TECHNISCHE UNIVERSITÄT
CHEMNITZ

Exercise sheet 1 – released on 15.10.2019

Exercise 1.1 (voting Exercise)
Let $\Omega := [0,1] \subset \mathbb{R}$. Show that:

(a) the space $X := C^0(\Omega)$ with $\langle f, g \rangle_X := \int_0^1 f(x)g(x)\,\mathrm{d}x$ for $f,\ g \in X$ is not a real-valued Hilbert space.

(b) the space $X := H^1(\Omega)$ with $\langle f, g \rangle_X := \int_0^1 \frac{\partial}{\partial x} f(x) \frac{\partial}{\partial x} g(x)\,\mathrm{d}x$ for $f,\ g \in X$ is not a real-valued Hilbert space.

Exercise 1.2 (written Exercise)[6 Points]
Prove Theorem 1.7 by following these steps.

(a) Use without proof that for $x \in X$ there exists a minimizing sequence $\{y_k\}_{k\in\mathbb{N}} \subset Y$ satisfying

$$\lim_{k\to\infty} \|x - y_k\|_X = \inf_{y\in Y} \|x - y\|_X .$$

Use the Parallelogram law to prove that this sequence converges and $\tilde{y} := \lim_{k\to\infty} y_k \in Y$. Conclude that $\|x - \tilde{y}\|_X = \inf_{y\in Y} \|x - y\|_X$ and show that $\tilde{y}$ is unique. Define $Px := \tilde{y}$.

(b) Show that for $x \in X$

$$\|x - Px\|_X = \inf_{y\in Y} \|x - y\|_X \quad \Leftrightarrow \quad \langle x - Px, y \rangle_X = 0, \quad \forall y \in Y.$$

(c) Using this equivalent characterization to show that P is linear.

(d) Show that for $x \in X$

$$\langle x - Px, y \rangle_X = 0, \quad \forall y \in Y \quad \Leftrightarrow \quad Px = \sum_{i=1}^{n} \langle x, \varphi_i \rangle_X \varphi_i.$$

Exercise 1.3 (voting Exercise)
Let $X = \mathbb{R}^n$, $A \in \mathbb{R}^{n\times n}$ and define the bilinear form $a : X \times X \to \mathbb{R}$ via $a(u,v) = \langle Au, v \rangle_X$. We denote with

$$\alpha := \inf_{u\in X\setminus\{0\}} \frac{a(u,u)}{\|u\|_X^2}$$

the *coercivity constant* of a and call a *coercive* if $\alpha > 0$. Show that:

(a) $a(u,u) = a_s(u,u) = \langle A_S u, u \rangle_X$, where a_s is defined in Definition 1.11 and $A_S := \frac{1}{2}(A + A^\intercal)$.

(b) the coercivity constant of a is $\min\{\lambda_i\}_{i=1}^n$, where $\{\lambda_i\}_{i=1}^n$ are the eigenvalues of A_S.

(c) a is coercive $\Leftrightarrow$ A_S is positive definite.

Explanation of terms

- For **written Exercises** hand in your elaboration/solution after one of the lectures or via Email to Dominik Garmatter until **28.10.2019**.
- For **programming Exercises** send your **commented** solution in `MATLAB`-Code via E-mail to Dominik Garmatter. Please, start the subject of your E-mail with **"MOR_201920_1:"**.
- For **voting Exercises** no written solution is requested. The solution for these exercises will be discussed/presented during the exercise course.
- All exercises of Exercise sheet 1 will be discussed during the exercise on 29.10.2019.