UNIVERSITY OF TECHNOLOGY
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CHEMNITZ

Nonlinear function approximation using different strategies

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Aim

High-dimensional setting

Unknown function: $f \in L_2(\mathcal{D}, \mu)$ is a d-dimensional function with basis expansion

$$f(\boldsymbol{x}) := \sum_{\boldsymbol{k}\in\mathbb{N}^d} c_{\boldsymbol{k}} \Phi_{\boldsymbol{k}}(\boldsymbol{x}).$$

Bounded orthonormal product basis: The domain $\mathcal{D}$, measure μ and basis $\Phi_{\boldsymbol{k}}$ of $L_2(\mathcal{D}, \mu)$ are of tensor-product structure and

$$\max_{\boldsymbol{k}\in\mathbb{N}^d} \|\Phi_{\boldsymbol{k}}\|_\infty =: B < \infty.$$

Black-box sampling: We can access the sample $f(\boldsymbol{x})$ for any $\boldsymbol{x}$ at any time.

Truncation and approximation

Desired approximation: We want to compute an approximate truncated basis expansion

$$S_I^{\mathcal{A}} f(\boldsymbol{x}) := \sum_{\boldsymbol{k}\in I} \hat{f}_{\boldsymbol{k}}\, \Phi_{\boldsymbol{k}}(\boldsymbol{x}),$$

where $\hat{f}_{\boldsymbol{k}} \approx c_{\boldsymbol{k}}$ and I is s-sparse, i.e., $|I| = s$.

Best s-term approximation:

"best" index set I $\iff$ the $c_{\boldsymbol{k}}$ with $\boldsymbol{k} \in I$ have the largest absolute values

Error:

$$\|S_I^{\mathcal{A}} f - f\| \leq \underbrace{\|S_I^{\mathcal{A}} f - S_I f\|}_{\text{coef. approx. error}} + \underbrace{\|S_I f - f\|}_{\text{truncation error}}$$

Main challenge

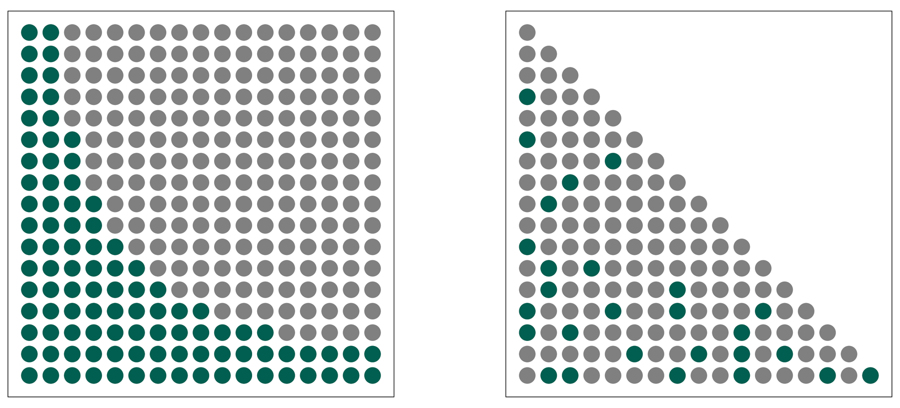
Quantities to compute:

- approx. coefficients $\hat{f}_{\boldsymbol{k}} \approx c_{\boldsymbol{k}}\ \forall \boldsymbol{k} \in I$
- suitable sparse index set $I \subset \mathbb{N}^d$

Necessary assumption:

$I \subset \mathbb{N}^d \overset{\text{assume}}{\longrightarrow} I \subset \Gamma$ with a (given) finite search space $\Gamma \subset \mathbb{N}^d$

Curse of dimensionality: Approximating all $c_{\boldsymbol{k}}$, $\boldsymbol{k} \in \Gamma$ is still computationally unfeasible.



Dimension-incremental Algorithm

Projected coefficients

Projected coefficient: For a fixed anchor $\tilde{\boldsymbol{x}}$, the projected coefficients for $\boldsymbol{k} \in K \subset \mathbb{N}^t$ are

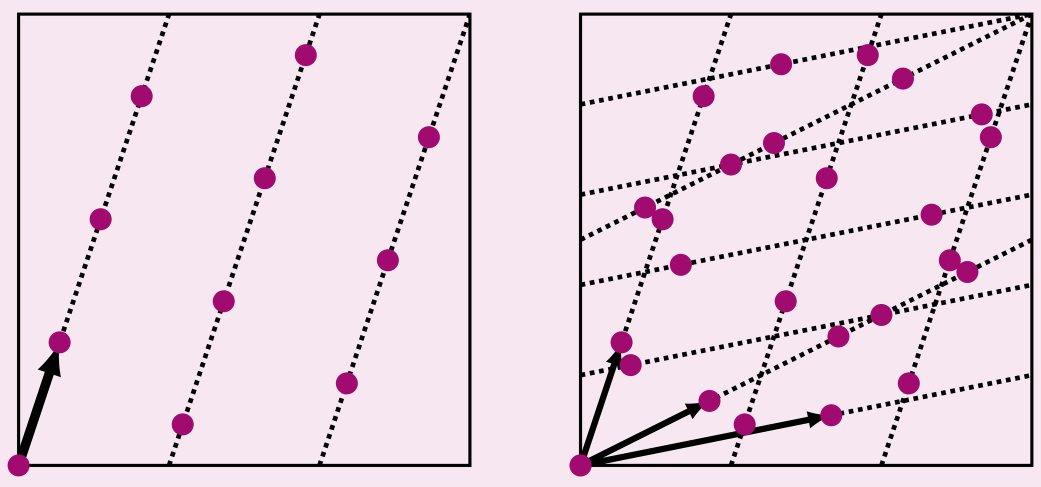
$$c_{\{1,\dots,t\},\boldsymbol{k}}(\tilde{\boldsymbol{x}}) := \left\langle f(\cdot, \tilde{\boldsymbol{x}}), \Phi_{\{1,\dots,t\},\boldsymbol{k}}(\cdot) \right\rangle.$$

Importance of $\boldsymbol{k}$: We expect $\boldsymbol{k}$ to be important, if $\left|c_{\{1,\dots,t\},\boldsymbol{k}}(\tilde{\boldsymbol{x}})\right|$ is large, because

$$c_{\{1,\dots,t\},\boldsymbol{k}}(\tilde{\boldsymbol{x}}) = \sum_{\boldsymbol{h}\in\mathbb{N}^{d-t}} c_{(\boldsymbol{k},\boldsymbol{h})} \Phi_{\{t+1,\dots,d\},(\boldsymbol{k},\boldsymbol{h})}(\tilde{\boldsymbol{x}}).$$

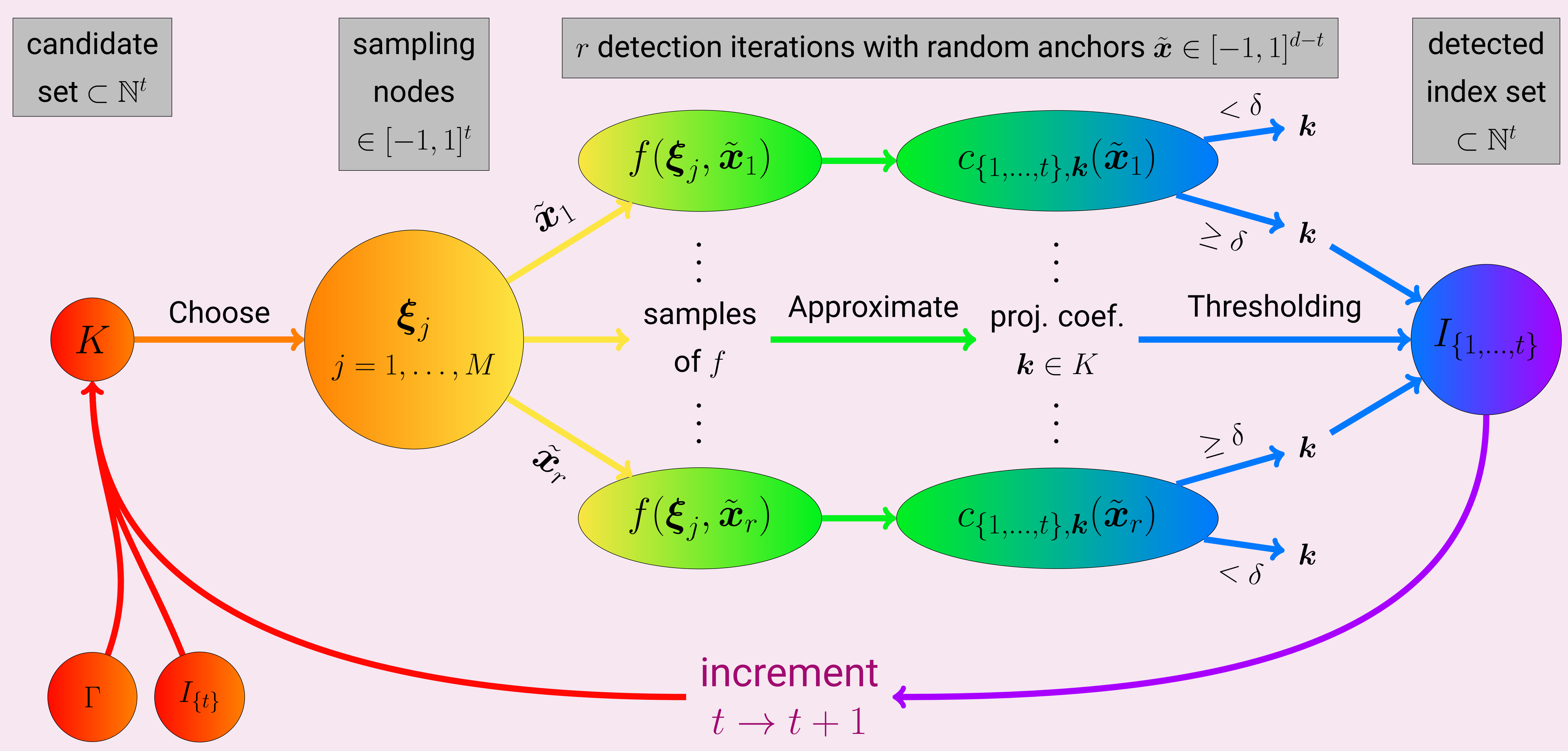
Approximation of the projected coefficients:

- Monte Carlo integration (MC, CMC)
- rank-1 lattices (R1L)
- subsampled rank-1 lattices [2] (ssR1L)
- multiple rank-1 lattices (mR1L, CmR1L)



(a) R1L (b) mR1L

The dimension-incremental algorithm (for e.g. $\mathcal{D} = [-1, 1]^d$)



Theorem[1]

Assumptions: We need mild assumptions on the parameters of the algorithm and the accuracy of the approximated projected coefficients.

Theoretical detection guarantee: Then, the output index set contains the "best" index set with high probability.

Related Work

CS approach in similar setting:
[Choi, Iwen, Krahmer '20]
[Choi, Iwen, Volkmer '21]

Polynomial approx. via LSQ and CS:
[Adcock, Brugiapaglia, Webster '22]

Sparse polynomial chaos expansions:
[Lüthen, Marelli, Sudret '21]

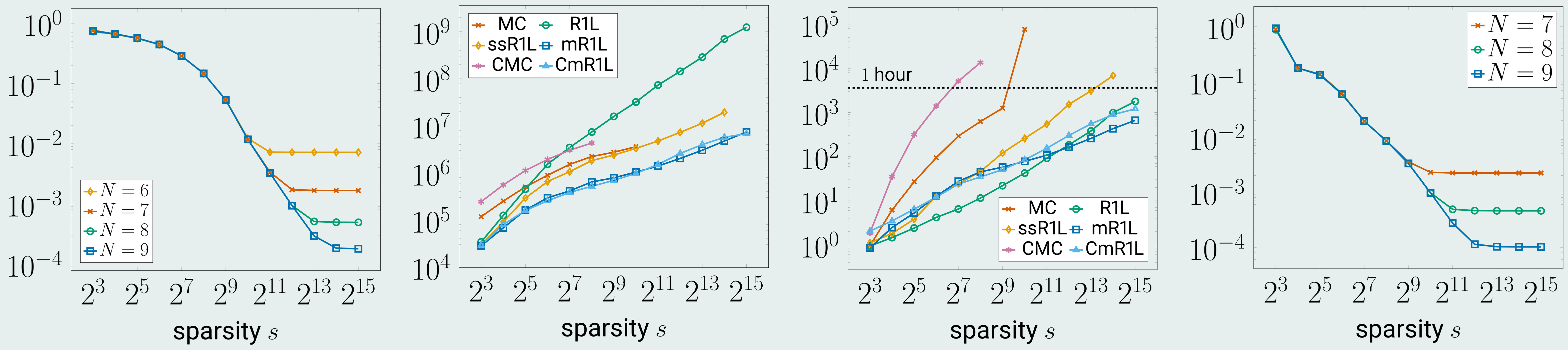
Numerics

10-dimensional periodic – Approximation of test functions – 9-dimensional non-periodic

"Fourier": Approximation of a periodic test function on $\mathbb{T}^{10}$ with the Fourier basis and the methods MC, R1L, ssR1L and mR1L.

Parameters:
s - sparsity
N - extension of search space Γ

"Chebyshev": Approximation of a non-periodic test function on $[-1,1]^9$ with the Chebyshev basis and the methods CMC and CmR1L.



(a) rel. L_2 error ("Fourier") (b) samples used ($N = 8$) (c) comp. time ($N = 8$) (d) rel. L_2 error ("Chebyshev")

[1] L. Kämmerer, D. Potts and F. Taubert. Nonlinear approximation in bounded orthonormal product bases. *Sampl. Theory Signal Process. Data Anal., 2023.*

[2] F. Bartel and F. Taubert. Nonlinear Approximation with Subsampled Rank-1 Lattices. *arXiv preprint, 2023.*

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