Constructive subsampling of finite frames with applications in optimal function recovery

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Motivation

What if...

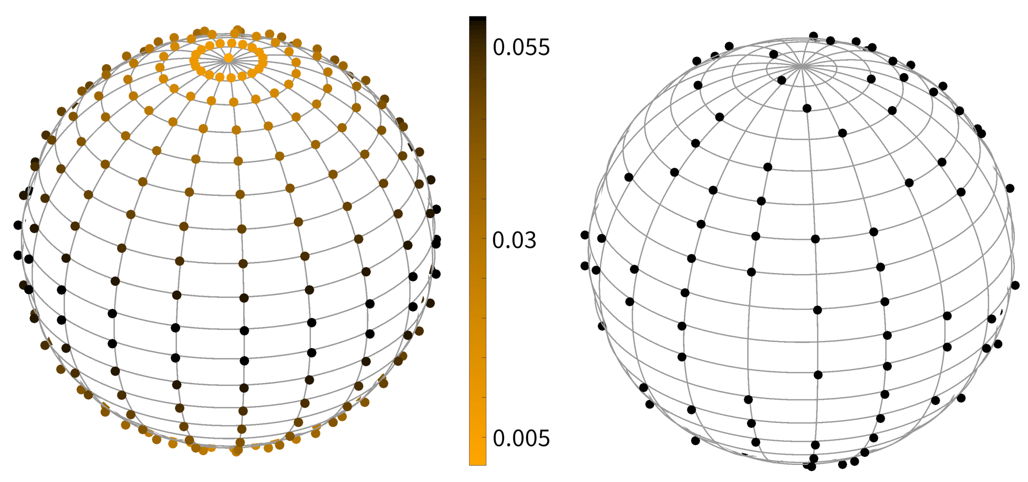
given: • an L_2-Marcinkiewicz-Zygmund inequality for V, i.e., points $\boldsymbol{x}^1, \dots, \boldsymbol{x}^M$ and weights $\omega_1, \dots, \omega_M$ satisfying

$$A\|f\|_{L_2}^2 \le \sum_{i=1}^{M} \omega_i |f(\boldsymbol{x}^i)|^2 \le B\|f\|_{L_2}^2 \quad \forall f \in V\,, \tag{MZ}$$

• $b > 1$

we find: $J \subset [M]$, $|J| \le \lceil bm \rceil$ with

$$A'\|f\|_{L_2}^2 \le \sum_{i \in J} |f(\boldsymbol{x}^i)| \quad \forall f \in V\,.$$



Least squares

For a function space V, points $\boldsymbol{X}_n = (\boldsymbol{x}^i)_{i \in J}$, and weights w_m, we define the **least squares approximation**

$$S_{V,w_m}^{\boldsymbol{X}_n} f = \arg\min_{g \in V} \sum_{i \in J} w_m(\boldsymbol{x}^i) |g(\boldsymbol{x}^i) - f(\boldsymbol{x}^i)|^2\,.$$

Theorem (function recovery) [1, 3]

Let $H(K)$ be a RKHS, $\sup_{\boldsymbol{x} \in D} K(\boldsymbol{x}, \boldsymbol{x}) < \infty$, $\mathrm{Id}\colon H(K) \hookrightarrow L_2(D, \nu)$ compact with singular values $\sigma_1 \ge \sigma_2 \ge \dots \ge 0$ and $1 + \frac{m}{10} \le b \le 2$. Then there is a function space $V \subset H(K)$ and a point set $\boldsymbol{X}_n$ with $|\boldsymbol{X}_n| \le \lceil bm \rceil$ with

$$\sup_{\|f\|_{H(K)} \le 1} \|f - S_{V,w_m}^{\boldsymbol{X}_n} f\|_{L_2(D,\nu)}^2 \le C \frac{\log(m/p)}{(b-1)^3} \Big(\sigma_{m+1}^2 + \frac{7}{m} \sum_{k=m+1}^{\infty} \sigma_k^2 \Big)$$

with probability exceeding $1 - \frac{3}{2}p$.

Equivalent formulation

Let $\eta_0, \dots, \eta_{m-1}$ be an ONB of a function space V and

$$\boldsymbol{L} = \begin{bmatrix} \eta_0(\boldsymbol{x}^1) & \dots & \eta_{m-1}(\boldsymbol{x}^1) \\ \vdots & \ddots & \vdots \\ \eta_0(\boldsymbol{x}^M) & \dots & \eta_{m-1}(\boldsymbol{x}^M) \end{bmatrix} = \begin{bmatrix} (\boldsymbol{y}^1)^* \\ \vdots \\ (\boldsymbol{y}^M)^* \end{bmatrix}.$$

Then we have the following equivalence:

$$\text{(MZ)} \Leftrightarrow A \le \sigma_{\min}^2(\boldsymbol{L}) \le \sigma_{\max}^2(\boldsymbol{L}) \le B$$
$\Leftrightarrow \boldsymbol{y}^1, \dots, \boldsymbol{y}^M$ **frame**, i.e.,

$$A\|\boldsymbol{a}\|_2^2 \le \sum_{i=1}^{M} |\langle \boldsymbol{a}, \boldsymbol{y}^i \rangle|^2 \le B\|\boldsymbol{a}\|_2^2 \quad \forall \boldsymbol{a} \in \mathbb{C}^m.$$

$\Rightarrow$ The problem of subsampling L_2-MZ inequalities is equivalent to subsampling of frames.

Frame subsampling

Theorem (existence) [1]

Let $\boldsymbol{y}^1, \dots, \boldsymbol{y}^M \in \mathbb{C}^m$ frame with $\|\boldsymbol{y}^i\|_2^2 \le \frac{m}{M}$ and $b > \frac{1642}{A}$. Then there exists $J \subset [M]$, $|J| \le bm$, with

$$12\|\boldsymbol{a}\|_2^2 \le \frac{1}{m} \sum_{i \in J} |\langle \boldsymbol{a}, \boldsymbol{y}^i \rangle|^2 \le 1642 \frac{B}{A} \|\boldsymbol{a}\|_2^2$$

for all $\boldsymbol{a} \in \mathbb{C}^m$.

- ☺ The desired subframe exists!
- ☹ The approach is non-constructive as it is based on the Kadison-Singer theorem equivalent to the Feichtinger conjecture.
- ☹ The oversampling factor b cannot be choosen close to one.

Theorem (weighted construction) [2]

Let $\boldsymbol{y}^1, \dots, \boldsymbol{y}^M \in \mathbb{C}^m$ form a 1-tight frame and $b > 1$. Then the BSS algorithm computes $J \subset [M]$ with $|J| \le \lceil bm \rceil$ and $s_i \ge 0$, s.t.

$$\|\boldsymbol{a}\|_2^2 \le \sum_{i \in J} s_i |\langle \boldsymbol{a}, \boldsymbol{y}^i \rangle|^2 \le \frac{(\sqrt{b}+1)^2}{(\sqrt{b}-1)^2} \|\boldsymbol{a}\|_2^2$$

for all $\boldsymbol{a} \in \mathbb{C}^m$.

- ☺ The approach is constructive. Starting with an empty frame, elements are carefully added whilst watching the bounds.
- ☹ It only works for 1-tight frames.
- ☹ We introduce further weights s_i.

Theorem (constructive unweighted subsampling) [3]

Let $\boldsymbol{y}^1, \dots, \boldsymbol{y}^M \in \mathbb{C}^m$ with $m \in \mathbb{N}_{\ge 10}$. Further, take $b \ge 1 + \frac{10}{m}$ and assume $M \ge \lceil bm \rceil$. By applying BSS to $\tilde{\boldsymbol{y}}^1, \dots, \tilde{\boldsymbol{y}}^M$, we obtain indices $J' \subset [M]$ with $|J'| \le \lceil bm \rceil$ such that

$$\frac{1}{M} \sum_{i=1}^{M} |\langle \boldsymbol{a}, \boldsymbol{y}^i \rangle|^2 \le \frac{432 b^3}{(b-1)^3} \frac{1}{m} \sum_{i \in J'} |\langle \boldsymbol{a}, \boldsymbol{y}^i \rangle|^2$$

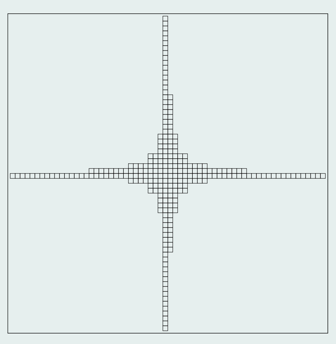
for all $\boldsymbol{a} \in \mathbb{C}^m$.

- ☺ Constructive unweighted subframe as desired.

Numerics

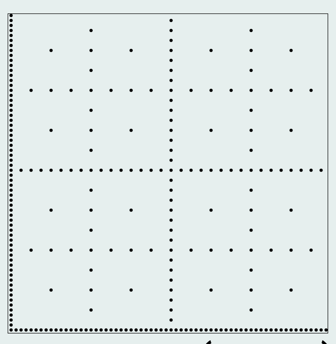
Subsampling of a Fourier matrix

frequencies I



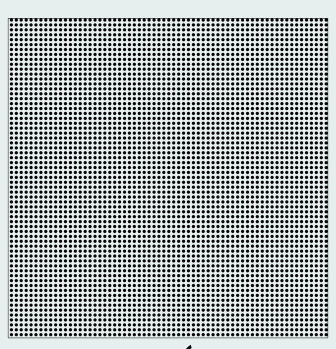
$m = 256$

sparse grid



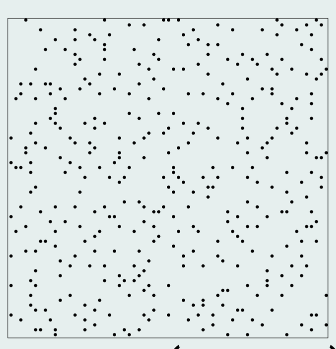
$M = 256$ $(b = 1)$
$A = 0.043$, $B = 16$

initial points



$M = 4225$ $(b \approx 16.5)$
$A = 1$, $B = 1$

BSS subsampling

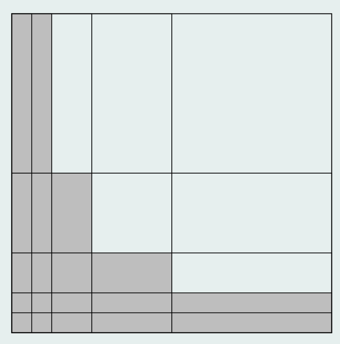


$n = 384$ $(b \approx 1.5)$
$A = 0.067$, $B = 2.582$

- Fourier matrix $\boldsymbol{Y} = [\exp(2\pi \mathrm{i} \langle \boldsymbol{k}, \boldsymbol{x} \rangle)]_{\boldsymbol{k} \in I, \boldsymbol{x} \in \boldsymbol{X}}$
- Sparse grids are exact for the dyadic hyperbolic cross with oversampling $b = 1$, i.e., $n = m$. But they suffer bad condition of $\boldsymbol{Y}$.
- BSS with a slight oversampling allows to remedy this bad condition.

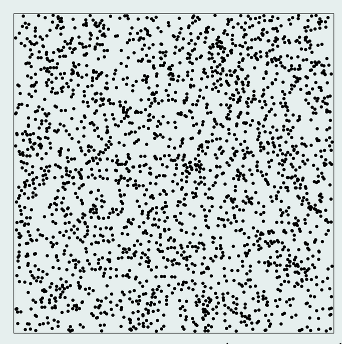
Subsampling of a wavelet matrix

frequencies I



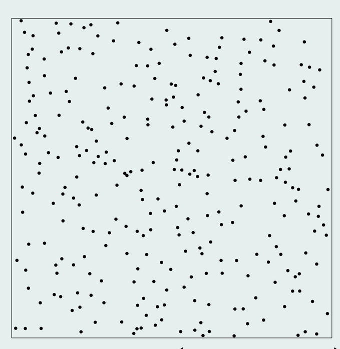
$m = 192$

initial points



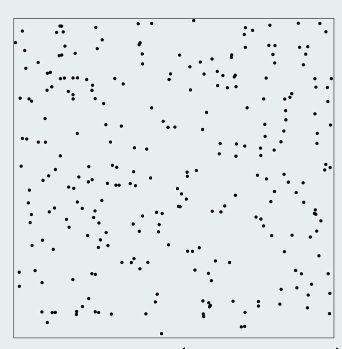
$M = 2400$ $(b = 2)$
$A = 0.017$, $B = 1.02$

BSS subsampling



$n = 288$ $(b \approx 1.5)$
$A = 0.003$, $B = 1.056$

random subsampling



$n = 288$ $(b \approx 1.5)$
$A = 0$, $B = 1.134$

- Wavelet matrix $\boldsymbol{Y}$ with the Chui-Wang wavelets $\varphi_{\boldsymbol{j},\boldsymbol{k}}$
- $\boldsymbol{X}$: $2\,400$ randomly drawn points
- BSS computes an invertible submatrix.
- For comparison: random subsampling fails.

[1] N. Nagel, M. Schäfer, and T. Ullrich. A new upper bound for sampling numbers. *Found. Comp. Math., 2021.*

[2] J. D. Batson, D. A. Spielman, and N. Srivastava. Twice-Ramanujan sparsifiers. *SIAM J. Comput., 2012.*

[3] F. Bartel, M. Schäfer, and T. Ullrich. Constructive subsampling of finite frames with applications in optimal function recovery. *arXiv preprint, 2022.*

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Software
https://github.com/felixbartel/BSSsubsampling.jl