On the reconstruction of functions from values at subsampled quadrature points

Felix Bartel   Lutz Kämmerer   Daniel Potts   Tino Ullrich

Approximation and geometry in high dimensions 2022

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Mathematik!
TU Chemnitz

Function recovery from samples

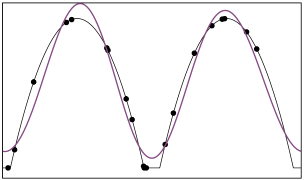
Given:

- $L_2 = L_2(D, \nu)$
- reproducing kernel Hilbert space $H(K) \hookrightarrow L_2$

Goal:

- find good points $\boldsymbol{X} = \{\boldsymbol{x}^1, \dots, \boldsymbol{x}^n\} \subset D$ and a sampling recovery operator $S_{\boldsymbol{X}} \colon H(K) \to L_2$ with small **worst-case error**

$$\sup_{\|f\|_{H(K)} \le 1} \|f - S_{\boldsymbol{X}} f\|_{L_2}$$



$f(\boldsymbol{x})$ ——

$(\boldsymbol{x}^i, f(\boldsymbol{x}^i))$ •

$(S_{\boldsymbol{X}} f)(\boldsymbol{x})$ ——

Reproducing kernel Hilbert space

- assume finite trace $\int_D K(\boldsymbol{x},\boldsymbol{x})\,\mathrm{d}\nu(\boldsymbol{x}) < \infty$
- embedding $\mathrm{Id}_{K,\nu}\colon H(K) \hookrightarrow L_2$ has the representation

$$\mathrm{Id}_{K,\nu}(f) = \sum_{k=0}^{\infty} \sigma_k \langle f, e_k\rangle_{H(K)} \eta_k$$

with $e_k = \sigma_k \eta_k$ and

  - **singular values** $\sigma_0 \geq \sigma_1 \geq \cdots \geq 0$
  - **right singular functions** $e_0, e_1, \dots$ forming an ONS in $H(K)$
  - **left singular functions** $\eta_0, \eta_1, \dots$ forming an ONS in L_2

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  - **left singular functions** $\eta_0, \eta_1, \dots$ forming an ONS in L_2
- Kolmogorov width

$$d_m(H(K)) := \inf_{\substack{\ell_0, \dots, \ell_{m-1} \colon H(K) \to \mathbb{C} \\ \varphi_0, \dots, \varphi_{m-1} \in L_2}} \sup_{f \in H(K)} \Big\| f - \sum_{k=0}^{m-1} \ell_k(f) \varphi_k \Big\|_{L_2}$$

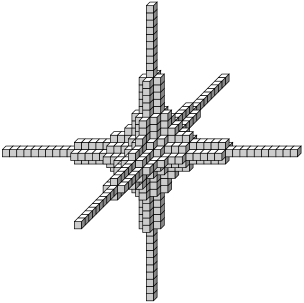
$$= \sup_{f \in H(K)} \Big\| f - \sum_{k=0}^{m-1} \langle f, \eta_k \rangle_{L_2} \eta_k \Big\|_{L_2} = \sigma_m$$

Sobolev spaces with dominating mixed smoothness on $\mathbb{T}^d$

- $H^s_{\text{mix}}(\mathbb{T}^d) = \{f \in L_2 : \|f\|_{H^s_{\text{mix}}} < \infty\}$ $(s > 1/2)$ with

$$\langle f, g\rangle_{H^s_{\text{mix}}} := \sum_{\boldsymbol{j}\in\{0,s\}^d} \langle D^{(\boldsymbol{j})} f, D^{(\boldsymbol{j})} g\rangle_{L_2}$$

- singular functions $\eta_{\boldsymbol{k}} = \exp(2\pi \mathrm{i}\langle \boldsymbol{k}, \cdot\rangle)$
- singular values $\sigma_{\boldsymbol{k}} = \prod_{j=1}^{d} (1 + (2\pi|k_j|)^{2s})^{-1/2}$
- Kolmogorov width $d_m(H^s_{\text{mix}}) = m^{-s}(\log m)^{(d-1)s}$



I_R^{hc}

Least squares

- assume $n \geq m$
- ansatz $f(\boldsymbol{x}) = \sum_{k=0}^{m-1} c_k \eta_k(\boldsymbol{x})$ with $\boldsymbol{c} = (c_0, \ldots, c_{m-1})^\mathsf{T}$ solving

$$\begin{pmatrix} \eta_0(\boldsymbol{x}^1) & \cdots & \eta_{m-1}(\boldsymbol{x}^1) \\ \vdots & \ddots & \vdots \\ \eta_0(\boldsymbol{x}^n) & \cdots & \eta_{m-1}(\boldsymbol{x}^n) \end{pmatrix} \begin{pmatrix} c_0 \\ \vdots \\ c_{m-1} \end{pmatrix} = \begin{pmatrix} f(\boldsymbol{x}^1) \\ \vdots \\ f(\boldsymbol{x}^n) \end{pmatrix}$$

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for a weight matrix $\boldsymbol{W} = \mathrm{diag}(\omega_1, \ldots, \omega_n)$

Least squares

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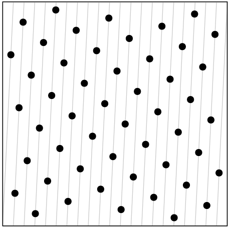
$$\Bigg\| \underbrace{\begin{pmatrix} \eta_0(\boldsymbol{x}^1) & \cdots & \eta_{m-1}(\boldsymbol{x}^1) \\ \vdots & \ddots & \vdots \\ \eta_0(\boldsymbol{x}^n) & \cdots & \eta_{m-1}(\boldsymbol{x}^n) \end{pmatrix}}_{=:\boldsymbol{L}} \begin{pmatrix} c_0 \\ \vdots \\ c_{m-1} \end{pmatrix} - \underbrace{\begin{pmatrix} f(\boldsymbol{x}^1) \\ \vdots \\ f(\boldsymbol{x}^n) \end{pmatrix}}_{=:\boldsymbol{f}} \Bigg\|_{\boldsymbol{W}}^2 \to \min$$

  for a weight matrix $\boldsymbol{W} = \mathrm{diag}(\omega_1, \ldots, \omega_n)$
- the solution is given by

$$\boldsymbol{c} = (\boldsymbol{L}^* \boldsymbol{W} \boldsymbol{L})^{-1} \boldsymbol{L}^* \boldsymbol{W} \boldsymbol{f} \quad \text{and} \quad (S_{\boldsymbol{X}} f)(\boldsymbol{x}) = \sum_{k=0}^{m-1} c_k \eta_k(\boldsymbol{x})$$

Structured points: rank-1 lattice

- $D = \mathbb{T}^d$, $\{\eta_0, \dots, \eta_{m-1}\} = \{\exp(2\pi\mathrm{i}\langle \boldsymbol{k}, \cdot\rangle\}_{\boldsymbol{k}\in I}$
- $\boldsymbol{X} = \Lambda(\boldsymbol{z}, n) = \left\{\frac{i}{n}\boldsymbol{z} \bmod \boldsymbol{1}, i = 0, \dots, n-1\right\}$
- [Nuyens '07], [Kämmerer, Potts, Volkmer '15]:
  - given I, there exist algorithms that find $\boldsymbol{z}$ and n such that $\boldsymbol{L}^*\boldsymbol{W}\boldsymbol{L} = \boldsymbol{I}$
  - multiplication with $\boldsymbol{L}$ can be carried out with the LFFT in $\mathcal{O}(n \log n)$



$z = (1, 21)^\mathsf{T}, n = 55$

Theorem

[Byrenheid, Kämmerer, T. Ullrich, Volkmer '17]

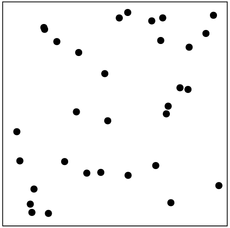
$$n^{-s/2} \lesssim \sup_{\|f\|_{H^s_{\text{mix}}} \le 1} \|f - S_{\boldsymbol{X}} f\|_{L_2} \lesssim n^{-s/2} (\log n)^{\frac{d-2}{2}s + \frac{d-1}{2}}$$

Random points

- $\boldsymbol{X} = \{\boldsymbol{x}^1, \ldots, \boldsymbol{x}^n\}$ drawn randomly i.i.d. w.r.t. Lebesgue measure such that

$$n \sim m \log m$$

- no fast matrix-vector multiplication due to lack of structure

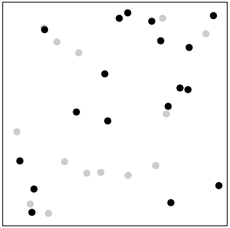


Theorem [M. Ullrich, Krieg '19], [Kämmerer, T. Ullrich, Volkmer '19]

$$n^{-s}(\log n)^{(d-1)s} \lesssim \sup_{\|f\|_{H^s_{\text{mix}}} \leq 1} \|f - S_{\boldsymbol{X}} f\|_{L_2} \lesssim n^{-s}(\log n)^{(d-1)s+s}$$

Subsampled random points

- $\boldsymbol{X} = \{\boldsymbol{x}^1, \ldots, \boldsymbol{x}^n\}$ drawn randomly i.i.d. w.r.t. Lebesgue measure such that $n \sim m \log m$
- there exist $\boldsymbol{X}' = \{\boldsymbol{x}^{i_1}, \ldots, \boldsymbol{x}^{i_{n'}}\} \subset \boldsymbol{X}$ subsampled points suitable for reconstruction with
$$n' \in \mathcal{O}(|I|)$$
- no fast matrix-vector multiplication



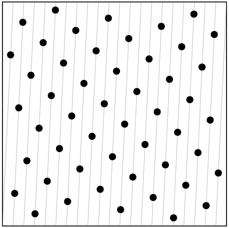
Theorem [Nagel, Schäfer, T. Ullrich '21], [B., Schäfer, T. Ullrich '22]

$$n^{-s}(\log n)^{(d-1)s} \lesssim \sup_{\|f\|_{H(K)} \le 1} \|f - S_{\boldsymbol{X}} f\|_{L_2} \lesssim n^{-s}(\log n)^{(d-1)s+1/2}$$

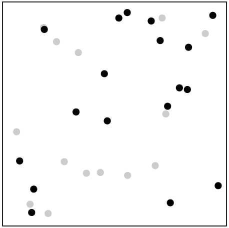
Theorem [Dolbeaut, Krieg, M. Ullrich '22]

$$\sup_{\|f\|_{H(K)} \le 1} \|f - S_{\boldsymbol{X}} f\|_{L_2} \sim n^{-s}(\log n)^{(d-1)s}$$

Goal: combine both advantages



structure for **fast algorithms**

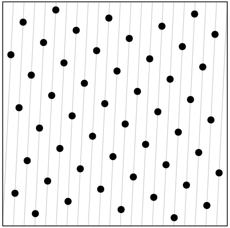


optimal rates

Approach (motivated by [Kunis, Rauhut '08])

1

$\boldsymbol{X}_{\mathsf{MZ}}$: good deterministic set of M points



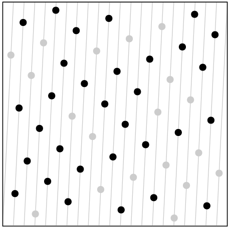
$\boldsymbol{X}_{\mathsf{MZ}} = \{\boldsymbol{x}^1, \ldots, \boldsymbol{x}^M\}$

$\boldsymbol{W}_{\mathsf{MZ}} \in [0, \infty)^{M \times M}$

random $\longrightarrow$ subsampling

2

$\boldsymbol{X}$: randomly subsample n points with $n \sim m \log m$



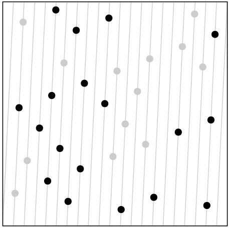
$\boldsymbol{X} = \{\boldsymbol{x}^i\}_{i \in J}$

$\boldsymbol{W} \in [0, \infty)^{|\boldsymbol{X}| \times |\boldsymbol{X}|}$

BSS $\longrightarrow$ subsampling

3

$\boldsymbol{X}'$: BSS subsample n' points with $n' \sim m$



$\boldsymbol{X}' = \{\boldsymbol{x}^i\}_{i \in J'}$

$\boldsymbol{W}' \in [0,\infty)^{|\boldsymbol{X}'| \times |\boldsymbol{X}'|}$

Starting points: MZ inequalities and exact quadrature

- points $\boldsymbol{X}_{\mathsf{MZ}}$ and weights $\omega_1, \dots, \omega_M$ fulfill a L_2 **Marcinkiewicz-Zygmund inequality** for $V \subset L_2$, iff

$$A\|g\|_{L_2}^2 \leq \sum_{i=1}^{M} \omega_i |g(\boldsymbol{x}^i)|^2 \leq B\|g\|_{L_2}^2 \quad \text{for all} \quad g \in V$$

- the full system matrix is well-conditioned

$$A \leq \sigma_{\min}^2(\boldsymbol{W}^{1/2}\boldsymbol{L}) \leq \sigma_{\max}^2(\boldsymbol{W}^{1/2}\boldsymbol{L}) \leq B$$

Starting points: MZ inequalities and exact quadrature

- **MZ inequalities are widly available**: [Mhaskar, Narcowich, Ward '01], [Keiner, Kunis, Potts '07], [Filbir, Mhaskar '11], [Müller-Gronbach, Novak, Ritter '12], [Temlyakov '18], [Gröchenig '20], [Filbir, Hielscher, Jahn, T. Ullrich '22], ...

Lemma

The MZ inequality on V with $A = B$ is equivalent to the exact integration

$$\sum_{i=1}^{M} \omega_i g(\boldsymbol{x}^i)\overline{h(\boldsymbol{x}^i)} = \int_D g(\boldsymbol{x})\overline{h(\boldsymbol{x})}\,\mathrm{d}\nu(\boldsymbol{x}) \quad \text{for all} \quad g, h \in V\,.$$

- **exact integration is widly available:** [Nuyens, Cools '06], [Kämmerer, Potts, Volkmer '15], [Trefethen '19], ...

Incorporate fast algorithms

- big point set $\boldsymbol{X}_{\mathsf{MZ}} = \{\boldsymbol{x}^1, \ldots, \boldsymbol{x}^M\}$ with $\mathcal{O}(M \log M)$ algorithm
- small point set $\boldsymbol{X} = \{\boldsymbol{x}^{j_1}, \ldots, \boldsymbol{x}^{j_n}\} \subset \boldsymbol{X}_{\mathsf{MZ}}$
- we may use the algorithm for the big point set

$$\boldsymbol{L}_{\boldsymbol{X}} = \boldsymbol{P}\boldsymbol{L}_{\boldsymbol{X}_{\mathsf{MZ}}} \quad \text{where} \quad \boldsymbol{P} = \begin{pmatrix} 1 & 0 & 0 & \ldots & 0 & 0 & 0 \\ 0 & 0 & 1 & \ldots & 0 & 0 & 0 \\ & \vdots & & & & \vdots & \\ 0 & 0 & 0 & \ldots & 0 & 1 & 0 \end{pmatrix} \begin{matrix} 1 \\ 2 \\ \vdots \\ n \end{matrix}$$

$$\begin{matrix} j_1 & & j_2 & & & j_n & \end{matrix}$$

- with $n \sim m \log m$ and $M \sim m^2$ we obtain the same complexity as for naive matrix-vector multiplication

$$\mathcal{O}(mn) = \mathcal{O}(M \log M)$$

Central theorem

Assumptions 1 of 2

Let

- $H(K)$ separable RKHS with $\sup_{\boldsymbol{x}\in D} K(\boldsymbol{x},\boldsymbol{x}) < \infty$ and $\int_D K(\boldsymbol{x},\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} < \infty$,
- σ_k and η_k singular values and functions of $\mathrm{Id}_{K,\nu}\colon H(K) \hookrightarrow L_2(\nu)$
- $I = \{0,\dots,m-1\} \subset I_M$
- $\boldsymbol{X}_{\mathsf{MZ}}$ points and $\boldsymbol{W}_{\mathsf{MZ}}$ weights fulfilling an L_2-Marcinkiewicz-Zygmund inequality with constants A and B for $V = \mathrm{span}\{\eta_k\}_{k\in I_M}$,

Central theorem

Assumptions 2 of 2

- $n := \left\lceil \frac{72B}{A} |I| \log |I| \right\rceil$,
- $\boldsymbol{X} = \{\boldsymbol{x}^i\}_{i \in J}$, $|J| = n$ points drawn i.i.d. from $\boldsymbol{X}_{\mathrm{MZ}}$ w.r.t. the **discrete density weights** $\varrho_i = \omega_i \varrho(\boldsymbol{x}^i)$ with

$$\varrho(\boldsymbol{x}^i) = \frac{\sum_{k \in I} |\eta_k(\boldsymbol{x}^i)|^2}{3 \sum_{j=1}^{M} \omega_j \sum_{k \in I} |\eta_k(\boldsymbol{x}^j)|^2} + \frac{\sum_{k \in I_M \setminus I} |e_k(\boldsymbol{x}^i)|^2}{3 \sum_{j=1}^{M} \omega_j \sum_{k \in I_M \setminus I} |e_k(\boldsymbol{x}^j)|^2} + \frac{\sum_{k \notin I_M} |e_k(\boldsymbol{x}^i)|^2}{3 \sum_{j=1}^{M} \omega_j \sum_{k \notin I_M} |e_k(\boldsymbol{x}^j)|^2},$$

- $b > 1 + \frac{10}{|I|}$.

- uses spectral properties of the embedding
- discrete version of [M. Ullrich, Krieg '19], [Kämmerer, T. Ullrich, Volkmer '19]
- $\varrho(\boldsymbol{x}^i)$ neglectable for BOS, i.e., $\sup_{x \in D} |\eta_k(x)| \leq B$ for all k

Central theorem

Theorem [B, Kämmerer, Potts, T. Ullrich '22]

Given the assumptions we construct $\boldsymbol{X}' \subset \boldsymbol{X}_{\mathsf{MZ}}$ with $|\boldsymbol{X}'| \le \lceil b|I|\rceil$ such that

$$\sup_{\|f\|_{H(K)}\le 1} \|f - S_I^{\boldsymbol{X}'} f\|_{L_2}^2 \le C_{A,B,b} \log|I| \Bigg(\sup_{k\notin I} \sigma_k^2 + \frac{1}{|I|} \sum_{k\in I_{\mathsf{MZ}}\setminus I} \sigma_k^2$$

$$+ \sup_{\|f\|_{H(K)}\le 1} \|f - S_{I_{\mathsf{MZ}}}^{\boldsymbol{X}_{\mathsf{MZ}}} f\|_{L_2}^2 + \frac{1}{|I|}\Big(\sum_{i=1}^{M} \omega_i\Big) \sup_{\|f\|_{H(K)}\le 1} \|f - P_{I_{\mathsf{MZ}}} f\|_{\ell_\infty(D)} \Bigg)$$

with probability larger than $1 - 6/|I|$ and $C_{A,B,b} = (\frac{B}{A})^2 (\frac{82\,b}{b-1})^3$.

Theorem for rank-1 lattices [B, Kämmerer, Potts, T. Ullrich '22]

Let
- $s > 1/2$,
- $I \subset I_{\mathsf{MZ}} \subset \mathbb{Z}^d$ hyperbolic cross frequency index sets, $|I| \geq 10$,
- $\boldsymbol{X}_{\mathsf{MZ}}$ reconstructing rank-1 lattice for I_{MZ} with M points,
- $b > 1 + \frac{10}{|I|}$.

Then we construct $\boldsymbol{X}' \subset \boldsymbol{X}_{\mathsf{MZ}}$ with $|\boldsymbol{X}'| \leq \lceil b|I| \rceil$ such that

$$\sup_{\|f\|_{H^s_{\mathsf{mix}}} \leq 1} \|f - S_I^{\boldsymbol{X}'} f\|_{L_2} \leq C_{d,s,b} \Big(|I|^{-s} (\log|I|)^{(d-1)s+1/2} + |I_{\mathsf{MZ}}|^{-s} (\log|I_{\mathsf{MZ}}|)^{(d-1)s+d/2} \Big)$$

with probability $1 - 6/|I|$.

Central theorem for rank-1 lattices

Theorem for rank-1 lattices [B, Kämmerer, Potts, T. Ullrich '22]

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Then we construct $\boldsymbol{X}' \subset \boldsymbol{X}_{\mathrm{MZ}}$ with $|\boldsymbol{X}'| \leq \lceil b|I| \rceil$ such that

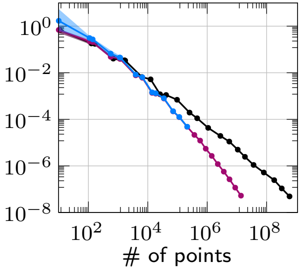
$$\sup_{\|f\|_{H^s_{\mathrm{mix}}} \leq 1} \|f - S_I^{\boldsymbol{X}'} f\|_{L_2} \leq C'_{d,s,b} \Big(|\boldsymbol{X}'|^{-s} (\log |\boldsymbol{X}'|)^{(d-1)s+1/2} + M^{-s/2} (\log M)^{\frac{(d-2)s+d}{2}} \Big)$$

with probability $1 - 6/|I|$.

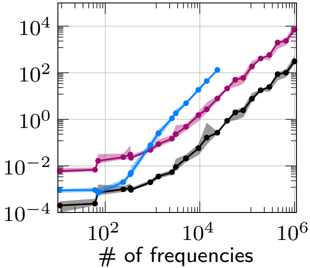
Numerical experiment for random subsampling

- $d = 5$, f tensorized bumb function in $H^s_{\text{mix}}(\mathbb{T}^5)$ for $s < 3/2$
- initial $\boldsymbol{X}_{\text{MZ}}$: fast probabilistic CBC rank-1 lattice construction [Kämmerer '20]

L_2-error



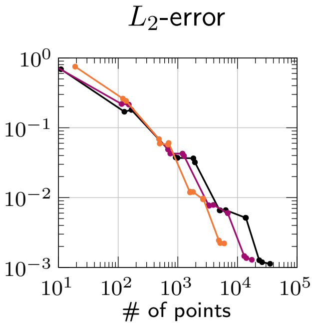
time in seconds

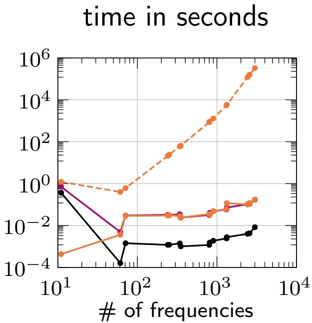


black: full rank-1 lattice
azure: continuously random
magenta: randomly subsampled

Numerical experiment for random + BSS subsampling

- $d = 5$, f tensorized bumb function in $H^s_{\text{mix}}(\mathbb{T}^5)$ for $s < 3/2$
- initial $\boldsymbol{X}_{\text{MZ}}$: fast probabilistic CBC rank-1 lattice construction [Kämmerer '20]

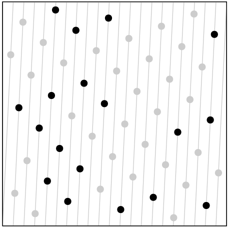




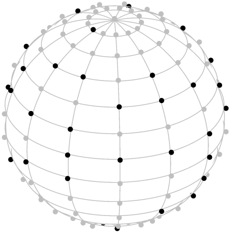
black: full rank-1 lattice
magenta: randomly subsampled
orange: BSS subsampled

Conclusion

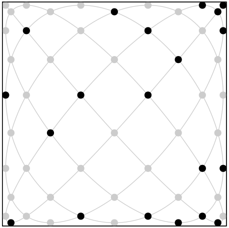
$\mathbb{T}^d$



$\mathbb{S}^d$



$[0,1]^d$



- we proposed **random + BSS** subsampling procedure for **MZ inequalities**
  - achieving the near optimal convergence rates ✓
  - making it possible to utilize fast algorithms ✓

- [B., Kämmerer, Potts, T. Ullrich] on **arXiv:2208.13597** "On the reconstruction of functions from values at subsampled quadrature points"