Nonlinear approximation with subsampled rank-1 lattices

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Approximation Theory and Beyond 2023

May 18th

Mathematik!
TU Chemnitz

Setting

Given:

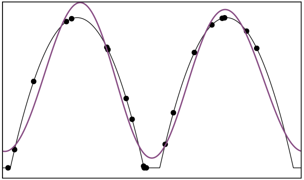
- $f\colon \mathbb{T}^d \to \mathbb{C}$

Goal: find approximation $S_{\boldsymbol{X}} f$ based on points $\boldsymbol{X} = \{\boldsymbol{x}^1, \dots, \boldsymbol{x}^n\} \subset \mathbb{T}^d$ with

- small error

$$\|f - S_{\boldsymbol{X}} f\|_{L_2}^2 = \int_{\mathbb{T}^d} |f(\boldsymbol{x}) - (S_{\boldsymbol{X}} f)(\boldsymbol{x})|^2 \,\mathrm{d}\boldsymbol{x}$$

- n small
- fast reconstruction



$f(\boldsymbol{x})$ ——
$(\boldsymbol{x}^i, f(\boldsymbol{x}^i))$ •
$(S_{\boldsymbol{X}} f)(\boldsymbol{x})$ ——

Least squares approximation

For
- $I = \{\boldsymbol{k}_1, \ldots, \boldsymbol{k}_m\} \subset \mathbb{Z}^d$ an frequency index set and
- $V = \operatorname{span}\{\exp(2\pi\mathrm{i}\langle \boldsymbol{k}, \cdot\rangle)\}_{\boldsymbol{k}\in I}$ approximation space

the **least squares approximation** is given by

$$S_{\boldsymbol{X}} f := \operatorname*{arg\,min}_{g\in V} \sum_{i=1}^{n} \Big| g(\boldsymbol{x}^i) - f(\boldsymbol{x}^i) \Big|^2 = \sum_{\boldsymbol{k}\in I} \hat{g}_{\boldsymbol{k}} \exp(2\pi\mathrm{i}\langle \boldsymbol{k}, \boldsymbol{x}\rangle)$$

with Fourier coefficients $\hat{\boldsymbol{g}} = (\boldsymbol{L}^*\boldsymbol{L})^{-1}\boldsymbol{L}^*\boldsymbol{f}$ and

$$\boldsymbol{L} = \begin{pmatrix} \exp(2\pi\mathrm{i}\langle \boldsymbol{k}_1, \boldsymbol{x}^1\rangle) & \cdots & \exp(2\pi\mathrm{i}\langle \boldsymbol{k}_m, \boldsymbol{x}^1\rangle) \\ \vdots & \ddots & \vdots \\ \exp(2\pi\mathrm{i}\langle \boldsymbol{k}_1, \boldsymbol{x}^n\rangle) & \cdots & \exp(2\pi\mathrm{i}\langle \boldsymbol{k}_m, \boldsymbol{x}^n\rangle) \end{pmatrix}.$$

Q: How to choose the points

$$\boldsymbol{X} = \left\{ \boldsymbol{x}^1, \ldots, \boldsymbol{x}^n \right\} \subset \mathbb{T}^d$$

Rank-1 lattices

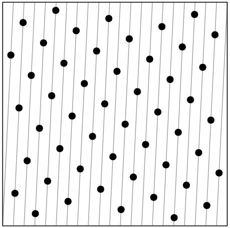
- **rank-1 lattice** [Sloan, Joe '94], [Nuyens '07], [Kämmerer, Potts, Volkmer '16]

$$\boldsymbol{X}_M := \left\{ \frac{1}{M}(i\boldsymbol{z} \bmod M\mathbb{1}) \in \mathbb{T}^d : i = 0, \dots, M-1 \right\}$$

- fast matrix-vector multiplication with FFT

- **reconstructing property** in I

$$\frac{1}{M} \sum_{i=1}^{M} \exp(2\pi \mathrm{i} \langle \boldsymbol{k} - \boldsymbol{l}, \boldsymbol{x}^i \rangle) = \delta_{\boldsymbol{l},\boldsymbol{k}} \quad \forall \boldsymbol{k}, \boldsymbol{l} \in I$$



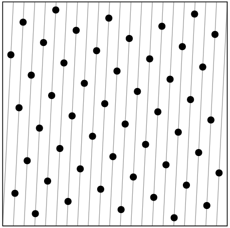
$\boldsymbol{z} = (1, 21)^{\mathsf{T}}, M = 55$

Rank-1 lattices

- for a reconstructing rank-1 lattice $\boldsymbol{X}_M$ on I, we have

$$\|f - S_{\boldsymbol{X}_M} f\|_{L_2}^2 = \|f - P_I f\|_{L_2}^2 + \sum_{\boldsymbol{k} \in I} \Big| \sum_{\substack{\boldsymbol{h} \in \mathbb{Z}^d \setminus \{\boldsymbol{0}\} \\ \langle \boldsymbol{h}, \boldsymbol{z} \rangle \equiv 0 \bmod M}} \hat{f}_{\boldsymbol{k}+\boldsymbol{h}} \Big|^2$$

- reconstruction property are approximately $|I|^2$ conditions blowing up the size M



$\boldsymbol{z} = (1, 21)^\mathsf{T}, M = 55$

Random points

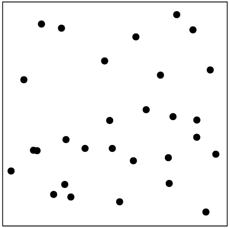
Theorem

For $I = \{\boldsymbol{k}_1, \dots, \boldsymbol{k}_m\} \subset \mathbb{Z}^d$ and i.i.d. uniform points $\boldsymbol{X} = \{\boldsymbol{x}^1, \dots, \boldsymbol{x}^n\} \subset \mathbb{T}^d$ with

$$n \geq 20 m \log m$$

we have with probability $1 - 2/n$

$$\|f - S_{\boldsymbol{X}} f\|_{L_2}^2 \leq 32 \|f - P_I f\|_{L_2}^2 \, .$$



☺ good sampling complexity

☹ no fast matrix-vector product

Randomly subsampled rank-1 lattices

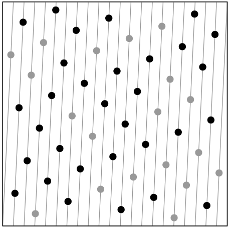
Theorem [B, Taubert '23]

Let
- $I = \{\boldsymbol{k}_1, \dots, \boldsymbol{k}_m\} \subset I_M \subset \mathbb{Z}^d$,
- $\boldsymbol{X}_M$ a reconstructing rank-1 lattice for I_M,
- $\boldsymbol{X} = \{\boldsymbol{x}^1, \dots, \boldsymbol{x}^n\} \subset \boldsymbol{X}_M$ random with

$$n \ge 24 m \log m\,.$$

Then with probability $1 - 2/n$

$$\|f - S_{\boldsymbol{X}} f\|_{L_2}^2 \le \left(3 + \sqrt{\frac{2|I_M \setminus I|}{9|I|}}\right)^2 \|f - P_I f\|_{L_2}^2 + 4\|f - P_{I_M} f\|_\infty^2\,.$$



$\boldsymbol{z} = (1, 21)^\mathsf{T}, M = 55,$
$n = 35$

☺ good sampling complexity

☺ fast matrix-vector product

Q: How to choose the frequencies

$$I = \left\{ \boldsymbol{k}_1, \ldots, \boldsymbol{k}_m \right\} \subset \mathbb{Z}^d$$

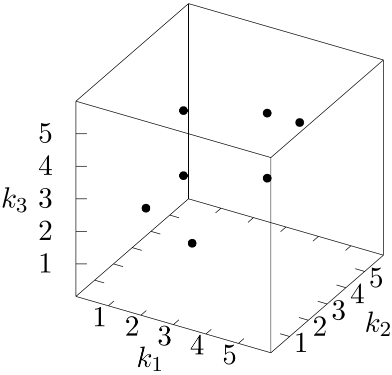
Best sparse approximation

Goal: find frequencies $I \subset \mathbb{Z}^d$ corresponding to the Fourier coefficients of largest magnitude $|\hat{f}_{\boldsymbol{k}}|$ such that

$$\|f - P_I f\|_{L_2}^2 = \sum_{\boldsymbol{k} \notin I} |\hat{f}_{\boldsymbol{k}}|^2$$

is smallest possible

Our approach: dimension-incremental SFT motivated by [Potts, Volkmer '16], [Kämmerer, Potts, Volkmer '20], [Gross, Iwen, Kämmerer, Volkmer], [Kämmerer, Potts, Taubert '22]



Projected Fourier coefficients

- **Q:** Is any frequency of the form $(7, \boldsymbol{l})$, $\boldsymbol{l} \in \mathbb{Z}^{d-1}$ important?
- define

$$\hat{f}_{\{1\},7}(\boldsymbol{\xi}) := \int_{\mathbb{T}} f(x, \boldsymbol{\xi}) \exp(-2\pi \mathrm{i} \langle 7, x \rangle) \, \mathrm{d}x$$
$$= \sum_{\boldsymbol{l} \in \mathbb{Z}^{d-1}} \hat{f}_{(7,\boldsymbol{l})} \exp(2\pi \mathrm{i} \langle \boldsymbol{l}, \boldsymbol{\xi} \rangle)$$

by Parseval's equality

$$\left\| \hat{f}_{\{1\},7} \right\|_{L_2}^2 = \sum_{\boldsymbol{l} \in \mathbb{Z}^{d-1}} \left| \hat{f}_{(7,\boldsymbol{l})} \right|^2$$

Projected Fourier coefficients

- **Q:** For $\boldsymbol{k} \in \mathbb{Z}^t$, is any frequency of the form $(\boldsymbol{k}, \boldsymbol{l})$, $\boldsymbol{l} \in \mathbb{Z}^{d-t}$ important?
- define for $\boldsymbol{k} \in \mathbb{T}^{d-t}$ the **projected Fourier coefficient**

$$\begin{aligned} \hat{f}_{\{1,\ldots,t\},\boldsymbol{k}}(\boldsymbol{\xi}) &:= \int_{\mathbb{T}^t} f(\boldsymbol{x}, \boldsymbol{\xi}) \exp(-2\pi\mathrm{i}\langle \boldsymbol{k}, \boldsymbol{x}\rangle) \,\mathrm{d}\boldsymbol{x} \\ &= \sum_{(\boldsymbol{k},\boldsymbol{l})\in\mathbb{Z}^d} \hat{f}_{(\boldsymbol{k},\boldsymbol{l})} \exp(2\pi\mathrm{i}\langle \boldsymbol{l}, \boldsymbol{\xi}\rangle) \end{aligned}$$

by Parseval's equality

$$\left\| \hat{f}_{\{1,\ldots,t\},\boldsymbol{k}} \right\|_{L_2}^2 = \sum_{\boldsymbol{l}\in\mathbb{Z}^{d-t}} \left| \hat{f}_{(\boldsymbol{k},\boldsymbol{l})} \right|^2$$

Projected Fourier coefficients

for fixed anchors $\boldsymbol{\xi}^1, \dots, \boldsymbol{\xi}^r \in \mathbb{T}^{d-t}$ we approximate

$$f(\cdot, \boldsymbol{\xi}^i) = \sum_{\boldsymbol{k} \in \mathbb{Z}^t} \hat{f}_{\{1,\dots,t\},\boldsymbol{k}}(\boldsymbol{\xi}^i) \exp(2\pi\mathrm{i}\langle \boldsymbol{k}, \cdot \rangle)$$

using subsampled rank-1 lattice $\boldsymbol{X}_M \subset \mathbb{T}^t$ in the first t components to obtain

$$\begin{gathered} \hat{g}_{\{1,\dots,t\},\boldsymbol{k}}(\boldsymbol{\xi}^1) \approx \hat{f}_{\{1,\dots,t\},\boldsymbol{k}}(\boldsymbol{\xi}^1) \\ \vdots \\ \hat{g}_{\{1,\dots,t\},\boldsymbol{k}}(\boldsymbol{\xi}^r) \approx \hat{f}_{\{1,\dots,t\},\boldsymbol{k}}(\boldsymbol{\xi}^r) \end{gathered}$$

to estimate $\|\hat{f}_{\{1,\dots,t\},\boldsymbol{k}}\|_{L_2}^2 \approx \sum_{i=1}^r |\hat{g}_{\{1,\dots,t\},\boldsymbol{k}}(\boldsymbol{\xi}^i)|^2$

Dimension-incremental SFT $d = 1$

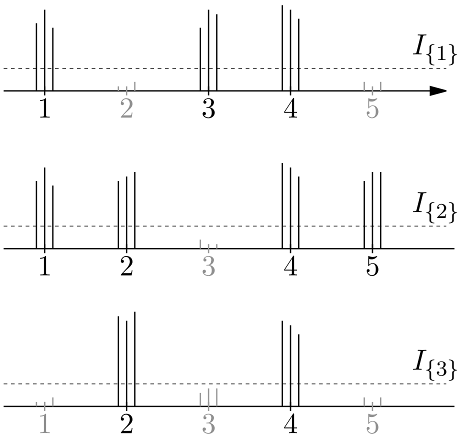
- candidate set $\Gamma = \{1, \ldots, 5\}^3$
- approximate all one-dimensional projected Fourier coefficients

$$\hat{g}_{\{1\},1}(\boldsymbol{\xi}^i), \ldots, \hat{g}_{\{1\},5}(\boldsymbol{\xi}^i)$$

$$\hat{g}_{\{2\},1}(\boldsymbol{\xi}^i), \ldots, \hat{g}_{\{2\},5}(\boldsymbol{\xi}^i)$$

$$\hat{g}_{\{3\},1}(\boldsymbol{\xi}^i), \ldots, \hat{g}_{\{3\},5}(\boldsymbol{\xi}^i)$$

- apply thresholding to obtain $I_{\{1\}}$, $I_{\{2\}}$, and $I_{\{3\}}$

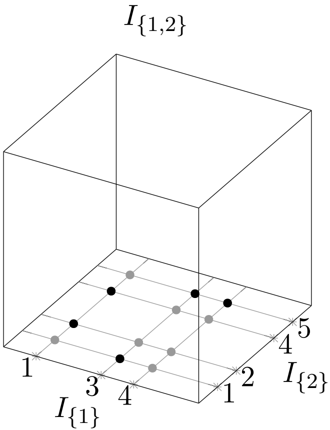


Dimension-incremental SFT $d = 2$

- approximate projected Fourier coefficients for

$$I_{\{1\}} \times I_{\{2\}} \subset \mathbb{Z}^2$$

- apply thresholding to obtain $I_{\{1,2\}}$

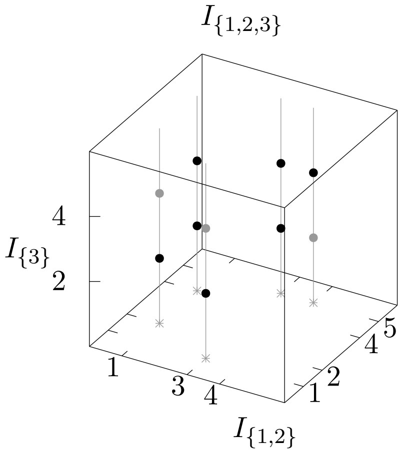


Dimension-incremental SFT $d = 3$

- approximate projected Fourier coefficients for

$$I_{\{1,2\}} \times I_{\{3\}} \subset \mathbb{Z}^3$$

- apply thresholding to obtain $I_{\{1,2,3\}}$



Theoretical detection guarantee of the dimension-incremental step

Theorem [B, Taubert '23]

Let
- $f\colon \mathbb{T}^d \to \mathbb{C}$, $\varepsilon, \delta > 0$, $I_\delta := \{\boldsymbol{k} \in \mathbb{Z}^d : |\hat{f}_{\boldsymbol{k}}| \geq \delta\}$,
- $\mathcal{P}_{\{1,\dots,t\}}(I_\delta) \subset I_{\{1,\dots,t\}} \subset I_{\{1,\dots,t\}}^M$,
- $\boldsymbol{X}^M$ reconstructing rank-1 lattice for $I_{\{1,\dots,t\}}^M$ with probability $1-\varepsilon$,
- $\boldsymbol{X} \subset \boldsymbol{X}^M$ an i.i.d. uniformly drawn subset with
$$|\boldsymbol{X}| \geq 12 |I_{\{1,\dots,t\}}| \left(\log |I_{\{1,\dots,t\}}| + \log\left(\frac{2r}{\varepsilon}\right)\right),$$
- $\boldsymbol{\xi}^1, \dots, \boldsymbol{\xi}^r \in \mathbb{T}^{d-t}$ be drawn i.i.d. uniformly random with
$$r \geq 4\left(|I_\delta| + \frac{1}{\delta^2}\left(\sum_{\boldsymbol{k}\notin I_\delta} |\hat{f}_{\boldsymbol{k}}|\right)^2\right)\left(\log |I_\delta| + \log \frac{1}{\varepsilon}\right),$$
- threshold
$$\delta' \leq \frac{\delta}{\sqrt{2}} - 4\|f - P_{I_\delta} f\|_{L_2} - 2\|f - \mathcal{P}_{I_{\{1,\dots,t\}}^M \times \mathbb{T}^{d-t}} f\|_\infty .$$

Theoretical detection guarantee of the dimension-incremental step

Theorem [B, Taubert '23]

Then we detect all important frequencies in dimension t with probability $1-3\varepsilon$, i.e.,

$$\max_{i=1,\dots,r} |\hat{g}_{\{1,\dots,t\},\boldsymbol{k}}(\boldsymbol{\xi}^i)| \geq \delta' \quad \forall \boldsymbol{k} \in \mathcal{P}_{\{1,\dots,t\}}(I_\delta)\,.$$

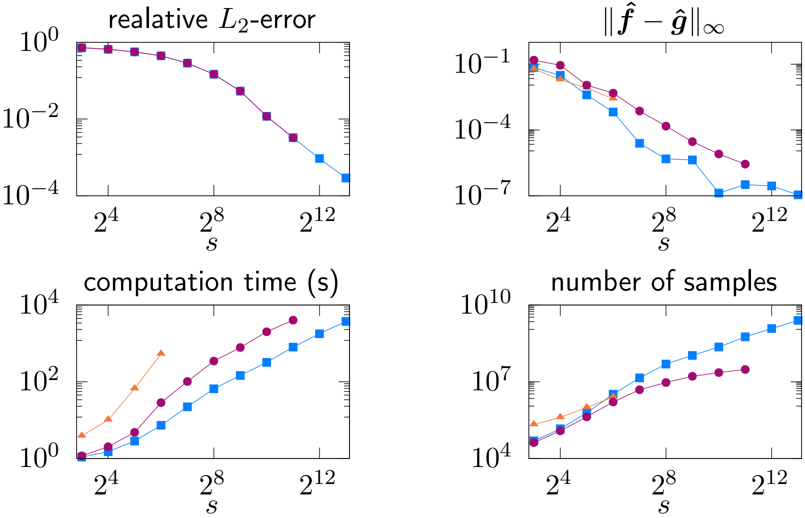
Numerical experiment

- 10-dimensional test function of B-splines

$$f(\boldsymbol{x}) \coloneqq \prod_{t\in\{1,3,8\}} N_2(x_t) + \prod_{t\in\{2,5,6,10\}} N_4(x_t) + \prod_{t\in\{4,7,9\}} N_6(x_t),$$

- candidate set $\Gamma = \{\boldsymbol{k} \in \mathbb{Z}^{10} : \prod_{t=1}^{10} \max\{1, |k_t|\} \leq 2^8\}$ with $|\Gamma| = 8\,827\,703\,433$
- $r = 5$ detection iterations
- target sparsities $s \in \{2^3, \ldots, 2^{13}\}$

Numerical experiment



■: rank-1 lattice, ●: subsampled rank-1 lattice, ▲: random points