

On the reconstruction of functions from values at subsampled quadrature points

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Workshop and Summer School on Applied Analysis

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Mathematik!
TU Chemnitz

Function recovery from samples

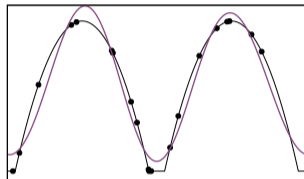
Given:

- $L_2 = L_2(D, \nu)$
- reproducing kernel Hilbert space $H(K) \hookrightarrow L_2$

Goal:

- find good points $\mathbf{X} = \{\mathbf{x}^1, \dots, \mathbf{x}^n\} \subset D$ and a sampling recovery operator $S_{\mathbf{X}}: H(K) \rightarrow L_2$ with small **worst-case error**

$$\sup_{\|f\|_{H(K)} \leq 1} \|f - S_{\mathbf{X}} f\|_{L_2}$$



$$\begin{array}{ll} f(x) & \text{—} \\ (x^i, f(x^i)) & \bullet \\ (S_{\mathbf{X}} f)(x) & \text{—} \end{array}$$

Least squares approximation

- let $\eta_0, \dots, \eta_{m-1} \in L_2$ with $n \geq m$
- ansatz $f(\mathbf{x}) = \sum_{k=0}^{m-1} c_k \eta_k(\mathbf{x})$ with $\mathbf{c} = (c_0, \dots, c_{m-1})^\top$ solving

$$\begin{pmatrix} \eta_0(\mathbf{x}^1) & \cdots & \eta_{m-1}(\mathbf{x}^1) \\ \vdots & \ddots & \vdots \\ \eta_0(\mathbf{x}^n) & \cdots & \eta_{m-1}(\mathbf{x}^n) \end{pmatrix} \begin{pmatrix} c_0 \\ \vdots \\ c_{m-1} \end{pmatrix} = \begin{pmatrix} f(\mathbf{x}^1) \\ \vdots \\ f(\mathbf{x}^n) \end{pmatrix}$$

Least squares approximation

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for a weight matrix $\mathbf{W} = \text{diag}(\omega_1, \dots, \omega_n)$

Least squares approximation

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for a weight matrix $W = \text{diag}(\omega_1, \dots, \omega_n)$

- the solution is given by

$$\mathbf{c} = (L^* W L)^{-1} L^* W \mathbf{f} \quad \text{and} \quad (S_X f)(\mathbf{x}) = \sum_{k=0}^{m-1} c_k \eta_k(\mathbf{x})$$

Reproducing kernel Hilbert space

- assume finite trace $\int_D K(\mathbf{x}, \mathbf{x}) \, d\nu(\mathbf{x}) < \infty$
- embedding $\text{Id}_{K,\nu}: H(K) \hookrightarrow L_2$ has the representation

$$\text{Id}_{K,\nu}(f) = \sum_{k=0}^{\infty} \sigma_k \langle f, e_k \rangle_{H(K)} \eta_k$$

with $e_k = \sigma_k \eta_k$ and

- **singular values** $\sigma_0 \geq \sigma_1 \geq \dots \geq 0$
- **right singular functions** e_0, e_1, \dots forming an ONS in $H(K)$
- **left singular functions** η_0, η_1, \dots forming an ONS in L_2

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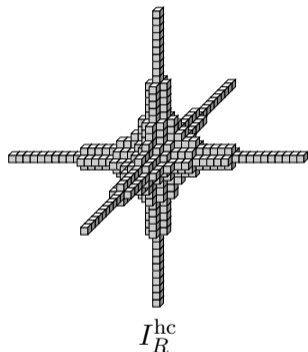
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 - **left singular functions** η_0, η_1, \dots forming an ONS in L_2
- Kolmogorov width

$$\begin{aligned} d_m(H(K)) &:= \inf_{\substack{\ell_0, \dots, \ell_{m-1}: H(K) \rightarrow \mathbb{C} \\ \varphi_0, \dots, \varphi_{m-1} \in L_2}} \sup_{\|f\|_{H(K)} \leq 1} \left\| f - \sum_{k=0}^{m-1} \ell_k(f) \varphi_k \right\|_{L_2} \\ &= \sup_{\|f\|_{H(K)} \leq 1} \left\| f - \sum_{k=0}^{m-1} \langle f, \eta_k \rangle_{L_2} \eta_k \right\|_{L_2} = \sigma_m \end{aligned}$$

- $H_{\text{mix}}^s(\mathbb{T}^d) = \{f \in L_2 : \|f\|_{H_{\text{mix}}^s} < \infty\}$ ($s > 1/2$)
with

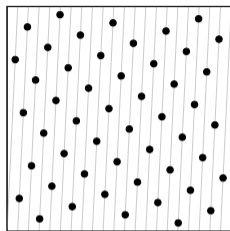
$$\langle f, g \rangle_{H_{\text{mix}}^s} := \sum_{j \in \{0, s\}^d} \langle D^{(j)} f, D^{(j)} g \rangle_{L_2}$$

- singular functions $\eta_{\mathbf{k}} = \exp(2\pi i \langle \mathbf{k}, \cdot \rangle)$
- singular values $\sigma_{\mathbf{k}} = \prod_{j=1}^d (1 + (2\pi |k_j|)^{2s})^{-1/2}$
- Kolmogorov width $d_m(H_{\text{mix}}^s) = m^{-s} (\log m)^{(d-1)s}$



Structured points: rank-1 lattice

- $D = \mathbb{T}^d$, $\{\eta_0, \dots, \eta_{m-1}\} = \{\exp(2\pi i \langle \mathbf{k}, \cdot \rangle)\}_{\mathbf{k} \in I}$
- $\mathbf{X} = \Lambda(\mathbf{z}, n) = \left\{ \frac{i}{n} \mathbf{z} \bmod \mathbf{1}, i = 0, \dots, n-1 \right\}$
- [Nuyens '07], [Kämmerer, Potts, Volkmer '15]:
 - given I , there exist algorithms that find \mathbf{z} and n such that $\mathbf{L}^* \mathbf{W} \mathbf{L} = \mathbf{I}$
 - multiplication with \mathbf{L} can be carried out with the LFFT in $\mathcal{O}(n \log n)$



$$\mathbf{z} = (1, 21)^\top, n = 55$$

Theorem

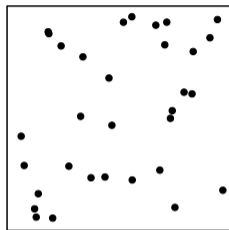
[Byrenheid, Kämmerer, T. Ullrich, Volkmer '17]

$$n^{-s} \lesssim \sup_{\|f\|_{H_{\text{mix}}^s} \leq 1} \|f - S_{\mathbf{X}} f\|_{L_2}^2 \lesssim n^{-s} (\log n)^{(d-2)s + (d-1)}$$

- $\mathbf{X} = \{\mathbf{x}^1, \dots, \mathbf{x}^n\}$ drawn randomly i.i.d. w.r.t. Lebesgue measure such that

$$n \sim m \log m$$

- no fast matrix-vector multiplication due to lack of structure



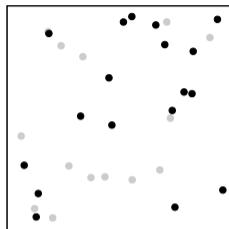
Theorem

[M. Ullrich, Krieg '19]

$$n^{-2s} (\log n)^{2(d-1)s} \lesssim \sup_{\|f\|_{H_{\text{mix}}^s} \leq 1} \|f - S_{\mathbf{X}} f\|_{L_2}^2 \lesssim n^{-2s} (\log n)^{2(d-1)s+2s}$$

Subsampled random points

- $\mathbf{X} = \{\mathbf{x}^1, \dots, \mathbf{x}^n\}$ drawn randomly i.i.d. w.r.t. Lebesgue measure such that $n \sim m \log m$
- there exist $\mathbf{X}' = \{\mathbf{x}^{i_1}, \dots, \mathbf{x}^{i_{n'}}\} \subset \mathbf{X}$ subsampled points suitable for reconstruction with
$$n' \in \mathcal{O}(|I|)$$
- no fast matrix-vector multiplication



Theorem

[Nagel, Schäfer, T. Ullrich '21], [B., Schäfer, T. Ullrich '22]

$$n^{-s}(\log n)^{2(d-1)s} \lesssim \sup_{\|f\|_{H(K)} \leq 1} \|f - S_{\mathbf{X}} f\|_{L_2}^2 \lesssim n^{-2s}(\log n)^{2(d-1)s+1}$$

Theorem

[Dolbeaut, Krieg, M. Ullrich '22]

$$\sup_{\|f\|_{H(K)} \leq 1} \|f - S_{\mathbf{X}} f\|_{L_2}^2 \sim n^{-2s}(\log n)^{2(d-1)s}$$

Theorem

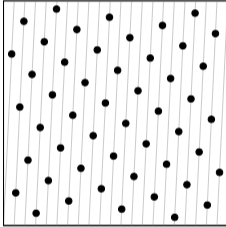
[B., Schäfer, T. Ullrich '22]

Let • $V_m \subset L_2$ m -dimensional function space and
• $1 + \frac{1}{m} < b$.

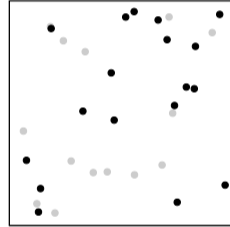
Then we can construct $\mathbf{x}^1, \dots, \mathbf{x}^n \in D$ with $n \leq \lceil bm \rceil$ such that

$$\|f\|_{L_2}^2 \leq C \frac{b^3}{(b-1)^3} \frac{1}{n} \sum_{i=1}^n |f(\mathbf{x}^i)|^2 \quad \text{for all } f \in V_m.$$

Goal: combine both advantages



structure for **fast algorithms**

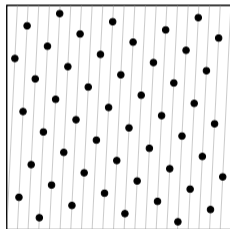


optimal rates

Approach (motivated by [Kunis, Rauhut '08])

1

\mathbf{X}_{MZ} : good
deterministic set of
 M points

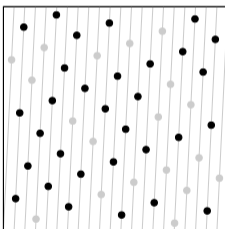


$$\mathbf{X}_{\text{MZ}} = \{\mathbf{x}^1, \dots, \mathbf{x}^M\}$$
$$\mathbf{W}_{\text{MZ}} \in [0, \infty)^{M \times M}$$

random
→
subsampling

2

\mathbf{X} : randomly
subsample n points
with $n \sim m \log m$

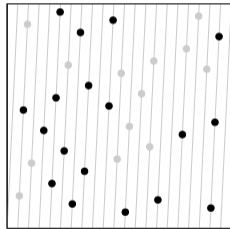


$$\mathbf{X} = \{\mathbf{x}^i\}_{i \in J}$$
$$\mathbf{W} \in [0, \infty)^{|\mathbf{X}| \times |\mathbf{X}|}$$

BSS
→
subsampling

3

\mathbf{X}' : BSS subsample
 n' points with
 $n' \sim m$



$$\mathbf{X}' = \{\mathbf{x}^i\}_{i \in J'}$$
$$\mathbf{W}' \in [0, \infty)^{|\mathbf{X}'| \times |\mathbf{X}'|}$$

- points \mathbf{X}_{MZ} and weights $\omega_1, \dots, \omega_M$ fulfill a L_2 **Marcinkiewicz-Zygmund inequality** for $V \subset L_2$, iff

$$A\|g\|_{L_2}^2 \leq \sum_{i=1}^M \omega_i |g(\mathbf{x}^i)|^2 \leq B\|g\|_{L_2}^2 \quad \text{for all } g \in V$$

- the full system matrix is well-conditioned

$$A \leq \sigma_{\min}^2(\mathbf{W}^{1/2}\mathbf{L}) \leq \sigma_{\max}^2(\mathbf{W}^{1/2}\mathbf{L}) \leq B$$

- **MZ inequalities are widely available:** [Mhaskar, Narcowich, Ward '01], [Keiner, Kunis, Potts '07], [Filbir, Mhaskar '11], [Müller-Gronbach, Novak, Ritter '12], [Temlyakov '18], [Gröchenig '20], [Filbir, Hielscher, Jahn, T. Ullrich to appear], ...

Lemma

The MZ inequality on V with $A = B$ is equivalent to the exact integration

$$\sum_{i=1}^M \omega_i g(\mathbf{x}^i) \overline{h(\mathbf{x}^i)} = \int_D g(\mathbf{x}) \overline{h(\mathbf{x})} d\nu(\mathbf{x}) \quad \text{for all } g, h \in V.$$

- **exact integration is widely available:** [Nuyens, Cools '06], [Kämmerer, Potts, Volkmer '15], [Trefethen '19], ...

Incorporate fast algorithms

- big point set $\mathbf{X}_{MZ} = \{\mathbf{x}^1, \dots, \mathbf{x}^M\}$ with $\mathcal{O}(M \log M)$ algorithm
- small point set $\mathbf{X} = \{\mathbf{x}^{j_1}, \dots, \mathbf{x}^{j_n}\} \subset \mathbf{X}_{MZ}$
- we may use the algorithm for the big point set

$$\mathbf{L}_{\mathbf{X}} = \mathbf{P} \mathbf{L}_{\mathbf{X}_{MZ}} \quad \text{where} \quad \mathbf{P} = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 & 0 \\ \vdots & & & & & \vdots & \\ 0 & 0 & 0 & \dots & 0 & 1 & 0 \end{pmatrix} \begin{matrix} 1 \\ 2 \\ \vdots \\ n \end{matrix}$$

$j_1 \qquad j_2 \qquad \qquad j_n$

- with $n \sim m \log m$ and $M \sim m^2$ we obtain the same complexity as for naive matrix-vector multiplication

$$\mathcal{O}(mn) = \mathcal{O}(M \log M)$$

Assumptions 1 of 2

Let

- $H(K)$ separable RKHS with $\sup_{\mathbf{x} \in D} K(\mathbf{x}, \mathbf{x}) < \infty$ and $\int_D K(\mathbf{x}, \mathbf{x}) d\mathbf{x} < \infty$,
- σ_k and η_k singular values and functions of $\text{Id}_{K,\nu}: H(K) \hookrightarrow L_2(\nu)$
- $I = \{0, \dots, m-1\} \subset I_M$
- \mathbf{X}_{MZ} points and \mathbf{W}_{MZ} weights fulfilling an L_2 -Marcinkiewicz-Zygmund inequality with constants A and B for $V = \text{span}\{\eta_k\}_{k \in I_M}$,

Assumptions 2 of 2

- $n := \left\lceil \frac{72B}{A} |I| \log |I| \right\rceil$,
- $\mathbf{X} = \{\mathbf{x}^i\}_{i \in J}$, $|J| = n$ points drawn i.i.d. from \mathbf{X}_{MZ} w.r.t. the **discrete density weights** $\varrho_i = \omega_i \varrho(\mathbf{x}^i)$ with

$$\varrho(\mathbf{x}^i) = \frac{\omega_i \sum_{k \in I} |\eta_k(\mathbf{x}^i)|^2}{3 \sum_{j=1}^M \omega_j \sum_{k \in I} |\eta_k(\mathbf{x}^j)|^2} + \frac{\omega_i \sum_{k \in I_{MZ} \setminus I} |e_k(\mathbf{x}^i)|^2}{3 \sum_{j=1}^M \omega_j \sum_{k \in I_{MZ} \setminus I} |e_k(\mathbf{x}^j)|^2} + \frac{\omega_i}{3}$$

- $b > 1 + \frac{1}{|I|}$.

- uses spectral properties of the embedding
- discrete version of [M. Ullrich, Krieg '19], [Kämmerer, T. Ullrich, Volkmer '19]
- $\varrho(\mathbf{x}^i)$ neglectable for BOS, i.e., $\sup_{x \in D} |\eta_k(x)| \leq B$ for all k

Theorem

[B, Kämmerer, Potts, T. Ullrich '22]

Given the assumptions we construct $\mathbf{X}' \subset \mathbf{X}_{\text{MZ}}$ with $|\mathbf{X}'| \leq \lceil b|I| \rceil$ such that

$$\sup_{\|f\|_{H(K)} \leq 1} \|f - S_I^{\mathbf{X}'} f\|_{L_2}^2 \leq C_{A,B,b} \log |I| \left(\sup_{k \notin I} \sigma_k^2 + \frac{1}{|I|} \sum_{k \in I_{\text{MZ}} \setminus I} \sigma_k^2 \right) + \sup_{\|f\|_{H(K)} \leq 1} \|f - P_{I_{\text{MZ}}} f\|_{\ell_\infty(D)}^2$$

with probability larger than $1 - 4/|I|$ and $C_{A,B,b} = 79\,616 \left(\frac{B}{A}\right)^2 \frac{(b+1)^2}{(b-1)^3}$.

Theorem for rank-1 lattices

[B, Kämmerer, Potts, T. Ullrich '22]

Let • $s > 1/2$,

- $I \subset I_{\text{MZ}} \subset \mathbb{Z}^d$ hyperbolic cross frequency index sets, $|I| \geq 3$,
- \mathbf{X}_{MZ} reconstructing rank-1 lattice for I_{MZ} with M points,
- $b > 1 + \frac{1}{|I|}$.

Then we construct $\mathbf{X}' \subset \mathbf{X}_{\text{MZ}}$ with $|\mathbf{X}'| = n \leq \lceil b|I| \rceil$ such that

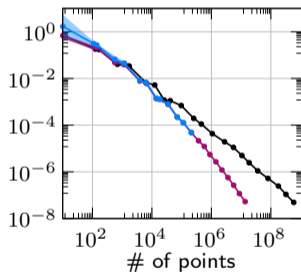
$$\sup_{\|f\|_{H_{\text{mix}}^s} \leq 1} \|f - S_I^{\mathbf{X}'} f\|_{L_2}^2 \leq C_{d,s,b} \log n \left(n^{-2s} (\log n)^{2(d-1)s} + M^{-s+1/2} (\log M)^{(d-1)(s+1)-s} \right)$$

with probability $1 - 4/|I|$.

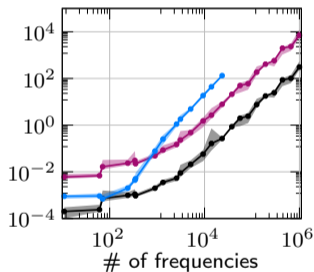
Numerical experiment for random subsampling

- $d = 5$, f tensorized bump function in $H_{\text{mix}}^s(\mathbb{T}^5)$ for $s < 3/2$
- initial \mathbf{X}_{MZ} : fast probabilistic CBC rank-1 lattice construction [Kämmerer '20]

L_2 -error



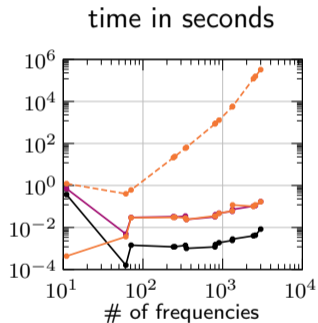
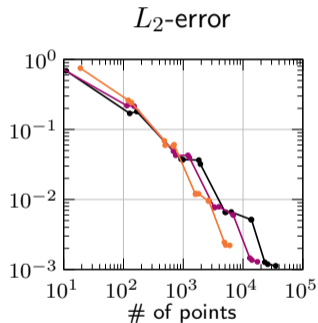
time in seconds



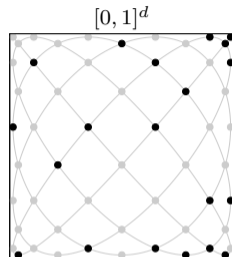
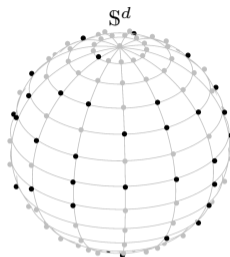
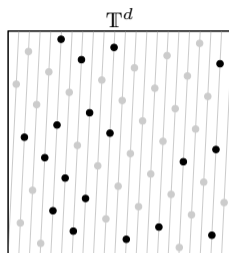
black: full rank-1 lattice
azure: continuously random
magenta: randomly subsampled

Numerical experiment for random + BSS subsampling

- $d = 5$, f tensorized bump function in $H_{\text{mix}}^s(\mathbb{T}^5)$ for $s < 3/2$
- initial \mathbf{X}_{MZ} : fast probabilistic CBC rank-1 lattice construction [Kämmerer '20]



black: full rank-1 lattice
magenta: randomly subsampled
orange: BSS subsampled



- we proposed **random + BSS** subsampling procedure for **MZ inequalities**
 - achieving the near optimal convergence rates ✓
 - making it possible to utilize fast algorithms ✓
- [B., Kämmerer, Potts, T. Ullrich '22] on **arXiv:2208.13597** “On the reconstruction of functions from values at subsampled quadrature points”