TECHNISCHE UNIVERSITÄT
CHEMNITZ

Exercise sheet 6 – released on 07.01.2020

Exercise 6.1 (voting Exercise)

Let $A_1, A_2 \in \mathbb{R}^{n\times n}$, $\tilde{B} \in \mathbb{R}^{n\times m}$, $x(t) : \mathbb{R} \to \mathbb{R}^n$, $u(t) : \mathbb{R} \to \mathbb{R}^m$, and $x_0, x_1 \in \mathbb{R}^n$. Write the following LTI control system of order 2

$$\ddot{x}(t) = A_1\dot{x}(t) + A_2x(t) + \tilde{B}u(t), \qquad \dot{x}(0) = x_1,\ x(0) = x_0,$$

where $\ddot{x}(t) := \frac{\partial^2}{\partial t \partial t}x(t)$, into an (equivalent) LTI control system of order 1.

Exercise 6.2 (written Exercise)[4 Points]

For every $A \in \mathbb{C}^{n\times n}$ there exists the Jordan normal form

$$S^{-1}AS = J = \mathrm{diag}(J_1, \dots J_g), \quad \text{with} \quad J_l := \begin{pmatrix} \lambda_l & 1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & \lambda_l & 1 \\ 0 & \cdots & 0 & \lambda_l \end{pmatrix},$$

where J_l are the *Jordan blocks* as described in Definition and Theorem 2.4. Show that for the matrix exponential

$$e^{At} = Se^{Jt}S^{-1} = S\mathrm{diag}(e^{J_1 t}, \dots, e^{J_g t})S^{-1}$$

holds and, letting d denote the dimension of a Jordan block J_l, show that

$$e^{J_l t} = \begin{pmatrix} 1 & t & \frac{t^2}{2!} & \cdots & \frac{t^{d-2}}{(d-2)!} & \frac{t^{d-1}}{(d-1)!} \\ 0 & 1 & t & \cdots & \frac{t^{d-3}}{(d-3)!} & \frac{t^{d-2}}{(d-2)!} \\ \vdots & \vdots & & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & t \\ 0 & 0 & 0 & \cdots & 0 & 1 \end{pmatrix} e^{\lambda_l t} \quad \text{for} \quad l = 1, \dots, g.$$

Tip: Try investigating the transformed variable $z(t) := S^{-1}x(t)$. For the Jordan block, try working with a suitable decomposition into a diagonal and a nilpotent matrix.

Exercise 6.3 (voting Exercise)

Prove Proposition 2.6: for any input $u \in \mathcal{U}_{ad} = C([t_0, t_f], \mathbb{R}^m)$ and initial state $x_0 \in \mathbb{R}^n$ the unique *state solution* of (LTI) is given by

$$x(t) = e^{A(t-t_0)}x_0 + \int_{t_0}^{t} e^{A(t-\tau)}Bu(\tau)\,\mathrm{d}\tau.$$

The corresponding *output-response* of (LTI) is then given by

$$y(t) = Ce^{A(t-t_0)}x_0 + \int_{t_0}^{t} Ce^{A(t-\tau)}Bu(\tau)\,\mathrm{d}\tau + Du(t).$$

Tip: make the Ansatz $x(t) = e^{At}z(t)$ and find a suitable function $z(t)$.

Exercise 6.4 (programming Exercise)[6 Points]

(a) Write a function `Matrix = exp_A_jordan(A,t)` that for a given matrix $A \in \mathbb{R}^{n\times n}$ and $t \in \mathbb{R}$ computes the matrix exponential $e^{At} \in \mathbb{R}^{n\times n}$ based on the Jordan normal form (use `help jordan` in Matlab and Exercise 6.2). Compare your results for

$$\bar{A} = \begin{pmatrix} 1 & 7 & 7 & -8 & 6 \\ 1 & 5 & 5 & -5 & 5 \\ 1 & 0 & 2 & -1 & 1 \\ 0 & 3 & 3 & -3 & 2 \\ -1 & -4 & -5 & 5 & -4 \end{pmatrix} \quad \text{and} \quad \bar{T} = 5$$

with the Matlab function `expm`.

(b) Consider for some $A \in \mathbb{R}^{n\times n}$ and the time horizon $[0,T]$ the ODE

$$\dot{x}(t) = Ax(t), \quad x(0) = x_0 \in \mathbb{R}^n. \tag{$*$}$$

Write a function `val = impl_Euler(A,T,x_0)` that approximates the solution of $(*)$ at the final time T using the implicit Euler-scheme and a suitable time grid. Apply your function to $\bar{A}$, $\bar{T}$, and $x_0 = (1,1,1,1,1)^\intercal$ and compare your results with the solution given via the matrix exponential.

(c) Repeat the experiments from (a) and (b) but replace $\bar{A}$ with the matrix given in `Sheet6_Data.m` on the webpage. What do you observe?

Explanation of terms

- For **written Exercises** hand in your elaboration/solution after one of the lectures or via Email to Dominik Garmatter until **20.01.2020**.
- For **programming Exercises** send your **commented** solution in `MATLAB`-Code via E-mail to Dominik Garmatter. Please, start the subject of your E-mail with **"MOR_201920_6:"**.
- For **voting Exercises** no written solution is requested. The solution for these exercises will be discussed/presented during the exercise course.
- All exercises of Exercise sheet 6 will be discussed during the exercise on 21.01.2020.