Exact L_2 Marcinkiewicz-Zygmund inequalities

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Mathematik!
TU Chemnitz

L_2 Marcinkiewicz-Zygmund (MZ) inequalities

Definition

[Marcinkiewicz, Zygmund 1937]

Let
- D be a domain,
- μ a measure on D,
- $L_2 = L_2(D, \mu)$ the space of square-integrable functions, and
- V an m-dimensional function space.

Points $\boldsymbol{x}^1, \dots, \boldsymbol{x}^n \in D$ and weights $\omega_1, \dots, \omega_n > 0$ fulfill an L_2 **MZ inequality** with constants $0 < (1-\varepsilon) \le (1+\varepsilon) < \infty$, iff

$$(1-\varepsilon)\|f\|_{L_2}^2 \le \sum_{i=1}^{n} \omega_i |f(\boldsymbol{x}^i)|^2 \le (1+\varepsilon)\|f\|_{L_2}^2 \quad \text{for all} \quad f \in V\,.$$

L_2 Marcinkiewicz-Zygmund (MZ) inequalities

Goal: Construct L_2 MZ inequalities with

- small number of points n compared to $\dim(V) = m$
- close constants, ideally $\varepsilon = 0$

Some existing work: [Mhaskar, Narcowich, Ward '01], [Nuyens, Cools '06], [Keiner, Kunis, Potts '07], [Filbir, Mhaskar '11], [Müller-Gronbach, Novak, Ritter '12], [Kämmerer, Potts, Volkmer '15], [Temlyakov '18], [Trefethen '19], [Gröchenig '20], [Filbir, Hielscher, Jahn, T. Ullrich '24], ...

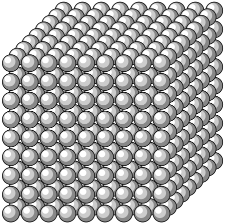
Examples of L_2 MZ inequalities

Example (equidistant points)

Let
- $L_2 = L_2(\mathbb{T}^d, \mathrm{d}\boldsymbol{x})$,
- $n \in \mathbb{N}$ be such that $\sqrt[d]{n} \in \mathbb{N}$, and
- $V = \operatorname{span}\{\exp(2\pi \mathrm{i}\langle \boldsymbol{k}, \cdot\rangle)\}_{\boldsymbol{k}\in I}$ with
- $I = \Big\{ -\frac{\sqrt[d]{n}}{2}, \dots, \frac{\sqrt[d]{n}}{2} - 1 \Big\}^d$.

Then $\boldsymbol{X} = \Big\{ \frac{1}{\sqrt[d]{n}} \boldsymbol{i} : \boldsymbol{i} \in \{1, \dots, \sqrt[d]{n}\}^d \Big\}$ fulfill

$$\|f\|_{L_2}^2 = \frac{1}{n} \sum_{\boldsymbol{x}\in\boldsymbol{X}} |f(\boldsymbol{x})|^2 \quad \text{for all} \quad f \in V \, .$$



point grid in $d = 3$

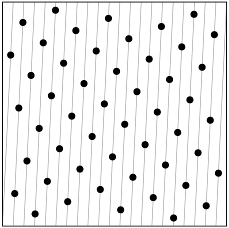
Examples of L_2 MZ inequalities

Example (rank-1 lattices) [Kämmerer, Potts, Volkmer 2015]

Let
- $L_2 = L_2(\mathbb{T}^d, \mathrm{d}\boldsymbol{x})$,
- $V = \operatorname{span}\{\exp(2\pi \mathrm{i}\langle \boldsymbol{k}, \cdot\rangle)\}_{\boldsymbol{k}\in I}$ with
- $I \subset [-\frac{|I|}{2}, \frac{|I|}{2}]^d \subset \mathbb{Z}^d$ symmetric.

Then, for some $|I| \le n \le |I|^2$, there exists a rank-1 lattice $\boldsymbol{X} = \{\frac{i}{n}\boldsymbol{z} \bmod \mathbb{1}, i = 0, \ldots, n-1\}$ such that

$$\|f\|_{L_2}^2 = \frac{1}{n} \sum_{\boldsymbol{x}\in\boldsymbol{X}} |f(\boldsymbol{x})|^2 \quad \text{for all} \quad f \in V\,.$$



$\boldsymbol{z} = (1, 21)^\top, n = 55$

Examples of L_2 MZ inequalities

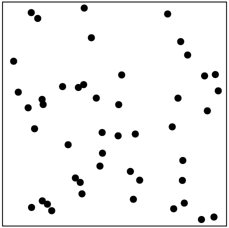
Example (random points)

[Tropp 2010][Cohen, Davenport, Leviatan 2017]

Let
- $V = \operatorname{span}\{\exp(2\pi \mathrm{i}\langle \boldsymbol{k}, \cdot \rangle)\}_{\boldsymbol{k}\in I}$ with $I \subset \mathbb{Z}^d$
- $\varepsilon > 0$, $$n \ge \frac{6}{\varepsilon^2}|I| \log |I|\,,$$
- $\boldsymbol{x}^1, \dots, \boldsymbol{x}^n$ drawn uniformly random.

Then with high probability it holds for all $f \in V$

$$(1-\varepsilon)\|f\|_{L_2}^2 \le \frac{1}{n}\sum_{i=1}^{n} |f(\boldsymbol{x}^i)|^2 \le (1+\varepsilon)\|f\|_{L_2}^2\,.$$



random points

Examples of L_2 MZ inequalities

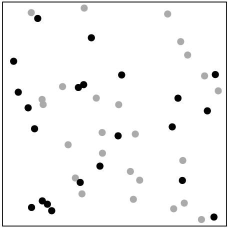
Example (subsampled points) [B., Schäfer, T. Ullrich 2023]

Let
- $V = \operatorname{span}\{\exp(2\pi \mathrm{i}\langle \boldsymbol{k}, \cdot\rangle)\}_{\boldsymbol{k}\in I}$ with $I \subset \mathbb{Z}^d$,
- $b > 1$, and $n = \lceil b|I| \rceil$.

Then there exist $\boldsymbol{x}^1, \dots, \boldsymbol{x}^n$ and $\omega_1, \dots, \omega_n > 0$ with

$$\|f\|_{L_2}^2 \le \sum_{i=1}^{n} \omega_i |f(\boldsymbol{x}^i)|^2 \le \left(\frac{\sqrt{b}+1}{\sqrt{b}-1}\right)^2 \|f\|_{L_2}^2$$

for all $f \in V$.



subsampled points

Q: How to construct general exact L_2-MZ inequalities?

Exact L_2 MZ inequalities

Theorem (connection to matrices)

Let
- $V = \mathrm{span}\{\exp(2\pi \mathrm{i}\langle \boldsymbol{k}, \cdot\rangle)\}_{\boldsymbol{k}\in I}$ with $I \subset \mathbb{Z}^d$,
- $\boldsymbol{x}^1, \dots, \boldsymbol{x}^n \in \mathbb{T}^d$, $\omega_1, \dots, \omega_n > 0$, and
- $\boldsymbol{F}(\boldsymbol{x}) = (\exp(2\pi \mathrm{i}\langle \boldsymbol{k}, \boldsymbol{x}\rangle))_{\boldsymbol{k}\in I} \in \mathbb{C}^{|I|}$.

Then

$$(1-\varepsilon)\|f\|_{L_2}^2 \le \sum_{i=1}^{n} \omega_i |f(\boldsymbol{x}^i)|^2 \le (1+\varepsilon)\|f\|_{L_2}^2 \quad \text{for all} \quad f \in V$$

iff

$$\Big\| \sum_{i=1}^{n} \omega_i \boldsymbol{F}(\boldsymbol{x}^i)\boldsymbol{F}(\boldsymbol{x}^i)^* - \boldsymbol{I} \Big\|_{2\to 2} \le \varepsilon \,.$$

Exact L_2 MZ inequalities

Problem of finding exact L_2 MZ inequalities in matrix language

Define
$$\mathcal{H} := \Big\{ \boldsymbol{H}(\boldsymbol{x}) = \boldsymbol{F}(\boldsymbol{x})\boldsymbol{F}(\boldsymbol{x})^* : \boldsymbol{x} \in \mathbb{T}^d \Big\} \subset \mathbb{C}^{m\times m} \,.$$

Then the set of L_2 MZ inequalities with weights adding up to one correspond to

$$\operatorname{conv} \mathcal{H} = \Big\{ \sum_{i=1}^{n} \omega_i \boldsymbol{F}(\boldsymbol{x}^i)\boldsymbol{F}(\boldsymbol{x}^i)^* : \boldsymbol{x}^1, \dots, \boldsymbol{x}^n \in \mathbb{T}^d, \omega_1 + \dots + \omega_n = 1, n \in \mathbb{N} \Big\} .$$

Goal: find zero of

$$\gamma\colon \operatorname{conv} \mathcal{H} \to [0,\infty),\ \boldsymbol{H} \mapsto \|\boldsymbol{H} - \boldsymbol{I}\|_{2\to 2} \,.$$

Exact L_2 MZ inequalities

Idea:

1. random L_2 MZ inequalities give
$$\forall \varepsilon > 0 \; \exists \boldsymbol{H}_\varepsilon \in \operatorname{conv} \mathcal{H} : \gamma(\boldsymbol{H}_\varepsilon) \le \varepsilon$$
2. continuity of γ and compactness of $\operatorname{conv} \mathcal{H}$ give
$$\exists \boldsymbol{H}^\star \in \operatorname{conv} \mathcal{H} : \gamma(\boldsymbol{H}^\star) = 0$$
3. Carathéodory gives
$$\boldsymbol{H}^\star = \sum_{i=1}^{m^2+1} \omega_i \boldsymbol{F}(\boldsymbol{x}^i) \boldsymbol{F}(\boldsymbol{x}^i)^*$$

Exact L_2 MZ inequalities

Theorem (exact MZ on $\mathbb{T}^d$) [B., Pozharska, Schäfer, T. Ullrich WIP]

Let
- $L_2 = L_2(\mathbb{T}^d, \mathrm{d}\boldsymbol{x})$,
- $V = \operatorname{span}\{\exp(2\pi \mathrm{i}\langle \boldsymbol{k}, \cdot\rangle)\}_{\boldsymbol{k}\in I}$ with $I \subset \mathbb{Z}^d$,

Then there exist $\boldsymbol{x}^1, \dots, \boldsymbol{x}^n \in \mathbb{T}^d$ and $\omega_1 + \dots + \omega_n = 1$ with $n \le |I|^2$ such that

$$\|f\|_{L_2}^2 = \sum_{i=1}^{n} \omega_i |f(\boldsymbol{x}^i)|^2 \quad \text{for all} \quad f \in V\,.$$

Exact L_2 MZ inequalities

Theorem (exact MZ on D) [B., Pozharska, Schäfer, T. Ullrich WIP]

Let
- D be a compact topological space,
- μ a probability measure,
- $V \subset C(D)$ with $\dim(V) = m$, and
- $n = \dim(\operatorname{span}\{f \cdot \overline{g} : f, g \in V\}) + 1 \leq m^2 + 1$.

Then there exist $\boldsymbol{x}^1, \ldots, \boldsymbol{x}^n \in D$ and $\omega_1 + \cdots + \omega_n = 1$ such that

$$\|f\|_{L_2}^2 = \sum_{i=1}^{n} \omega_i |f(\boldsymbol{x}^i)|^2 \quad \text{for all} \quad f \in V \, .$$

Exact L_2 MZ inequalities

Theorem (rank-1 lattices) [Kämmerer, Potts, Volkmer 2015]

Let
- $L_2 = L_2(\mathbb{T}^d, \mathrm{d}\boldsymbol{x})$,
- $V = \operatorname{span}\{\exp(2\pi\mathrm{i}\langle \boldsymbol{k}, \cdot\rangle)\}_{\boldsymbol{k}\in I}$ with
- $I \subset [-\frac{|I|}{2}, \frac{|I|}{2}]^d \subset \mathbb{Z}^d$ symmetric.

Then, for some $|I| \le n \le |I|^2$, there exists $\boldsymbol{X} = \{\frac{i}{n}\boldsymbol{z} \bmod \mathbb{1}, i = 0, \dots, n-1\}$, a rank-1 lattice, such that

$$\|f\|_{L_2}^2 = \frac{1}{n}\sum_{\boldsymbol{x}\in\boldsymbol{X}} |f(\boldsymbol{x})|^2 \quad \text{for all} \quad f \in V\,.$$

Exact L_2 MZ inequalities

Theorem (rank-1 lattices) [Kämmerer, Potts, Volkmer 2015]

Let
- $L_2 = L_2(\mathbb{T}^d, \mathrm{d}\boldsymbol{x})$,
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$$\|f\|_{L_2}^2 = \frac{1}{n}\sum_{\boldsymbol{x}\in\boldsymbol{X}} |f(\boldsymbol{x})|^2 \quad \text{for all} \quad f \in V\,.$$

[B., Pozharska, Schäfer, T. Ullrich WIP]: There exist points $\boldsymbol{x}^1, \dots, \boldsymbol{x}^{|I|^2} \in \mathbb{T}^d$ and weights $\omega_1 + \dots + \omega_{|I|^2} = 1$ such that

$$\|f\|_{L_2}^2 = \sum_{i=1}^{|I|^2} \omega_i |f(\boldsymbol{x}^i)|^2 \quad \text{for all} \quad f \in V\,.$$

Exact L_2 MZ inequalities

Theorem (Tchakaloff) [Putinar 1997]

Let
- D compact topological space,
- μ probability Borel measure,
- real-valued $V \subset C(D)$ with $\dim(V) = m$.

Then there are points $\{\boldsymbol{x}^1, \ldots, \boldsymbol{x}^{m+1}\} \subset D$ and $\omega_1 + \cdots + \omega_{m+1} = 1$ such that

$$\int_D f(x) \, \mathrm{d}\mu(x) = \sum_{i=1}^{m+1} \omega_i f(\boldsymbol{x}^i) \quad \text{for all} \quad f \in V\,.$$

Exact L_2 MZ inequalities

Theorem (Tchakaloff) [Putinar 1997]

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[B., Pozharska, Schäfer, T. Ullrich WIP]: able to prove the same and extension to complex case with $n = 2(m+1)$ points.

Exact L_2 MZ inequalities

Theorem (spherical designs) [Bondarenko, Radchenko, and Viazovska 2011]

Let
- $D = \mathbb{S}^d = \{x \in \mathbb{R}^{d+1} : \|\boldsymbol{x}\| = 1\}$, μ the surface measure,
- $V = \operatorname{span}\{\boldsymbol{x}^{\boldsymbol{k}} = x_1^{k_1} \cdots x_d^{k_d} : \|\boldsymbol{k}\|_1 \le l\}$ with $\dim(V) \sim l^d$.

Then there are $\boldsymbol{x}^1, \dots, \boldsymbol{x}^n \in \mathbb{S}^d$ with $n \le C_d l^d$, $C_d > 0$ such that

$$\int_{\mathbb{S}^d} f(\boldsymbol{x}) \, \mathrm{d}\mu(\boldsymbol{x}) = \frac{1}{n} \sum_{i=1}^{n} f(\boldsymbol{x}^i) \quad \text{for all} \quad f \in V\,.$$

Exact L_2 MZ inequalities

Theorem (spherical designs) [Bondarenko, Radchenko, and Viazovska 2011]

Let
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Then there are $\boldsymbol{x}^1, \dots, \boldsymbol{x}^n \in \mathbb{S}^d$ with $n \le C_d l^d$, $C_d > 0$ such that

$$\int_{\mathbb{S}^d} f(\boldsymbol{x}) \, \mathrm{d}\mu(\boldsymbol{x}) = \frac{1}{n} \sum_{i=1}^{n} f(\boldsymbol{x}^i) \quad \text{for all} \quad f \in V \,.$$

[B., Pozharska, Schäfer, T. Ullrich WIP]: For $m \ge d$ there exist points $\boldsymbol{x}^1, \dots, \boldsymbol{x}^n \in \mathbb{S}^d$ and weights $\omega_1 + \cdots + \omega_n = 1$ with $n \le 6(\frac{6}{d})^d l^d$ such that

$$\int_{\mathbb{S}^d} f(\boldsymbol{x}) \, \mathrm{d}\mu(\boldsymbol{x}) = \sum_{i=1}^{n} \omega_i f(\boldsymbol{x}^i) \quad \text{for all} \quad f \in V \,.$$

Connection to frames and D-optimal designs

Definition

Let
- D be a domain and
- μ a measure on D.

A map $\Psi(x)\colon D \to \mathbb{C}^m$ is a **(continuous) frame** with bounds $0 < A \le B < \infty$, iff

$$A\|\boldsymbol{a}\|_2^2 \le \int_D |\langle \boldsymbol{a}, \Psi(x)\rangle|^2 \,\mathrm{d}\mu(x) \le B\|\boldsymbol{a}\|_2^2 \quad \text{for all} \quad \boldsymbol{a} \in \mathbb{C}^m\,.$$

Connection to frames and D-optimal designs

Definition

Further,

- $\Psi(x)\colon D \to \mathbb{C}^m$ is called **Parseval**, iff $A = B = 1$;
- $\Psi(x)\colon D \to \mathbb{C}^m$ is called **scalable**, iff there exists a Borel measure ν such that

$$\|\boldsymbol{a}\|_2^2 = \int_D |\langle \boldsymbol{a}, \Psi(x)\rangle|^2 \,\mathrm{d}\nu(x) \quad \text{for all} \quad \boldsymbol{a} \in \mathbb{C}^m\,;$$

- $\Psi(x)\colon D \to \mathbb{C}^m$ is called **n-scalable**, iff there exist $x_1, \dots, x_n \in D$ and $\omega_1, \dots, \omega_n$ such that

$$\|\boldsymbol{a}\|_2^2 = \sum_{i=1}^{n} \omega_i |\langle \boldsymbol{a}, \Psi(x_i)\rangle|^2 \quad \text{for all} \quad \boldsymbol{a} \in \mathbb{C}^m\,.$$

Connection to frames and D-optimal designs

Theorem [B., Pozharska, Schäfer, T. Ullrich WIP]

Let
- D be a compact topological space,
- $\Psi\colon D \to \mathbb{C}^m$ continuous,
- $\mathcal{H} = \{\Psi(x)\Psi(x)^* : x \in D\} \subset \mathbb{C}^{m\times m}$, $\tilde{\mathcal{H}} = \{\frac{\Psi(x)\Psi(x)^*}{\|\Psi(x)\|_2^2} : x \in D\} \subset \mathbb{C}^{m\times m}$,
- $n = \dim(\operatorname{span}\mathcal{H}) \le m^2$.

T.f.a.e.

1. $\boldsymbol{I} \in \operatorname{conv}\mathcal{H}$;
2. Ψ is Parseval-scalable by a probability measure;
3. Ψ is $(n+1)$-Parseval-scalable with scalars adding up to one;
4. $\max\limits_{\boldsymbol{H}\in\operatorname{conv}\tilde{\mathcal{H}}} \det(\boldsymbol{H}) = 1$.