal$ is the controllability Gramian of $\left[\hat{A}, \hat{B}, \hat{C}, \hat{D}\right]$,

(b) Q is the observability Gramian of $[A, B, C, D] \;\Leftrightarrow\; \hat{Q} := T^{-\intercal} Q T^{-1}$ is the observability Gramian of $\left[\hat{A}, \hat{B}, \hat{C}, \hat{D}\right]$.

Exercise 7.3 (programming Exercise)[6 Points]
Write a `MATLAB` function

```
[A_11, B_1, C_1, D] = Balanced_Truncation(A,B,C,D,r)
```

which for a given system $[A, B, C, D] \in \Sigma_{n,m,p}$ and a desired reduced state dimension $r \leq n$ realized Algorithm 2 of the lecture notes and outputs the reduced system $[A_{11}, B_1, C_1, D] \in \Sigma_{r,m,p}$ (the `MATLAB` function `lyap` solves Lyapunov equations).

(a) Verify your implementation: using the system

$$A = \begin{pmatrix} 1 & 3 \\ -1 & -2 \end{pmatrix}, \quad B = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad C = \begin{pmatrix} 0 & 1 \end{pmatrix}, \quad D = 0$$

from Example 2.12, generate the balanced system and compare your results with Example 2.16.

(b) Use the system given in `Sheet7_Data.mat` to generate a reduced system for $r = 30$. Furthermore, `Sheet7_Data.mat` contains a control `u` that is constant in time and an initial state `x0`. Use these, to write functions

```
out = yfull(t)     and     out = yred(t)
```

that evaluate the full and reduced output at time $t > 0$. Sample the interval $[0, 1]$ with a suitable grid and evaluate both functions on the grid. Compare the results! What happens if you choose a different r? Interpret your results in view of the matrix Σ.

Tip: The formula derived before Definition 2.7 might be helpful.

Explanation of terms

- For **written Exercises** hand in your elaboration/solution after one of the lectures or via Email to Dominik Garmatter until **03.02.2020**.
- For **programming Exercises** send your **commented** solution in `MATLAB`-Code via E-mail to Dominik Garmatter. Please, start the subject of your E-mail with **"MOR_201920_7:"**.
- For **voting Exercises** no written solution is requested. The solution for these exercises will be discussed/presented during the exercise course.
- All exercises of Exercise sheet 7 will be discussed during the exercise on 04.02.2020.