TECHNISCHE UNIVERSITÄT
CHEMNITZ

Exercise sheet 5 – released on 10.12.2019

Exercise 5.1 (written Exercise)[6 Points]
For $\{u_i\}_{i=1}^n \subset X$, let

$$R : X \to X : u \mapsto Ru := \frac{1}{n}\sum_{i=1}^{n} \langle u_i, u\rangle_X \, u_i$$

be the operator defined in Theorem 1.64 with the orthonormal set $\{\varphi_i\}_{i=1}^{n'}$ of $0 < n' \leq n$ eigenvectors with real eigenvalues $\lambda_1 \geq \cdots \geq \lambda_{n'} > 0$.

(a) Verify that the coordinates of the data with respect to the POD-basis are uncorrelated, i.e., for all $j,\ k = 1, \ldots n'$ with $j \neq k$ it holds

$$\sum_{i=1}^{n} \langle u_i, \varphi_j\rangle_X \langle u_i, \varphi_k\rangle_X = 0.$$

(b) Verify that the mean squared coordinates of the data are exactly the eigenvalues, i.e.,

$$\frac{1}{n}\sum_{i=1}^{n} \langle u_i, \varphi_j\rangle_X^2 = \lambda_j \quad \text{for } j = 1, \ldots, n'.$$

(c) Defining $\bar{X} := \operatorname{span}(\{u_i\}_{i=1}^n)$, show that $R : \bar{X} \to \bar{X}$ is bijective and that $\{\varphi_i\}_{i=1}^{n'}$, $\{\sqrt{\lambda_i}\}_{i=1}^{n'}$ are the principal axes and principal axes sections of the ellipsoid $\mathcal{E} := \{u \in X \mid \langle u, R^{-1}u\rangle_X = 1\}$.

Tip: Use without proof that $w \in X$ is a principal axis of $\mathcal{E}$, if for all $\psi \in \bar{X}$ with $\langle \psi, w\rangle_X = 0$ it holds for all $u \in \operatorname{span}(w)$ with $\psi + u \in \mathcal{E}$ that

$$\left\langle \psi + u, R^{-1}(\psi + u)\right\rangle_X = 1 \Leftrightarrow \left\langle \psi - u, R^{-1}(\psi - u)\right\rangle_X = 1.$$

Exercise 5.2 (voting Exercise)
Prove Theorem 1.67 (b) for $m = 1$: let $\{u_i\}_{i=1}^n \subset X$ and define for a finite dimensional, closed subspace $Y \subset X$ the *mean square projection error*

$$J(Y) := \frac{1}{n}\sum_{i=1}^{n} \|u_i - P_Y u_i\|_X^2\,,$$

where $P_Y \in \mathcal{L}(X, Y)$ denotes the orthogonal projection onto Y. Then, show that

$$J(X_{POD,1}) = \inf_{\substack{Y \subset X \\ \dim(Y)=1}} J(Y),$$

where $X_{POD,1}$ was defined in Theorem 1.64.

Exercise 5.3 (voting Exercise)
Let $\{u_i\}_{i=1}^n \subset X$ be linearly independent with Gramian matrix $\mathbf{K}_u = \left(\langle u_i, u_j \rangle_X\right)_{i,j=1}^n \in \mathbb{R}^{n\times n}$, which has the Cholesky-factorization $\mathbf{K}_u = \mathbf{L}\mathbf{L}^\intercal$ (such that $\mathbf{L}$ is a lower triangular matrix with positive diagonal). Defining $c_{ij} := \left(\mathbf{L}^{-\intercal}\right)_{ij}$, show that $\{\varphi_1, \ldots, \varphi_n\}$ with $\varphi_j := \sum_{i=1}^j c_{ij} u_i$ is the Gram-Schmidt orthonormalized set of $\{u_i\}_{i=1}^n$, i.e.,

(a) $\{\varphi_1, \ldots, \varphi_n\}$ form an orthonormalized system,

(b) $\mathrm{span}(\{u_i\}_{i=1}^n) = \mathrm{span}(\{\varphi_i\}_{i=1}^n)$.

Exercise 5.4 (programming Exercise)[6 Points]
Find on the homepage the file `Sheet5_Data.mat` - it contains the `model` for a 3×3 Thermalblock as well as corresponding `detailed_data` (including an arbitrary reduced basis).

(a) Write a function `[RB, Training_err] = greedy_RB(model, detailed_data, S_train, Nmax)`, which for a given model and a set of training parameters $S_{train} \subset \mathcal{P}$ constructs a reduced basis of size `Nmax` using the Greedy-Algorithm (Algorithm 1 of the lecture notes) with the error estimator $\Delta_N(\mu)$ as error indicator. As a second output, the function should provide the sequence of the training error $\varepsilon_N = \max_{\mu\in S_{train}} \Delta(X_{GRE,N}, \mu)$.

(b) Write a function `eps_test = max_test_estimator(model, reduced_data, S_test)`, which for a given reduced model and a set of test parameters $S_{test} \in \mathcal{P}$ computes the sequence of the test error $\varepsilon_{test} := \max_{\mu\in S_{test}} \Delta(X_{GRE,N}, \mu)$.

(c) Randomly generate $S_{train} \neq S_{test} \subset \mathcal{P} := [0.1, 2]^9$ with $|S_{train}| = |S_{test}| = 100$, compute a reduced basis of size 40 using your function `greedy_RB` and the data `Sheet5_Data.mat`. Using this reduced basis, compute the test error with your function `max_test_estimator` and plot both errors over the size of the reduced basis. Interpret your results.

Explanation of terms

- For **written Exercises** hand in your elaboration/solution after one of the lectures or via Email to Dominik Garmatter until **06.01.2020**.

- For **programming Exercises** send your **commented** solution in `MATLAB`-Code via E-mail to Dominik Garmatter. Please, start the subject of your E-mail with **"MOR_201920_5:"**.

- For **voting Exercises** no written solution is requested. The solution for these exercises will be discussed/presented during the exercise course.

- All exercises of Exercise sheet 5 will be discussed during the exercise on 07.01.2020.